1. Show that of all the rectangles with a given area, the one with smallest perimeter is a square.

\[ A = x y \Rightarrow y = \frac{A}{x} \]

Perimeter \( P(x) = 2x + 2y = 2x + 2 \frac{A}{x} \)

\[ P'(x) = 2 - 2 \frac{A}{x^2} = \frac{2(x^2 - A)}{x^3} \]

\[ P'(x) = 0 \text{ when } x^2 - A = 0 \Rightarrow x = \sqrt{A} \]

\[ \Rightarrow \text{ minimum} \]

Sides of rectangle are \( \sqrt{A} \), \( A/\sqrt{A} = \sqrt{A} \), so is square.

2. Show that of all the rectangles with a given perimeter, the one with the greatest area is a square.

\[ p = 2x + 2y \Rightarrow 2y = p - 2x \Rightarrow y = \frac{1}{2}p - x \]

Area \( A(x) = x \left( \frac{1}{2}p - x \right) = \frac{1}{2}px - x^2 \)

\[ A'(x) = \frac{1}{2}p - 2x \]

\[ A'(x) = 0 \text{ when } \frac{1}{2}p - 2x = 0 \Rightarrow x = \frac{1}{4}(p) \]

\[ A''(x) = -2 < 0, \text{ so crit. } p+ \text{ is abs. max.} \]

Sides are \( \frac{1}{4}p \) and \( \frac{1}{2}p - \frac{1}{4}p = \frac{1}{4}p \), so is square.

3. Show that of all the isosceles triangles with a given perimeter, the one with the greatest area is equilateral.

\[ s^2 = h^2 + \left( \frac{b}{2} \right)^2 \Rightarrow h^2 = s^2 - \frac{b^2}{4} \Rightarrow h = \sqrt{s^2 - \frac{b^2}{4}} \]

Area \( A = \frac{1}{2}bh = \frac{1}{2}b \sqrt{s^2 - \frac{b^2}{4}} \)

Perimeter \( p = 2s + b \Rightarrow s = \left( p - b \right)/2 \)

\[ \Rightarrow A(b) = \frac{1}{2}b \sqrt{\left( p - b \right)^2/4 - b^2/4} = \frac{b}{4} \sqrt{p^2 - 2pb} \]

Product rule

\[ A'(b) = \frac{1}{4\sqrt{p^2 - 2pb}} \left( \frac{2b^2}{4} - \frac{bp^2}{4\sqrt{p^2 - 2pb}} \right) = \frac{p^2 - 2pb - bp}{4\sqrt{p^2 - 2pb}} = \frac{p^2 - 3pb}{4\sqrt{p^2 - 2pb}} \]

\[ A'(b) = 0 \text{ when } p^2 - 3pb = 0 \Rightarrow p^2 = 3pb \Rightarrow p = 3b \Rightarrow b = \frac{1}{3}p \]

Sides are \( \frac{1}{3}p, \frac{1}{3}p, \frac{1}{3}p \), so is equilateral.
4. A rectangular storage container with an open top is to have a volume of 10 m$^3$. The length of its base is twice the width. Material for the base costs $10 per square meter. Find the cost of materials for the cheapest such container.

$$V = lwh \Rightarrow 10 = (2w)(w)h \Rightarrow h = \frac{5}{w^2}$$

Cost: $10(\text{area of base}) + 6(\text{area of sides})$

$$C(w) = 10(2w)(w) + 6\left[2lhw + 2(wh)\right]$$
$$= 20w^2 + 36wh$$
$$= 20w^2 + 36w\left(\frac{5}{w^2}\right) = 20w^2 + \frac{180}{w}$$

$$C'(w) = 40w - \frac{180}{w^2} = \frac{40w^3 - 180}{w^2} = \frac{40\left(w^3 - \frac{9}{2}\right)}{w^2}$$

$$C'(w) = 0 \text{ when } w^3 - \frac{9}{2} = 0 \Rightarrow w = \sqrt[3]{\frac{9}{2}}$$

Therefore, minimum cost is

$$C\left(\sqrt[3]{\frac{9}{2}}\right) = 20\left(\sqrt[3]{\frac{9}{2}}\right)^2 + \frac{180}{\sqrt[3]{\frac{9}{2}}} \approx \$163.54$$

5. Find the point on the curve $y = \sqrt{x}$ that is closest to the point $(3,0)$. Hint: If the distance is minimized, the square of the distance will be minimized.

$$d = \sqrt{(x-3)^2 + (\sqrt{x}-0)^2} \quad \text{(Pythagorean)}, \text{ point } (x, \sqrt{x})$$

$$S = (x-3)^2 + (\sqrt{x}-0)^2 = (x-3)^2 + x$$

$$S' = 2(x-3) + 1 = 2x - 5$$

$$S' = 0 \text{ when } 2x - 5 = 0 \Rightarrow 2x = 5 \Rightarrow x = \frac{5}{2}$$

$$S'' = 2 > 0, \text{ so critical point is minimum. }$$

Thus, the minimum distance to $(3,0)$ occurs at

$$\left(\frac{5}{2}, \sqrt[3]{\frac{5}{2}}\right)$$
6. If you are offered one slice from a round pizza (in other words, a sector of a circle) and the slice must have a perimeter of 32 inches, what diameter pizza will reward you with the largest slice? Recall that the arc length of a circle with radius \( r \) and angle \( \theta \) is \( r \theta \).

\[
\text{perimeter} = 32 = 2r + r\theta \Rightarrow \theta = \frac{32 - 2r}{r}
\]

\[
\text{area} = A = \frac{1}{2} r^2 \theta \quad \text{(if whole circle is } \pi r^2, \text{ portion is } \frac{\theta}{2} r^2) = \frac{1}{2} r^2 \left( \frac{32 - 2r}{r} \right) = r(16 - r) = 16r - r^2, \quad 0 \leq r \leq 16
\]

\[
A'(r) = 16 - 2r \Rightarrow A'(r) = 0 \text{ when } r = 8 - \text{ why?}
\]

\[
A''(r) = -2 < 0 \quad \text{so crit pt. is maximum.}
\]

largest slice when \( r = 8 \) \Rightarrow \text{diameter} = 16 \text{ in}

7. Find an equation of the line through the point \((3,5)\) that cuts off the least area from the first quadrant.

find slope \( m \) (then use pt-slope). \[
y - 5 = m(x - 3) \]

\[
y = mx + (5 - 3m)
\]

\[
y_{\text{int}} = 5 - 3m
\]

\[
x_{\text{int}} = -\frac{5}{m} + 3
\]

\[
A(m) = \frac{1}{2}(5 + 3m)(-\frac{5}{m} + 3)
\]

\[
= 15 - \frac{25}{2m} - \frac{9}{2} m
\]

\[
A'(m) = \frac{25}{2m^2} - \frac{9}{2} \quad A'(m) = 0 \text{ when } \frac{25}{2m^2} = \frac{9}{2} \Rightarrow m^2 = \frac{25}{9}
\]

\[
m = -\frac{5}{3} \quad \text{since slope is negative.}
\]

\[
A''(m) = \frac{-25}{m^3} \quad > 0 \quad \text{so obs. min at crit. point.}
\]

equation of line is \[
y - 5 = -\frac{5}{3}(x - 3) \text{ or } y = -\frac{5}{3}x + 10
\]
8. A piece of wire 10m long is cut into two pieces. One piece is bent into a square and the other is bent into a circle. How should the wire be cut so that the total area enclosed by the two shapes is a) maximum? b) minimum?

\[
\text{Total area: } A(x) = \left( \frac{x}{4} \right)^2 + \pi \left( \frac{10 - x}{2\pi} \right)^2 = \frac{x^2}{16} + \frac{(10 - x)^2}{4\pi}
\]

\[
A'(x) = \frac{x}{8} - \frac{10 - x}{2\pi} = \left( \frac{1}{2\pi} + \frac{1}{8} \right)x - \frac{5}{\pi} \quad \Rightarrow \quad A''(x) = \left( \frac{1}{2\pi} + \frac{1}{8} \right) > 0 \quad \text{so critical point is minimum.}
\]

Area minimum when \( x = \frac{40}{4 + \pi} \).

Area max when \( x = 0 \) or \( x = 10 \):

\[
A(0) = \frac{25}{\pi}, \quad A(10) = \left( \frac{10}{4} \right)^2 = (2.5)^2 = 6.25
\]

\( A(0) > A(10), \) so \( \text{max at } x = 0 \).

9. Find the area of the largest trapezoid that can be inscribed in a circle of radius 1 and whose base is a diameter of the circle.

\[
\text{area} = A = \frac{1}{2} h \left( B + b \right) = h \left( \frac{B + b}{2} \right)
\]

\( h = y, \ B = 2, \ b = 2x \)

\[
A = \frac{1}{2} y \left( 2 + 2x \right) = y(1 + x)
\]

\( f(x, y) \) on circle rad 1 \( \Rightarrow \ x^2 + y^2 = 1 \Rightarrow y^2 = \frac{1-x^2}{4} \)

Area maximized when square of area maximized:

\[
T = A^2 = y^2 (1 + x)^2 = (1 - x^2)(1 + x)^2
\]

\[
T' = \begin{cases} \frac{1}{2} \left( 1 + x \right)^2 \left[ (1 - x^2) + (1 + x) \right] = -2 \left( 1 + x \right) \left[ 2x^2 + x - 1 \right] \\ -2 \left( 1 + x \right)(2x - 1)(x + 1) \end{cases}
\]

\( T' = 0 \) when \( 1 + x = 0 \) or \( 2x - 1 = 0 \) \( \Rightarrow x = -1 \) or \( x = \frac{1}{2} \).

\[
\text{max area} = A = y \left( 1 + x \right) = \sqrt{1 - \left( \frac{1}{2} \right)^2} = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}
\]

\[
\text{max area} = A = y \left( 1 + x \right) = \sqrt{1 - \left( \frac{1}{2} \right)^2} = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}
\]