Chapter 2: Limits and Derivatives

2.1 The Tangent and Velocity Problems

2.2 The Limit of a Function

Limits: \( \lim_{x \to a} f(x) = L \)

One-sided Limits: \( \lim_{x \to a^-} f(x) = L \)

\( x = a \) is a vertical asymptote of \( y = f(x) \) if at least one one-sided limit as \( x \) approaches \( a \) is \( \pm \infty \).

2.3 Calculating Limits Using the Limit Laws

If \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) exist (particularly not \( \pm \infty \)):

\[
\begin{align*}
\lim_{x \to a} [f(x) + g(x)] &= \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \\
\lim_{x \to a} [f(x) - g(x)] &= \lim_{x \to a} f(x) - \lim_{x \to a} g(x) \\
\lim_{x \to a} [cf(x)] &= c \lim_{x \to a} f(x) \text{ where } c \text{ is a constant} \\
\lim_{x \to a} [f(x)g(x)] &= \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) \\
\lim_{x \to a} \left[ \frac{f(x)}{g(x)} \right] &= \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \text{ if } \lim_{x \to a} g(x) \neq 0 \\
\lim_{x \to a} [f(x)]^n &= \left[ \lim_{x \to a} f(x) \right]^n \text{ where } n \text{ is a positive integer} \\
\lim_{x \to a} \sqrt[n]{f(x)} &= \sqrt[n]{\lim_{x \to a} f(x)} \text{, } n \text{ is a pos. int. (if } n \text{ is even, assume } \lim_{x \to a} f(x) > 0) \\
\textbf{Squeeze Theorem:} \text{ If } f(x) \leq g(x) \leq h(x) \text{ when } x \text{ is near } a \text{ (except possibly at } a \text{) and } \lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L, \text{ then } \lim_{x \to a} g(x) = L.
\end{align*}
\]

2.5 Continuity

A function \( f(x) \) is continuous at \( a \) if \( \lim_{x \to a} f(x) = f(a) \).

Polynomials, exponentials, logarithms, roots, trig functions, inverse trig functions and rational functions are all continuous at each point in their domains.

If \( f, g \) are continuous at \( a, c \) constant, then \( f + g, f - g, cf, fg, \frac{f}{g} \) if \( g(a) \neq 0 \) are continuous at \( a \).

If \( g \) is continuous at \( a \) and \( f \) is continuous at \( g(a) \), then \( f \circ g \) is continuous at \( a \).

\textbf{Intermediate Value Theorem:} Suppose that \( f \) is continuous on \([a, b]\) and let \( N \) be any number between \( f(a) \) and \( f(b) \), where \( f(a) \neq f(b) \). Then there exists a number \( c \) in \((a, b)\) such that \( f(c) = N \).

2.6 Limits at Infinity: Horizontal Asymptotes

\( y = L \) is a horizontal asymptote of the curve \( y = f(x) \) if \( \lim_{x \to \infty} f(x) = L \) or \( \lim_{x \to -\infty} f(x) = L \).

2.7 Derivatives and Rates of Change

The tangent line to \( y = f(x) \) at \((a, f(a))\) is the line through \((a, f(a))\) whose slope is equal to \( f'(a) \), the derivative of \( f \) at \( a \).

\textbf{Point-slope formula:} The equation of a line with slope \( f'(a) \) at the point \((a, f(a))\) is \( y - f(a) = f'(a)(x - a) \)

The following terms mean the same thing: the derivative, the slope of the line tangent to the curve, and the instantaneous rate of change.
2.8 The Derivative as a Function

Definition of the derivative: \( f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \) (provided the limit exists)

Differentiation is the process of taking a derivative, and a function \( f \) is differentiable at \( a \) if \( f'(a) \) exists.

If \( f \) is differentiable at \( a \) then \( f \) is continuous at \( a \).

Given \( y = f(x) \), we can denote its derivative as \( f'(x) \), \( y' \), \( \frac{d}{dx} f(x) \), or \( \frac{dy}{dx} \) (Leibniz notation).
Chapter 3: Differentiation Rules

3.1 Derivatives of Polynomials and Exponential Functions
\[
\frac{d}{dx}(c) = 0 \quad \text{(where } c \text{ is any constant)}
\]
\[
\frac{d}{dx}(x^n) = nx^{n-1} \quad \text{(where } n \text{ is any constant)}
\]
\[
\frac{d}{dx}(e^x) = e^x
\]
\[
\frac{d}{dx}(a^x) = a^x \ln a
\]

If \( f \) and \( g \) are both differentiable and \( c \) is constant, then
\[
\frac{d}{dx}[cf(x)] = cf'(x)
\]
\[
\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)
\]

3.2 The Product and Quotient Rules

If \( f \) and \( g \) are both differentiable, then
\[
\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)
\]
\[
\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}
\]

3.3 Derivatives of Trigonometric Functions
\[
\frac{d}{dx}(\sin x) = \cos x
\]
\[
\frac{d}{dx}(\cos x) = -\sin x
\]
\[
\frac{d}{dx}(\tan x) = \sec^2 x
\]
\[
\frac{d}{dx}(\csc x) = -\csc x \cot x
\]
\[
\frac{d}{dx}(\sec x) = \sec x \tan x
\]
\[
\frac{d}{dx}(\cot x) = -\csc^2 x
\]

3.4 The Chain Rule

If \( g \) is differentiable at \( x \) and \( f \) is differentiable at \( g(x) \), then the composite \( F = f \circ g \) defined by
\[
F(x) = f(g(x))
\]
is differentiable at \( x \), and
\[
F'(x) = f'(g(x)) \cdot g'(x)
\]
Also written as
\[
\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)
\]
or
\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}
\]

3.5 Implicit Differentiation

*Implicit Differentiation:* Differentiate both sides of the equation \( y = f(x) \) with respect to \( x \) and then solve the resulting equation for \( y' = \frac{dy}{dx} \).
\[
\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}
\]
\[
\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}
\]
\[
\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}
\]
\[
\frac{d}{dx}(\arccsc x) = -\frac{1}{x\sqrt{x^2-1}}
\]
\[
\frac{d}{dx}(\arcsec x) = \frac{1}{x\sqrt{x^2-1}}
\]
\[
\frac{d}{dx}(\arccot x) = \frac{1}{1+x^2}
\]

3.6 Derivatives of Logarithmic Functions
\[
\frac{d}{dx}(\ln x) = \frac{1}{x}
\]
\[
\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}
\]
\[
\frac{d}{dx}(\ln |x|) = \frac{1}{x}
\]

*Logarithmic Differentiation:* Take natural logarithms of both sides of an equation \( y = f(x) \) and use the logarithm laws to simplify, and then differentiate implicitly with respect to \( x \) to solve for \( y' \).

3.7 Rates of Change in the Natural and Social Sciences

If \( s(t) \) is the position function of a particle at time \( t \), then \( v = s'(t) \) represents the instantaneous velocity, and \( a = v'(t) = s''(t) \) represents the acceleration.
3.8 Exponential Growth and Decay

The differential equation \( \frac{dy}{dx} = ky \) only has solutions of the form \( Ce^{kx} \), where \( C \) is a constant. More precisely, the solutions are \( y(x) = y(0)e^{kx} \).

Here we say that “the growth/decay rate is proportional to the size/mass” or “the relative growth/decay rate is constant.”

The half-life is the time required for half of a quantity to decay.

3.9 Related Rates

Formulas to know:
- Pythagorean theorem \( a^2 + b^2 = c^2 \)
- Similar triangles \( \frac{a}{b} = \frac{a'}{b'} \)
- Trig identities: \( \sin^2 x + \cos^2 x = 1 \), \( \tan^2 x + 1 = \sec^2 x \), \( 1 + \cot^2 x = \csc^2 x \), \( \sin(2x) = 2 \sin x \cos x \), \( \cos(2x) = \cos^2 x - \sin^2 x \)
- Evaluation of trigonometric functions at special angles (\( \pi \), \( \pi/2 \), 0, etc.)
- Relationship of each of the six trigonometric functions to the hypotenuse and the opposite and adjacent sides of a right triangle (SOH-CAH-TOA)
- Circumference (2\( \pi r \)) and diameter (2\( \pi \)) of a circle
- Areas of rectangle \( A = l \cdot w \), circle \( A = \pi r^2 \), triangle \( A = \frac{1}{2} b \cdot h \)
- Volumes of box \( V = l \cdot w \cdot h \), sphere \( V = \frac{4}{3} \pi r^3 \), cone \( V = \frac{1}{3} \pi r^2 h \)
- Distance between points \( (x_1, y_1) \) and \( (x_2, y_2) \) is \( D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \)

Problem-solving strategies:
1. Read the problem and draw a diagram, introducing notation for quantities.
2. Express the given information and the required rate in terms of derivatives, and write an equation that relates the various quantities of the problem (try to eliminate variables by substitution using geometry).
3. Use the chain rule to differentiate both sides of the equation with respect to \( t \).
4. Substitute the given information into the resulting equation and solve for the unknown rate.

3.10 Linear Approximations and Differentials

Linear approximation/Tangent line approximation: \( f(x) \approx L(x) = f(a) + f'(a)(x - a) \)

3.11 Hyperbolic Functions

\[
\begin{align*}
\sinh x &= \frac{e^x - e^{-x}}{2} \\
\cosh x &= \frac{e^x + e^{-x}}{2} \\
\tanh x &= \frac{\sinh x}{\cosh x} \\
\csch x &= \frac{1}{\sinh x} \\
\sech x &= \frac{1}{\cosh x} \\
\coth x &= \frac{\cosh x}{\sinh x}
\end{align*}
\]

\[
\begin{align*}
\frac{d}{dx} (\sinh x) &= \cosh x \\
\frac{d}{dx} (\cosh x) &= \sinh x \\
\frac{d}{dx} (\tanh x) &= \text{sech}^2 x \\
\frac{d}{dx} (\text{csch } x) &= -\text{csch } x \coth x \\
\frac{d}{dx} (\sech x) &= -\text{sech } x \tanh x \\
\frac{d}{dx} (\coth x) &= -\text{csch } x \coth x
\end{align*}
\]
Chapter 4: Applications of Differentiation

4.1 Maximum and Minimum Values

**Extreme Value Theorem**: If \( f \) is continuous on a closed interval \([a, b]\), then \( f \) attains an absolute maximum value \( f(c) \) and an absolute minimum value \( f(d) \) at some numbers \( c \) and \( d \) in \([a, b]\).

A critical number of a function \( f \) is a number \( c \) in the domain of \( f \) such that either \( f'(c) = 0 \) or \( f'(c) \) does not exist.

If \( f \) has a local maximum or minimum at \( c \), then \( c \) is a critical number of \( f \).

**Closed Interval Method**: To find the absolute maximum and minimum values of a continuous function \( f \) on a closed interval \([a, b]\), compare the values of \( f \) at the critical numbers of \( f \) in \((a, b)\) and the values of \( f \) at the endpoints of the interval.

4.2 The Mean Value Theorem

**Rolle’s Theorem**: Let \( f \) be a function that is continuous on the closed interval \([a, b]\), differentiable on the open interval \((a, b)\), and let \( f(a) = f(b) \). Then there is a number \( c \) in \((a, b)\) such that \( f'(c) = 0 \).

**Mean Value Theorem**: Let \( f \) be a function that is continuous on the closed interval \([a, b]\) and differentiable on the open interval \((a, b)\). Then there is a number \( c \) in \((a, b)\) such that

\[ f'(c) = \frac{f(b) - f(a)}{b - a}, \text{ or equivalently, } f(b) - f(a) = f'(c)(b - a). \]

4.3 How Derivatives Affect the Shape of a Graph

**Increasing/Decreasing Test**: If \( f'(x) > 0 \) on an interval, then \( f \) is increasing on that interval. If \( f'(x) < 0 \) on an interval, then \( f \) is decreasing on that interval.

**First Derivative Test**: Suppose \( c \) is a critical number of a continuous function \( f \).

1. If \( f' \) changes from positive to negative at \( c \), then \( f \) has a local maximum at \( c \).
2. If \( f' \) changes from negative to positive at \( c \), then \( f \) has a local minimum at \( c \).
3. If \( f' \) has the same sign on either side of \( c \), then \( f \) has no local max or min at \( c \).

**Concavity Test**: If \( f''(x) > 0 \) on an interval, then \( f \) is concave upward on that interval. If \( f''(x) < 0 \) on an interval, then \( f \) is concave downward on that interval.

A point on a curve \( y = f(x) \) is an inflection point if \( f \) is continuous there and the curve changes from concave up to concave down or from concave down to concave up, i.e. \( f'' \) changes sign.

**Second Derivative Test**: Suppose \( f'' \) is continuous near \( c \), and \( f''(c) = 0 \). If \( f''(c) > 0 \), then \( f \) has a local minimum at \( c \). If \( f''(c) < 0 \), then \( f \) has a local maximum at \( c \).

4.4 Indeterminate Forms and L’Hospital’s Rule

**L’Hospital’s Rule**: If we have an indeterminate form \( \frac{0}{0} \) or \( \frac{\pm \infty}{\pm \infty} \), then \( \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \).

<table>
<thead>
<tr>
<th>Indeterminate form</th>
<th>Technique to determine limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 ) ( \pm \infty )</td>
<td>Use l'Hospital’s Rule.</td>
</tr>
<tr>
<td>( 0^1 ) ( \pm \infty )</td>
<td>Rewire the product as a quotient and then use l'Hospital’s Rule.</td>
</tr>
<tr>
<td>( 0 \cdot \pm \infty )</td>
<td>Use the common denominator, the conjugate, trigonometric identities, or other algebraic techniques to rewrite the difference as a quotient and then see if l’Hospital’s Rule is applicable.</td>
</tr>
<tr>
<td>( \infty - \infty )</td>
<td></td>
</tr>
</tbody>
</table>
4.7 Optimization Problems

Steps for solving optimization problems.

1. Read the problem and draw a diagram, introducing notation for quantities.
2. Find an equation for the quantity $Q$ that is to be maximized or minimized.
3. Use the given information to find relationships among the variables, so that you can eliminate all but one of the variables in your equation from step 2, getting a function $Q = f(x)$.
4. Use maximization and minimization methods to find the absolute maximum or minimum value of $f$ (if the domain of $f$ is a closed interval, check endpoints as well.)

First Derivative test for Absolute Extreme Values: Suppose $c$ is a critical number of continuous $f$.

- If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum of $f$.
- If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(x)$ is the absolute minimum of $f$.

4.8 Newton’s Method

Newton’s Method: Approximate root of $f(x)$ by picking close initial value $x_1$, and progressively approximating $x_2, x_3, \ldots$ by finding the $x$-intercept of the line tangent to $f$ at $x_n$: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

4.8 Antiderivatives

A function $F$ is called an antiderivative of $f$ on interval $I$ if $F'(x) = f(x)$ for all $x$ in $I$.

The most general antiderivative of $f$ on $I$ is $F(x) + C$ where $C$ is an arbitrary constant.

<table>
<thead>
<tr>
<th>Function</th>
<th>Particular antiderivative</th>
<th>Function</th>
<th>Particular antiderivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>$cf(x)$</td>
<td>$cF(x)$</td>
<td>$\sec^2 x$</td>
<td>$\tan x$</td>
</tr>
<tr>
<td>$f(x) + g(x)$</td>
<td>$F(x) + G(x)$</td>
<td>$\csc^2 x$</td>
<td>$- \cot x$</td>
</tr>
<tr>
<td>$c$ (constant)</td>
<td>$cx$</td>
<td>$\sec x \tan x$</td>
<td>$\sec x$</td>
</tr>
<tr>
<td>$x^n$ ($n \neq 1$)</td>
<td>$\frac{x^{n+1}}{n+1}$</td>
<td>$\csc x \cot x$</td>
<td>$- \csc x$</td>
</tr>
<tr>
<td>$\frac{1}{x}$</td>
<td>$\ln</td>
<td>x</td>
<td>$</td>
</tr>
<tr>
<td>$e^x$</td>
<td>$e^x$</td>
<td>$\frac{1}{\sqrt{1-x^2}}$</td>
<td>$\arcsin x$</td>
</tr>
<tr>
<td>$b^x$ ($b \neq 1, b &gt; 0$)</td>
<td>$\frac{b^x}{\ln b}$</td>
<td>$\frac{1}{x\sqrt{x^2-1}}$</td>
<td>$\arccsc x$</td>
</tr>
<tr>
<td>$\sin x$</td>
<td>$- \cos x$</td>
<td>$\sinh x$</td>
<td>$\cosh x$</td>
</tr>
<tr>
<td>$\cos x$</td>
<td>$\sin x$</td>
<td>$\cosh x$</td>
<td>$\sinh x$</td>
</tr>
</tbody>
</table>
Chapter 5: Integrals

5.1 Areas and Distances

The area $A$ of the region $S$ that lies under the graph of the continuous function $f$ is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} [f(x_1^*)\Delta x + [f(x_2^*)\Delta x + \cdots + [f(x_n^*)\Delta x$$

$x_1^*, \ldots, x_n^*$ can be any sample points from each interval, sometimes we use left/right endpoints of the subinterval, lower/upper sum chooses $f(x_i^*)$ to be minimum/maximum value of $f$ on the $i$th subinterval.

The distance traveled is equal to the area under the graph of the velocity function.

5.2 The Definite Integral

If $f$ is a function defined for $a \leq x \leq b$, we divide $[a, b]$ into $n$ subintervals of equal width $\Delta x = (b-a)/n$. Let $x_1^*, \ldots, x_n^*$ be any sample points in these subintervals. Then the definite integral of $f$ from $a$ to $b$ is

$$\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that $f$ is integrable on $[a,b]$.

The definite integral $\int_a^b f(x) \, dx$ can be interpreted as the area under the curve $y = f(x)$ from $a$ to $b$. If $f$ is continuous on $[a,b]$, or if $f$ has only a finite number of jump discontinuities, then $f$ is integrable on $[a,b]$.

If $f$ is integrable on $[a,b]$, then $\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x$, where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.

Properties of the Definite Integral: Assuming $f$ and $g$ are continuous,

1. $\int_b^a f(x) \, dx = - \int_a^b f(x) \, dx$
2. $\int_a^a f(x) \, dx = 0$
3. $\int_a^b c \, dx = c(b-a)$ (c is a constant)
4. $\int_a^b cf(x) \, dx = c \int_a^b f(x) \, dx$
5. $\int_a^b [f(x) + g(x)] \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx$
6. $\int_a^b [f(x) - g(x)] \, dx = \int_a^b f(x) \, dx - \int_a^b g(x) \, dx$
7. $\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx$
8. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x) \, dx \geq 0$.
9. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x) \, dx \geq \int_a^b g(x) \, dx$. 
10. If \( m \leq f(x) \leq M \) for \( a \leq x \leq b \), then \( m(b - a) \leq \int_a^b f(x) \, dx \leq M(b - a) \).

Properties of Summation Notation:
\[
\sum_{i=1}^{n} c = nc \quad (c \text{ is a constant}) \quad \sum_{i=1}^{n} ca_i = c \sum_{i=1}^{n} a_i \quad \sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i
\]
\[
\sum_{i=1}^{n} i = \frac{n(n + 1)}{2} \quad \sum_{i=1}^{n} i^2 = \frac{n(n + 1)(2n + 1)}{6}
\]

5.3 The Fundamental Theorem of Calculus

**Fundamental Theorem of Calculus**: Suppose \( f \) is continuous on \([a, b]\).

1. If \( g(x) = \int_a^x f(t) \, dt \), then \( g'(x) = f(x) \).

2. \( \int_a^b f(x) \, dx = F(b) - F(a) \) where \( F \) is any antiderivative of \( f \).

5.4 Indefinite Integrals and the Net Change Theorem

The indefinite integral is notation used for an antiderivative: \( \int f(x) \, dx = F(x) \) means \( F'(x) = f(x) \).

**Net Change Theorem**: The integral of a rate of change is the net change.

5.5 The Substitution Rule

**The Substitution Rule**: If \( u = g(x) \) is a differentiable function whose range is an interval \( I \) and \( f \) is continuous on \( I \), then
\[
\int f(g(x))g'(x) \, dx = \int f(u) \, du
\]

**Substitution for Definite Integrals**: If \( g' \) is continuous on \([a, b]\) and \( f \) is continuous on the range of \( u = g(x) \), then
\[
\int_a^b f(g(x))g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du
\]

If \( f \) is an even function, then \( \int_{-a}^{a} f(x) \, dx = 2 \int_0^{a} f(x) \, dx \).

If \( f \) is an odd function, then \( \int_{-a}^{a} f(x) \, dx = 0 \).

Trigonometric Identities to know (Section 7.2):
\[
\tan(x) = \frac{\sin(x)}{\cos(x)} \quad \cot(x) = \frac{\cos(x)}{\sin(x)} \quad \sec(x) = \frac{1}{\cos(x)} \quad \csc(x) = \frac{1}{\sin(x)}
\]
\[
\sin^2(x) + \cos^2(x) = 1 \quad \tan^2(x) + 1 = \sec^2(x) \quad \cot^2(x) + 1 = \csc^2(x)
\]
\[
\sin(2x) = 2 \sin(x) \cos(x) \quad \cos(2x) = \cos^2(x) - \sin^2(x)
\]
\[
\cos^2(x) = \frac{1}{2} + \frac{1}{2} \cos(2x) \quad \sin^2(x) = \frac{1}{2} - \frac{1}{2} \cos(2x)
\]
Chapter 6: Applications of Integration

6.1 Areas Between Curves
The area between the curves \( y = f(x) \), \( y = g(x) \), and between \( x = a \) and \( x = b \) is

\[
A = \int_a^b |f(x) - g(x)| \, dx
\]

6.2 Volumes
Let \( S \) be a solid that lies between \( x = a \) and \( x = b \). If the cross-sectional area of \( S \) in the plane through \( x \) and perpendicular to the \( x \)-axis is \( A(x) \), where \( A \) is continuous, then the volume of \( S \) is

\[
V = \int_a^b A(x) \, dx
\]

In other words,
\[
V = \int_{x_{\text{min}}}^{x_{\text{max}}} \text{(cross-sectional area of the slice at } x) \, dx.
\]
(Note the above can be done with respect to \( y \) as well.)

Washer Method: If the cross-section is a washer, we compute the area of the washer by subtracting the area of the inner disk from the area of the outer disk:

\[
A = \pi \text{(outer radius)}^2 - \pi \text{(inner radius)}^2
\]

6.3 Volumes by Cylindrical Shells
Cylindrical Shell Method: Instead of using cross-sections, use the height parallel to the axis of rotation to determine the height of a cylinder with outer surface area \( 2\pi rh \), where \( h \) is the height and \( x \) is the distance from the axis of rotation.

The surface has volume \( V = \int_a^b 2\pi rh \, dx \), where \( r \) and \( h \) are in terms of \( x \).

If we revolve about a horizontal \( y = c \) line, the washer method integrates with respect to \( x \), while the cylindrical shells method integrates with respect to \( y \).
If we revolve about a vertical \( x = d \) line, the washer method integrates with respect to \( y \), while the cylindrical shells method integrates with respect to \( x \).

6.5 Average Value of a Function
The average value of the function \( f \) on the interval \([a, b]\) is

\[
f_{\text{ave}} = \frac{1}{b - a} \int_a^b f(x) \, dx
\]

The Mean Value Theorem for Integrals: If \( f \) is continuous on \([a, b]\), then there exists a number \( c \) in \([a, b]\) such that

\[
f(c) = f_{\text{ave}} = \frac{1}{b - a} \int_a^b f(x) \, dx
\]
or, equivalently,

\[
\int_a^b f(x) \, dx = f(c)(b - a)
\]
▷ Chapter 2 Notes:

- **Possible Questions:**
  - Evaluate the limit of a given function.
  - From a graph of a function, determine the limit of the function.
  - Find horizontal or vertical asymptotes of a function using limits.
  - Find constants that make a particular piecewise function continuous.
  - Show there exists roots or certain values of a function using the intermediate value theorem.
  - Find the limit of a function using the Squeeze Theorem.
  - Find the equation of the tangent line to a function at a given point.
  - From a graph of a function, sketch the graph of its derivative.
  - Find the derivative of a function using the definition of the derivative.

- **Notation notes, and things to be careful of:**
  - Make sure you write limits correctly. Do not drop the limit until you get rid of the limiting variable. Also, make sure to write = in each line.
  - Be careful when you say \( f(x) = \) a function where you’ve canceled out terms, the functions may not be equal.
  - If a limit is \( \pm \infty \), it technically does not exist. Infinity is not a number.
  - Always make sure to figure out if a function goes to \(+ \infty\) or \(- \infty\) when taking the limit using \( 0^\pm \).
  - Do not say something equals something that is either undefined or indeterminate. This includes indeterminate forms and numbers divided by 0. (We fudge this rule a bit when we say a limit = \( \pm \infty \).)
  - Although there are several equivalent definitions of the derivative, your life will be easier if you use the \( h \to 0 \) definition in practice.

  **DO NOT:**
  
  \[
  \lim_{x \to a} f(x) = 0 \quad \lim_{x \to a} f(x) = \frac{0}{0} \quad \lim_{x \to a} f(x) = \infty \quad \lim_{x \to a} f(x) = \frac{\text{nonzero #}}{0^\pm}
  \]

▷ Chapter 3 Notes:

- **Possible Questions:**
  - Find the derivative of a function using the derivative rules.
  - Find the equation of the line tangent to a curve at a given point.
  - Find the equations of lines that are parallel/perpendicular to the tangent line at a particular point.
  - Find the \( x \)-values at which the tangent line of \( f(x) \) is parallel/perpendicular to a given line.
  - Use implicit/logarithmic differentiation to find a derivative.
  - Given a position function of a physical object (projectiles, particles, etc), answer questions about its position, velocity, and acceleration.
    
    e.g. What is its velocity/acceleration after \( t \) seconds? When does it hit the ground? What is its velocity/acceleration when it is at a certain position? How far did it travel? When does it reach maximum height, and what is the max height? When is it at rest?
  - If an experimental quantity has constant growth/decay rate (perhaps half-life),
    
    Find a formula for the amount at time \( t \).
    
    Find the relative growth/decay rate given some data points.
Find the amount after a certain amount of time has passed.
Find the rate of growth/decay after a certain amount of time has passed.
Find the time at which the amount is a certain number.
Solve a related rates word problem.
Use a linear approximation to obtain a good approximation for a certain value.
Given \( f(x) \), use a linear approximation to approximate \( f(a) \) for some decimal value \( a \), e.g. 1.9 or 7.1.
Evaluate a limit that involves a hyperbolic trig function.
Find the derivative of a function that involves a hyperbolic trig function.

- **Notation notes, and things to be careful of:**
  Use derivative notation correctly, especially Leibniz notation!
  Know how to simplify using log and exponent rules, and be on the lookout for "disguised constants."
  Know how to recognize when to use which derivative rule.
    - **Product, quotient, chain rule** - recognize a product, a quotient, or a composite function
    - **Implicit differentiation** - if the function is not written as \( y = f(x) = \) something involving only \( x \)
    - **Logarithmic differentiation** - often will have powers involving \( x \) or complicated products or quotients.
  Remember your logarithm and exponent rules!
  Note that the derivatives \( \frac{d}{dx}(x^n) = nx^{n-1} \) and \( \frac{d}{dx}(a^x) = a^x \ln a \) are different.
  Make sure you are careful with related rates questions, especially with which variables you are treating as constants.
  Simplify a related rates expression before implicitly differentiating - use geometry to replace the quantities you don’t know with expressions involving quantities you do know.
  When using linear approximation, choose an \( a \)-value at which it’s easy to evaluate the function and its derivative, but is also very close to the value you’re trying to approximate.

▷ **Chapter 4 Notes:**

- **Possible Questions:**
  Find the critical numbers of a function.
  Find the absolute maximum and absolute minimum values of a given function on an interval.
  Verify that a function satisfies the Mean Value Theorem or Rolle’s Theorem, and find all numbers \( c \) that satisfy the conclusion of the theorem.
  Find the intervals on which a function is increasing or decreasing.
  Find the local maximum and minimum values of a function.
  Find the intervals of concavity and the inflection points of a function.
  Evaluate the limit of a given function (using l’Hospital’s rule).
  Solve an optimization word problem.
  Use Newton’s method to approximate the root of a function (if this is on there, will be easy enough not to require a calculator).
  Find a general antiderivative of a function.
  Find a function given its (second) derivative and a point on the function (and a point on its derivative).
  Given information about velocity or acceleration, solve things about position.
• Notation notes, and things to be careful of:
  Note that a critical point only requires \(f'(x) = 0\), but an inflection point requires the sign of \(f''(x)\) to change (these are not the same concept for first and second derivative).
  When you use l’Hospital’s rule, write down where you used it! You can write ”by l’Hospital’s rule,” or just put an \(H\) over the \(=\).
  For optimization problems, make sure that you show why the value you found is an absolute maximum/minimum.
  Know statements of Extreme Value Theorem, Intermediate Value Theorem, Rolle’s Theorem, and Mean Value Theorem.
  Know how the algorithm for Newton’s Method works (problems on test will not require calculator).

▷ Chapter 5 Notes:

• Possible Questions:
  Find the area under the curve of a given function.
  Given a velocity function, find the distance traveled.
  Find the value of a Riemann sum with a certain number of intervals.
  Set up the limit representation of a Riemann sum using various kinds of sample points \(x_i\).
  Evaluate a limit that involves summation notation.
  Use property 10 of integrals above to bound an integral from above and below.
  Find the value of a definite integral using areas and properties of definite integrals (i.e. without using the Fundamental Theorem of Calculus).
  Evaluate an integral.
  Using the Fundamental Theorem of Calculus, find the derivative (or second derivative) of a function.
  Find the general indefinite integral of a function.
  Evaluate an integral.
  Use the Net Change Theorem to solve a problem involving applications such as velocity, acceleration, or rates of growth.
  Do a \(u\)-substitution problem, perhaps involving trigonometric functions.

• Notation notes, and things to be careful of:
  Know how to interpret and properties of summation notation.
  Don’t forget the \(C\) in indefinite integrals!
  Terms are getting confusing: know the difference between the terms undefined, indeterminate, and indefinite.
  Know the difference between definite and indefinite integrals, and definitely do not mix them when doing computations!
  Don’t forget the \(C\) in indefinite integrals, and don’t add \(C\) when doing definite integrals!
  Keep your eye out for even/odd functions on intervals symmetric about 0.
  When doing definite integrals with \(u\)-substitution:
    DO: Use = signs wherever appropriate
    DO: Simplify to a reasonable amount
    DON’T: Forget to substitute the limits of integration
DON’T: Include both $x$’s and $u$’s in a single expression (including limits for $x$’s and $u$’s)

DON’T: Substitute $x$ back in for $u$ at the end - changing the bounds makes this step unnecessary.

Remember ‘back-substitution:’ after doing $u$-substitution, express the leftover $x$’s in terms of $u$.

Remember your trigonometric identities!

Some integrals may involve other algebraic skills such as completing the square or polynomial long division.

▷ **Chapter 6 Notes:**

- **Possible Questions:**
  - Sketch the region enclosed by given curves and find its area.
  - Solve an area/volume question using parameters.
  - Find the volume of a solid of revolution (by using washers or cylindrical shells).
  - Find the volume of a solid with cross-sections that are a certain geometric shape (rectangles, triangles, semi-circles, etc.)
  - Find the average value of a function on a given interval.

- **Notation notes, and things to be careful of:**
  - Remember to find the points of intersection when finding areas between curves.
  - Sometimes, you may have to break the area up into the sum of two or more definite integrals.
  - When using the washer method, always check your inner and outer radii by plugging in points to make sure your formulas are what you want them to be.
  - Areas/volumes can’t be negative! If you are getting a negative number, you likely have switched some terms in your integral.