Math 221  Practice Final Exam  Fall 2017

Name ___________________________  NetID: _______________________
                  UIN: ___________________

- 3 hours
- No calculators
- Show sufficient work

- Turn off all electronic devices and put away all items except a pen/pencil and an eraser.
- Remove hats and sunglasses.
- You must show sufficient work to justify each answer.
- While the test is in progress, we will not answer questions concerning the test material.
- Do not leave early unless you are at the end of a row.

150 points total.
1. (5 points each) Evaluate the limits without the use of derivatives. Show sufficient justification for each answer. For infinite limits, you must state if it is $\infty$ or $-\infty$.

(a) $\lim_{x \to \infty} \frac{(2x+1)^3}{6-5x^3} = \lim_{x \to \infty} \frac{8x^3 + 12x^2 + 6x + 1}{6 - 5x^3} = \lim_{x \to \infty} \frac{\left(8x^3 + 12x^2 + 6x + 1\right) \cdot \frac{1}{x^3}}{(6 - 5x^3) \cdot \frac{1}{x^3}} = \lim_{x \to \infty} \frac{8 + \frac{12}{x} + \frac{6}{x^2} + \frac{1}{x^3}}{\frac{6}{x^3} - 5} = \frac{8}{-5} = \frac{-8}{5}

(b) $\lim_{x \to 0} \left(\frac{2}{x} - \frac{32}{x^3 + 16x}\right) = \lim_{x \to 0} \left(\frac{2(x+16)}{x(x+16)} - \frac{32}{x(x+16)}\right) = \lim_{x \to 0} \left(\frac{2x + 32 - 32}{x(x+16)}\right) = \lim_{x \to 0} \left(\frac{2x}{x(x+16)}\right) = \lim_{x \to 0} \frac{2}{x+16} = \frac{2}{16} = \frac{1}{8}$
2. (10 points) Find all the asymptotes (both horizontal and vertical) of the function

\[ f(x) = \frac{2x^2 - 2}{x^2 - 6x + 5} \]

\[ \underline{\text{Vertical:}} \]

\[ f(x) = \frac{2(x^2 - 1)}{(x-5)(x-1)} = \frac{2 (x-1)(x+1)}{(x-5)(x-1)} \]

\[ \text{denom. is 0 at } x = 5 \text{ and } x = 1 \]

\[ \lim_{x \to 5^+} f(x) = \lim_{x \to 5^-} \frac{2x^2 - 2}{(x-5)(x-1)} \to 0^+ \cdot 0^- = \infty \]

\[ \Rightarrow \text{asymptote at } x = 5 \]

\[ \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} \frac{2 (x-1)(x+1)}{(x-5)(x-1)} \to 0^- \cdot 4 = 0^- = -\infty \]

\[ \Rightarrow \text{not an asymptote} \]

\[ \underline{\text{Horizontal:}} \]

\[ \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{2x^2 - 2}{x^2 - 6x + 5} = \lim_{x \to \infty} \frac{2 - \frac{2}{x^2}}{1 - \frac{6}{x} + \frac{5}{x^2}} = 2 \]

\[ \begin{aligned} &\text{Same for } x \to -\infty. \\ &\end{aligned} \]

\[ \underline{\text{Asymptotes:}} \]

\[ \text{Vertical at } x = 5 \]

\[ \text{Horizontal at } y = 2 \]
3. (10 points) Fill in the missing information in the following theorems.

**Intermediate Value Theorem** Suppose that

- \( f \) is \text{continuous} on \([a, b]\) and

- \( N \) is any number between \( f(a) \) and \( f(b) \), where \( f(a) \neq f(b) \)

Then there exists a number \( c \) in \((a, b)\) such that \( f(c) = N \).

**Mean Value Theorem** Let \( f \) be a function that satisfies the following two hypotheses.

(1) \( f \) is \text{continuous} on the closed interval \([a, b]\).

(2) \( f \) is \text{differentiable} on the open interval \((a, b)\).

Then there is a number \( c \) in \((a, b)\) such that \( f'(c) = \frac{f(b) - f(a)}{b - a} \).

**Mean Value Theorem For Integrals**

If \( f \) is \text{continuous} on \([a, b]\),

Then there is a number \( c \) in \([a, b]\) such that \( f(c) = \frac{1}{b-a} \int_a^b f(x) \, dx \).

**Fundamental Theorem of Calculus, Part 2**

If \( f \) is \text{continuous} on \([a, b]\), then \( \int_a^b f(x) \, dx = F(b) - F(a) \)

where \( F' \) is any \text{antiderivative} of \( f \).
4. (5 points) Find \( g'(x) \) given that \( g(x) = 5x^4 - 3\sqrt{x^2} + \sec^2(5x) + \frac{\ln x}{x} \).

\[
g'(x) = 20x^3 - \frac{2}{3}x^{-\frac{1}{3}} + 2\sec^2(5x)\sec(5x)\tan(5x)
+ \frac{x \cdot \frac{1}{x} - \ln(x)}{x^2}
\]

5. (5 points) Evaluate the limit.

\[
\lim_{x \to 1^+} x^{1/(x^4-1)} = \text{indeterminate } 1^\infty
\]

\[
= \lim_{x \to 1^+} e^{\ln(x^{1/4})}
= \lim_{x \to 1^+} \frac{1}{x^4-1} \ln x
= e^{\lim_{x \to 1^+} \frac{\ln x}{x^4-1}} = e^0 = 1
\]

\[
H \lim_{x \to 1^+} \frac{\sqrt{x}}{4x^3} = \frac{1}{4}
\]
6. (10 points) Let \( f(x) = 3x^2 + 6x + 10 \). Use the definition of a derivative as a limit to prove that \( f'(x) = 6x + 6 \). Show each step in your calculation and be sure to use proper terminology in each step of your proof.

\[
\begin{align*}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{3(x+h)^2 + 6(x+h) + 10 - 3x^2 - 6x - 10}{h} \\
&= \lim_{h \to 0} \frac{3x^2 + 6x + 3h^2 + 6x + 6h + 10 - 3x^2 - 6x - 10}{h} \\
&= \lim_{h \to 0} \frac{6xh + 3h^2 + 6h}{h} \\
&= \lim_{h \to 0} 6x + 3h + 6 \\
&= 6x + 6.
\end{align*}
\]
7. (10 points) Find the equation of a line perpendicular to the tangent line of

\[ f(x) = \frac{\sin(2x) \cos(x)}{1 - \sin^2(x)} \]

at \( x = \pi/3 \).

\[
\sin(2x) = 2 \sin x \cos x \\
1 - \sin^2 x = \cos^2 x
\]

\[
f(x) = \frac{2 \sin x \cos x \cos x}{\cos^2 x} = 2 \sin x
\]

\[
f'(x) = 2 \cos x
\]

\[
f'\left(\frac{\pi}{3}\right) = 2 \cos\left(\frac{\pi}{3}\right) = 2 \cdot \frac{1}{2} = 1
\]

perpendicular slope is -1

point: \( x = \pi/3 \)

\[
f(x) = 2 \sin \left(\frac{\pi}{3}\right) = 2 \frac{\sqrt{3}}{2} = \sqrt{3}
\]

\[
y - \sqrt{3} = -1 \left( x - \frac{\pi}{3} \right)
\]

\[
y = -x + \frac{\pi}{3} + \sqrt{3}
\]
8. (10 points) The half-life of titanium-44 is 63 years. What was the original mass in mg of a sample if it has decayed to 100 mg after 50 years?

\[ A = c e^{kt} \]

**Original mass** \( A(0) = c e^{k \cdot 0} = c \)

**Original is 1 \( \Rightarrow \text{half is} \frac{1}{2} \)**

\[ \frac{1}{2} = 1 e^{k \cdot 63} \]

\[ \frac{1}{2} = e^{63k} \]

\[ \ln \left( \frac{1}{2} \right) = k \]

\[ k = \frac{\ln \left( \frac{1}{2} \right)}{63} \]

**\( A = ce \frac{\ln \left( \frac{1}{2} \right)}{63} \cdot 50 \)**

\[ 100 = ce \frac{\ln \left( \frac{1}{2} \right)}{63} \cdot 50 \]

\[ \Rightarrow c = \frac{100}{e \frac{\ln \left( \frac{1}{2} \right)}{63} \cdot 50} \text{ mg} \]
9. (5 points) Determine a formula for \( f(x) \), given that

- \( f''(x) = 16e^{2x} + 12x + 11 \sin x \)
- \( f'(0) = 8 \)
- \( f(0) = 25 \)

\[
f'(x) = 8e^{2x} + 6x^2 - 11 \cos x + C
\]
\[
f'(0) = 8 + 0 - 11 + C = 8 \implies C = 11
\]
\[
f'(x) = 8e^{2x} + 6x^2 - 11 \cos x + 11
\]
\[
f(x) = 4e^{2x} + 2x^3 - 11 \sin x + 11x + D
\]
\[
f(0) = 4 + 0 - 0 + 0 + D = 25 \implies D = 21
\]

\[
\Rightarrow f(x) = 4e^{2x} + 2x^3 - 11 \sin x + 11x + 21
\]

10. (5 points) Use logarithmic differentiation to find the derivative of

\[
y = \frac{e^{-x} \cos^2 x}{x^2 + x + 1}
\]

\[
\ln y = \ln \left( \frac{e^{-x} \cos^2 x}{x^2 + x + 1} \right)
\]
\[
\ln y = \ln(e^{-x}) + 2 \ln |\cos x| - \ln(x^2 + x + 1)
\]
\[
\frac{1}{y} y' = -1 + 2 \frac{1}{\cos x} (\sin x) - \frac{1}{x^2 + x + 1} (2x + 1)
\]
\[
y' = y \left( -1 - 2 \tan x - \frac{2x + 1}{x^2 + x + 1} \right)
\]
\[
y' = \frac{e^{-x} \cos^2 x}{x^2 + x + 1} \left( -1 - 2 \tan x - \frac{2x + 1}{x^2 + x + 1} \right)
\]
11. (10 points) Gravel is being dumped from a conveyor belt at a rate of 20 cubic feet per minute. It forms a pile in the shape of a tall cone whose base diameter is half its height. How fast is the height increasing when the pile is 4 ft high?

\[
V = \frac{1}{3} \pi r^2 h
\]

\[
d = \frac{1}{2} h
\]

\[
2r = \frac{1}{2} h
\]

\[
r = \frac{1}{4} h
\]

\[
V = \frac{1}{3} \pi \left( \frac{1}{4}h \right)^2 h
\]

\[
V = \frac{1}{16} \pi h^3
\]

Implicit differentiation:

\[
\frac{dV}{dt} = \frac{1}{16} \pi \left( 2h^2 \frac{dh}{dt} \right)
\]

@ h = 4

\[
\frac{dV}{dt} = \frac{1}{16} \pi \left( 2 \cdot 4^2 \frac{dh}{dt} \right)
\]

\[
20 = \frac{1}{16} \pi \cdot 32 \frac{dh}{dt}
\]

\[
\frac{20}{\pi} = \frac{dh}{dt} \text{ ft/min}
\]
12. (10 points) A function \( f(x) \) is differentiable everywhere and has the following second derivative.

\[
 f''(x) = \frac{(2x^2 - 50)(x + 6)(x^2 + 9)(x - 4)^4}{11e^{-x}}
\]

Find the intervals of concavity for \( f(x) \) and state each \( x \)-value at which the graph of \( f(x) \) has an inflection point.

\[ 11e^{-x} \text{ always positive} \]

\[
 f''(x) = \frac{2(x - 5)(x + 3)(x + 6)^7(x^2 + 9)(x - 4)^4}{11e^{-x}} = 0 \quad \text{when} \quad x = 5, \quad x = -3, \quad x = -6, \quad x = 4.
\]

\[ f''(x) \]

\[
\begin{array}{c|cccc}
 & -6 & -5 & 4 & 5 \\
\hline
(-\infty, -6) & - & - & + & + \\
(-6, -5) & - & - & + & + \\
(-5, 4) & - & + & + & + \\
(4, 5) & - & + & + & + \\
(5, \infty) & + & + & + & + \\
\end{array}
\]

\[ f''(x) \]

\[
\begin{array}{c|cccc}
 & x - 5 & x + 3 & (x + 6)^7 & (x^2 + 9) & (x - 4)^4 \\
\hline
(-\infty, -6) & - & - & + & + & - \\
(-6, -5) & - & - & + & + & + \\
(-5, 4) & - & + & + & + & - \\
(4, 5) & - & + & + & + & - \\
(5, \infty) & + & + & + & + & + \\
\end{array}
\]

Concave up on \((-6, -5) \cup (5, \infty)\)

Concave down on \((-\infty, -6) \cup (-5, 5)\)

Inflection points at \( x = -6, -5, 5 \)
13. (10 points) Find the points on the ellipse $4x^2 + y^2 = 4$ that are the farthest away from the point (1,0).

$$d = \sqrt{(x-1)^2 + (y-0)^2}$$

Maximize distance - want eq. only w/ distance in relation to $x$.

$$d = \sqrt{x^2 - 2x + 1 + y^2}$$

$$\Rightarrow 4x^2 + y^2 = 4$$

$$\Rightarrow y^2 = 4 - 4x^2$$

Distance is maximized when square is maximized.

$$S = d^2 = -3x^2 - 2x + 5$$

$$S' = -6x - 2$$

$$S'' = 0 \text{ when } x = -\frac{1}{3}$$

$$S'' = -6 < 0, \text{ so by second derivative test is a maximum at } -\frac{1}{3}.$$ 

$$y\text{-values: } y = \pm \sqrt{4 - 4\left(-\frac{1}{3}\right)^2} = \pm \sqrt{\frac{32}{9}} = \pm \frac{4}{3} \sqrt{2}$$

$$\Rightarrow \text{ points are } \left(-\frac{1}{3}, \frac{4}{3}\sqrt{2}\right), \left(-\frac{1}{3}, -\frac{4}{3}\sqrt{2}\right)$$
14. (10 points) The area between the $x$-axis and the graph of $f(x) = x^3 + 4x$ on the interval $[2, 5]$ can be written as the limit of a Riemann sum in many different ways. Fill in the missing information for the limits so that the only variables appearing are $n$ and $k$. Do not evaluate these limits.

(a) Using the limit of a left Riemann sum,

\[
\text{Area} = \lim_{n \to \infty} \sum_{k=0}^{n-1} \left[ \left( 2 + k \cdot \frac{3}{n} \right)^3 + 4 \left( 2 + k \cdot \frac{3}{n} \right) \right] \cdot \frac{3}{n}
\]

(b) Using the limit of a left Riemann sum,

\[
\text{Area} = \lim_{n \to \infty} \sum_{k=1}^{n} \left[ \left( 2 + (k-1) \cdot \frac{3}{n} \right)^3 + 4 \left( 2 + (k-1) \cdot \frac{3}{n} \right) \right] \cdot \frac{3}{n}
\]

(c) Using the limit of a right Riemann sum,

\[
\text{Area} = \lim_{n \to \infty} \sum_{k=0}^{n-1} \left[ \left( 2 + (k+1) \cdot \frac{3}{n} \right)^3 + 4 \left( 2 + (k+1) \cdot \frac{3}{n} \right) \right] \cdot \frac{3}{n}
\]

(d) Using the limit of a right Riemann sum,

\[
\text{Area} = \lim_{n \to \infty} \sum_{k=1}^{n} \left[ \left( 2 + k \cdot \frac{3}{n} \right)^3 + 4 \left( 2 + k \cdot \frac{3}{n} \right) \right] \cdot \frac{3}{n}
\]

(e) Using the limit of a midpoint Riemann sum,

\[
\text{Area} = \lim_{n \to \infty} \sum_{k=0}^{n-1} \left[ \left( 2 + (k+\frac{1}{2}) \cdot \frac{3}{n} \right)^3 + 4 \left( 2 + (k+\frac{1}{2}) \cdot \frac{3}{n} \right) \right] \cdot \frac{3}{n}
\]

(f) Using the limit of a midpoint Riemann sum,

\[
\text{Area} = \lim_{n \to \infty} \sum_{k=1}^{n} \left[ \left( 2 + (k-\frac{1}{2}) \cdot \frac{3}{n} \right)^3 + 4 \left( 2 + (k-\frac{1}{2}) \cdot \frac{3}{n} \right) \right] \cdot \frac{3}{n}
\]
15. (10 points) Suppose that \( f \) is an even function which satisfies the following:

- \( \int_{-5}^{5} f(x) \, dx = 8 \Rightarrow \int_{0}^{5} f(x) \, dx = 4 \)
- \( \int_{0}^{16} f(x) \, dx = 14 \)

Evaluate the following quantities

(a) \( \int_{5}^{16} (6f(x) + 10) \, dx \)

\[
= 6 \int_{5}^{16} f(x) \, dx + \int_{5}^{16} 10 \, dx \\
= 6 \left( \int_{5}^{16} f(x) \, dx - \int_{0}^{5} f(x) \, dx \right) + \left[ 10x \right]_{5}^{16} \\
= 6 (14 - 4) + 160 - 50 \\
= 60 + 110 = 170
\]

(b) \( \int_{4}^{0} 20xf(x^2) \, dx \)

\[
= -\int_{0}^{4} 20x f(x^2) \, dx \\
\quad \text{\( u = x^2 \)} \\
\quad \text{\( du = 2x \, dx \)} \\
= -10 \int_{0}^{16} f(u) \, du \\
= -10 (14) = -140
\]
16. (5 points each) Evaluate the following integrals.

(a) \[ \int \cot^5 x \csc^3 x \, dx \]

\[ u = \csc x \quad du = -\csc x \cot x \, dx \]

\[ \sin^2 x + \cos^2 x = 1 \]

\[ \Rightarrow \quad \text{divide by } \sin^2 x \]

\[ 1 + \cot^2 x = \csc^2 x \]

\[ \csc^2 x - 1 = \cot^2 x \]

\[ \begin{align*}
= & \int \cot^4 x \csc^2 x \cot x \csc x \, dx \\
= & -\int (\cot^2 x)^2 \csc^2 x \, (-\csc x \cot x \, dx) \\
= & -\int (u^2 - 1)^2 u^2 \, du \\
= & -\int (u^4 - 2u^3 + u) u^2 \, du \\
= & -\int u^6 - 2u^4 + u^2 \, du \\
= & -\frac{u^7}{7} + \frac{2u^5}{5} - \frac{u^3}{3} + C \\
\end{align*} \]

\[ \begin{align*}
\frac{\csc^7 x}{7} + \frac{2\csc^5 x}{5} - \frac{\csc^3 x}{3} + C
\end{align*} \]

(b) \[ \int \frac{14x^{1.5}}{\sqrt{3}x^7 + 2} + (x^7 + 2)^8 \tan(x^7 + 2) \, dx \]

\[ u = x^7 + 2 \quad x^7 = u - 2 \]

\[ du = 7x^6 \, dx \]

\[ \alpha = 1 \Rightarrow u = 1 \quad \alpha = \frac{7}{\sqrt{3}} \Rightarrow u = -1 \]

\[ \int_{-1}^{1} \left( \frac{2(u^2 - 2)}{u} + u^8 \tan u \right) \, du \quad \tan \text{ is odd, } u^8 \text{ is even} \]

\[ \Rightarrow u^8 \tan u \text{ is odd} \]

\[ 2 \int_{-1}^{1} \left( \frac{u^2 - 2}{u} \right) \, du + \int_{-1}^{1} u^8 \tan u \, du \]

\[ \tan \text{ is odd, } u^8 \text{ is even} \]

\[ \Rightarrow u^8 \tan u \text{ is odd} \]

\[ = 2 \int_{-1}^{1} \left( \frac{u^2}{u} - \frac{2}{u} \right) \, du \]

\[ = 2 \int_{-1}^{1} \left( 1 - \frac{2}{u} \right) \, du = 2 \left[ u - 2 \ln |u| \right]_{-1}^{0} \]

\[ = 2 \left[ 0 - 2 \ln |1| - (-1 - 2 \ln |1|) \right] \]

\[ = 2 \left[ 2 \right] = 4 \]
17. (10 points) Let \( R \) be the finite region in the first quadrant bounded by the graphs of \( y = 4x + 1 \) and \( y = x^2 + 1 \) and below the horizontal line \( y = 5 \). Set up, but do not evaluate, definite integrals which represent the given quantities. Use proper notation.

(a) The area of \( R \). Integrate with respect to \( x \).

\[
A = \int_0^1 (4x+1-x^2+1) \, dx + \int_1^2 (5-x^2+1) \, dx
\]

(b) The volume of the solid with base \( R \) whose cross-sections perpendicular to the \( x \)-axis are isosceles right triangles, with the hypotenuse lying in the \( xy \)-plane.

\[
A(y) = \frac{1}{2}bh = \frac{1}{4}b^2
\]

\[
V = \int_1^5 \frac{1}{4} \left( \sqrt{y-1} - \frac{y-1}{4} \right)^2 \, dy
\]

(c) The volume of the solid obtained when \( R \) is revolved around the vertical line \( x = 3 \). Integrate with respect to \( x \).

\[
V = \int_0^1 2\pi (3-x)(4x+1-x^2+1) \, dx + \int_1^2 2\pi (3-x)(5-x^2+1) \, dx
\]

(d) The volume of the solid obtained when \( R \) is revolved around the vertical line \( x = 3 \). Integrate with respect to \( y \).

\[
V = \int_{y_{\text{min}}}^{y_{\text{max}}} [\pi (3-y^2) - \pi (\sqrt{y-1})^2] \, dy
\]
18. Bonus 5 points: Evaluate the following indefinite integral:

\[ \int x^2 \sin x \, dx \]

\[ u = x^2 \quad \rightarrow \quad du = 2x \, dx \]
\[ dv = \sin x \, dx \quad \rightarrow \quad v = -\cos x \]

\[ \int x^2 \sin x \, dx = -x^2 \cos x - \int -2x \cos x \, dx \]

\[ = -x^2 \cos x + 2 \int x \cos x \, dx \]

\[ u = x \quad \rightarrow \quad du = dx \]
\[ dv = \cos x \, dx \quad \rightarrow \quad v = \sin x \]

\[ = -x^2 \cos x + 2 \left( x \sin x - \int \sin x \, dx \right) \]

\[ = -x^2 \cos x + 2x \sin x + 2 \cos x + C \]