An optimal L^p-bound on the Krein spectral shift function

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Let $\xi_{A,B}$ be the Krein spectral shift function for a pair of operators A, B, with $C = A - B$ trace class. Then

$$\int F(|\xi_{A,B}(\lambda)|)\, d\lambda \leq \int F(|\xi_{|C|,0}(\lambda)|)\, d\lambda$$
$$= \sum_{j=1}^{\infty} \Big[F(j) - F(j-1)\big]\mu_j(C),$$

where F is any non-negative convex function on $[0, \infty)$ with $F(0) = 0$ and $\mu_j(C)$ are the singular values of C.

The Krein spectral shift function

Let A, B be bounded self-adjoint operators such that their difference $A - B$ is trace class. The Krein spectral shift function $\xi_{A,B}$ for the pair A, B is determined by

$$\text{tr}(f(A) - f(B)) = \int f'(\lambda)\xi_{A,B}(\lambda)\, d\lambda$$

for all functions $f \in C_0^\infty(\mathbb{R})$ and $\xi(\lambda) = 0$ if $|\lambda|$ is large enough.

The two bounds

$$\int |\xi_{A,B}(\lambda)|\, d\lambda \leq \text{tr}(|A - B|) \qquad (1)$$

and

$$|\xi_{A,B}(\lambda)| \leq n \qquad \text{if } A - B \text{ is rank } n \qquad (2)$$

are well known

Theorem 1 (Combes,Hislop, and Nakamura) *One has the L^p-bound*

$$\|\xi_{A,B}\|_p := \Big(\int |\xi_{A,B}(\lambda)|^p \, d\lambda \Big)^{1/p} \leq \sum_{j=1}^{\infty} \mu_j(C)^{1/p}$$

for $1 \leq p < \infty$. Note that this bound includes the endpoint cases (1) and (2) for $p = 1$ and and in the limit $p \to \infty$, respectively.

Proof: Write $C := A - B = \sum_{j=1}^{\infty} \mu_j(C) \langle \phi_j, . \rangle \psi_j$ and $B_n := B + \sum_{j=1}^{n} \mu_j(C) \langle \phi_j, . \rangle \psi_j$.

Then ζ_{B_{n+1},B_n} is the spectral shift function of a *rank one pair*. Hence

$$\int |\zeta_{B_{n+1},B_n}|^p = \int |\zeta_{B_{n+1},B_n}|^{p-1} |\zeta_{B_{n+1},B_n}| \leq \int |\zeta_{B_{n+1},B_n}| \leq \mu_n.$$

Use the triangle inequality

$$\Big\| \zeta_{A,B} \Big\|_p = \Big\| \sum \zeta_{B_{n+1},B_n} \Big\|_p \leq \sum \Big\| \zeta_{B_{n+1},B_n} \Big\|_p$$

to sum this up. ■

A special spectral shift function

Let C be a positive trace class operator with eigenvalues μ_j. The spectral shift function for the pair $C, 0$ is simply given by

$$\xi_{C,0}(\lambda) = n \text{ if } \mu_{n+1} \leq \lambda < \mu_n$$
$$\xi_{C,0}(\lambda) = 0 \text{ if } \lambda < 0 \text{ or } \lambda \geq \mu_1.$$

In particular, $\xi_{C,0}$ enjoys the following important properties:

- $\xi_{C,0}$ takes only values in $\mathbb{N}_0$ (or $\mathbb{Z}$ if C is not non-negative).
- For any non-negative function F on $[0, \infty)$ with $F(0) = 0$, we have

$$\int F(|\xi_{C,0}(\lambda)|)\, d\lambda = \sum_{j=1}^{\infty} F(j)\Big(\mu_j - \mu_{j+1}\Big).$$

- In addition, if F is monotone increasing, then

$$\int F(|\xi_{C,0}(\lambda)|)\, d\lambda = \sum_{j=1}^{\infty} \Big[F(j) - F(j-1)\Big]\mu_j.$$

Main Result

The above example $\zeta_{C,0}$ is an extreme case:

Theorem 2 (Barry Simon, 100DM)
Let F be a non-negative convex function on $[0,\infty)$ vanishing at zero. Given a non-negative compact operator C with singular values $\mu_j(C)$,

$$\int F(|\xi_{A,B}(\lambda)|)\, d\lambda \le \int F(|\xi_{C,0}(\lambda)|)\, d\lambda = \sum_{j=1}^{\infty} \Big[F(j) - F(j-1)]\mu_j(C)$$

for all pairs of bounded operators A, B with $\sum_{j=n}^{\infty} \mu_j(|A-B|) \le \sum_{j=n}^{\infty} \mu_j(C)$ for all $n \in \mathbb{N}$. In particular, this is the case if $|A-B| \le C$.

Corollary 3 *In terms of the singular values μ_j of the difference $A-B$, we have the L^p-bound*

$$\|\xi_{A,B}\|_p \le \|\xi_{|A-B|,0}\|_p = \Big(\sum_{n=1}^{\infty} \Big[n^p - (n-1)^p\Big]\mu_n \Big)^{1/p}.$$

Remark:

$$\Big(\sum_{n=1}^{\infty}\Big[n^p-(n-1)^p\Big]\mu_n\Big)^{1/p}\le\sum_{n=1}^{\infty}\mu_n^{1/p}.$$

Proof: With $\mu(n):=\mu_n-\mu_{n+1}\ge 0$ rewrite

$$\Big(\sum_{n=1}^{\infty}\Big[n^p-(n-1)^p\Big]\mu_n\Big)^{1/p}=\Big(\sum_{n=1}^{\infty}n^p\mu(n)\Big)^{1/p}$$

The right-hand side is the l^p-norm of the function $n\to n^p$ in the weighted l^p-space $l^p(\mu)$. Write $n=1+(n-1)$ and use Minkowski's inequality to get

$$\begin{aligned}
&\Big(\sum_{n=1}^{\infty}n^p\mu(n)\Big)^{1/p}\le\\
&\quad\Big(\sum_{n=1}^{\infty}\mu(n)\Big)^{1/p}+\Big(\sum_{n=2}^{\infty}(n-1)^p\mu(j)\Big)^{1/p}\\
&\quad=\mu_1^{1/p}+\Big(\sum_{n=2}^{\infty}(n-1)^p\mu(n)\Big)^{1/p}\le\dots\\
&\quad\le\sum_{n=1}^{N}\mu_n^{1/p}+\Big(\sum_{n=N}^{\infty}(n-N)^p\mu(n)\Big)^{1/p}.
\end{aligned}$$

■

The Proof

Let $m_f(t) := |\{\lambda : |f(\lambda)| > t\}|$. We will write $m_{A,B}$ for the distribution function of $\xi_{A,B}$.

Lemma 4 (Basic Lemma) *With $C = A - B$, we have for all $n \in \mathbb{N}_0$*

$$\int_n^\infty m_{A,B}(t)\,dt \le \sum_{j=n+1}^\infty \mu_j(C) = \int_n^\infty m_{|C|,0}(t)\,dt.$$

Proof: Set $(x-s)_+ := \sup\{0, x-s\}$. Then

$$\int_s^\infty m_f(t)\,dt = \int (|f(\lambda)| - s)_+\,d\lambda \qquad (3)$$

for all $s \ge 0$. Write

$$|\xi_{A,B}| = |\xi_{A,B+C_n} + \xi_{B+C_n,B}| \le |\xi_{A,B+C_n}| + n,$$

with $C_n := \sum_{j=1}^n \mu_j(C)\langle \phi_j, .\rangle \psi_j$. Thus

$$(|\xi_{A,B}(\lambda)| - n)_+ \le |\xi_{A,B+C_n}(\lambda)|.$$

Using (3), we get

$$\begin{aligned}\int_n^\infty m_{A,B}(t)\,dt &= \int (|\xi_{A,B}(\lambda)| - n)_+\,d\lambda \\ &\le \int |\xi_{A,B+C_n}(\lambda)|\,d\lambda = \mathrm{tr}\Big(C - C_n\Big).\end{aligned}$$

■

Lemma 5 *For any non-negative, convex function F on $[0,\infty)$ which vanishes at zero, there exists a non-negative, locally finite measure ν_F on $[0,\infty)$ such that*

$$F(t) = \int_0^\infty (t-u)_+ \nu_F(du) \quad \textit{for all } t \geq 0.$$

F is strictly convex if and only if ν_F is strictly positive, that is, $\nu_F([a,b]) > 0$ for all $0 \leq a < b$.

Proof: Let F' be the left derivative of F, $F'(0) := 0$. Define ν_F by

$$\nu_F([a,b)) := F'(b) - F'(a).$$

Then $F'(s) = \nu_F([0,s))$. Calculate

$$\int_0^\infty (t-u)_+ \nu_F(du) = \int_{[0,t)} \int_u^t ds\, \nu_F(du)$$

$$= \int_0^t \nu_F([0,s))\, ds = \int_0^t F'(s)\, ds = F(t).$$

■

Lemma 5 gives

$$\int F(|f(\lambda)|)\,d\lambda = \int_0^\infty \int (|f(\lambda)| - u)_+ \,d\lambda \nu_F(du)$$
$$= \int_0^\infty \underbrace{\int_u^\infty m_f(u)\,du}_{=:\mathbf{Q_f(u)}} \nu_F(du)$$

Hence we have

Lemma 6 *Let F be any non-negative, convex function F on $[0,\infty)$ which vanishes at zero. Given two functions f and g, $Q_f \le Q_g$ implies*

$$\int F(|f(\lambda)|)\,d\lambda \le \int F(|g(\lambda)|)\,d\lambda.$$

Moreover, if F is strictly convex and $Q_f < Q_g$ on a set of positive Lebesgue measure, then the inequality above is strict.

Lemma 7 *Suppose that g takes only values in $\mathbb{N}_0$. Then the inequality $Q_f(n) \leq Q_g(n)$ for $n \in \mathbb{N}_0$ implies*

$$Q_f(t) \leq Q_g(t) \qquad \textit{for all } t \geq 0.$$

Proof: Q_f and Q_g are convex AND Q_g is linear on $[n, n+1]$. The claim follows from convexity.

■

Proof of the Theorem: Given A and B, let $D = |A - B|$ and C be any non-negative trace class operator with

$$\sum_{j=n}^{\infty} \mu_j(D) \leq \sum_{j=n}^{\infty} \mu_j(C) \qquad \text{for all } n \in \mathbb{N}.$$

The Basic Lemma shows

$$Q_{\xi_{A,B}}(n) \leq Q_{\xi_{|D|,0}}(n) \leq Q_{\xi_{C,0}}(n) \quad \text{for all } n \in \mathbb{N}_0. \tag{4}$$

Lemma 7 then implies that (4) extends from $\mathbb{N}_0$ to all positive real n. Once one has that, Lemma 6 proves

$$\int F(|\xi_{A,B}(\lambda)|)\, d\lambda \leq \int F(|\xi_{C,0}(\lambda)|)\, d\lambda.$$

Fubini-Tonelli implies summation by parts*

$$\sum_{j=1}^{\infty} F(j)\Big(\mu_j - \mu_{j+1}\Big)$$

$$= \sum_{j=1}^{\infty} \sum_{n=1}^{j} \Big(F(n) - F(n-1)\Big)\Big(\mu_j - \mu_{j+1}\Big)$$

$$= \sum_{1 \le n \le j} \Big(\underbrace{F(n) - F(n-1)}_{\ge 0}\Big)\Big(\underbrace{\mu_j - \mu_{j+1}}_{\ge 0}\Big)$$

$$= \sum_{n=1}^{\infty} \Big(F(n) - F(n-1)\Big) \sum_{j=n}^{\infty} \Big(\mu_j - \mu_{j+1}\Big)$$

$$= \sum_{n=1}^{\infty} \Big(F(n) - F(n-1)\Big)\mu_n,$$

since $\sum_{j=n}^{\infty}\Big(\mu_j - \mu_{j+1}\Big)$ telescopes to μ_n.

*Or Riemann integral = Lebesgue integral!

Remark:

If

$$\mu_n = \frac{1}{n^p \ln(n+1)^\alpha}$$

then

$$\sum_{n=1}^{\infty} \mu_n^{1/p} < \infty$$

if and only if $\alpha > p$.

Whereas

$$\sum_{n=1}^{\infty} \Big(n^p - (n-1)^p\Big)\mu_n < \infty$$

if and only if $\alpha > 1$.