Variational Estimates for Discrete Schrödinger Operators with Potentials of Indefinite Sign

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The discrete Schrödinger operator

$$(Hu)(n) = \sum_{|n-n'|=1} u(n') + V(n)u(n) \qquad (1)$$

with bounded potential $V : \mathbb{Z}^d \to \mathbb{R}$ on $l^2(\mathbb{Z}^d)$.

The free (discrete) Schrödinger operator, H_0, corresponds to the case $V = 0$.
Note that $-2d \leq H_0 \leq 2d$.

Remark: The discretization of the Laplacian corresponds to $H_0 + 2d$.

Basic properties:

- $\sigma(H_0) = \sigma_{\text{ess}}(H_0) = \sigma_{\text{ac}}(H_0) = [-2d, 2d]$.
- If $V = 0$, then $\sigma(H) = \sigma(H_0) = [-2d, 2d]$.
- If $\lim_{|n|\to\infty} V(n) = 0$ (i.e., V is compact) then

  $$\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H_0) = [-2d, 2d]$$

  (Weyl's theorem).

Question: Is there a converse to the above, and if so, how does the answer depend on the dimension?

Motivation:

- (Killip-Simon:) Let $d = 1$. If $\sigma(H) \subset [-2, 2]$, then $V = 0$.

- (Rakhmanov) Let J be a general half-line Jacobi matrix on $l^2(\mathbb{N})$,

$$(Ju)(n) = a_n u(n+1) + b_n u(n) + a_{n-1} u(n-1) \quad (2)$$

where $a_n > 0$ and $\mathbb{N} = \{1, 2, \dots\}$. Suppose that $[-2, 2]$ is the essential support of the a.c. part of the spectral measure and also the essential spectrum. Then $\lim_{n\to\infty} |a_n - 1| + |b_n| = 0$, that is, J is a compact perturbation of J_0, the Jacobi matrix with $a_n = 1$, $b_n = 0$.

The result: There is a converse, iff $d \leq 2$.

Theorem 1. *Let $d \leq 2$.*
a) If $\sigma(H) \subset [-2d, 2d]$, then $V = 0$.
b) If $\sigma_{\text{ess}}(H) \subset [-2d, 2d]$, then $V(n) \to 0$ as $n \to \infty$.
(Converse to Weyl's theorem.)

Theorem 2. *Let $d \geq 3$. There exists a potential V with $\limsup_{n\to\infty} V(n) > 0$ such that $\sigma_{\text{ess}}(H) = [-2d, 2d]$.*

Remark:

- It is easy to see that for $d \geq 3$ there are potentials $V \not\equiv 0$ such that $\sigma(H) = \sigma(H_0)$ (choose V to be a small enough bump).
- Of course, if V has a fixed sign, both results in Theorem 1 are easily proven by a simple variational argument using very simple test functions!
- The proof of Theorem 2 is done by constructing a (sparse!) potential $V \geq 0$ such that the Birman-Schwinger operator $\sqrt{V}(2d - H_0)^{-1}\sqrt{V}$ has norm strictly less than one. This is possible, since $(2d - H_0)^{-1}(n, m) \sim |n - m|^{-(d-2)}$ as $|n - m| \to \infty$.

More results:

- Let $d = 1$. Let H be an arbitrary one-dimensional discrete Schrödinger operator. Then $\sup \sigma_{\text{ess}}(H) - \inf \sigma_{\text{ess}}(H) \geq 4$ with equality if and only if $V(n) \to V_\infty$ a constant as $|n| \to \infty$.
- On $l^2(\mathbb{N})$ (half-line): If $\sigma(H) = [-2, 2]$ (i.e., no bound states), then

$$|V(n)| \leq 2n^{-1/2}$$

- If $|V(n)| \geq Cn^{-\alpha}$ and H has only finitely many bound states, then $\alpha \geq 1$.
- If $|V(n)| \geq Cn^{-\alpha}$ with $\alpha = 1$ and $C > 1$ or $\alpha < 1$ and $C > 0$, then H has infinitely many bound states.
- Let $|V(n)| \geq Cn^{-\alpha}$ with $\alpha < 1$. Then

$$\sum_j (|E_j| - 2d)^\gamma = \infty \quad \text{for } \gamma < \frac{1-\alpha}{2\alpha},$$

  where E_j are the eigenvalues of H outside $[-2d, 2d]$. In particular, the eigenvalue sum $\sum_j (|E_j| - 2)^{1/2}$ diverges if $\alpha < 1$.

The variational estimate

Recall: $H_0 u(n) = \sum_{|j|=1} u(n+j)$.

Then $-2d \leq H_0 \leq 2d$ and

$$\langle \varphi, (2d - H_0)\varphi \rangle = \frac{1}{2} \sum_n \sum_{|l|=1} |\varphi(n+l) - \varphi(n)|^2$$

Let U be given by $U\varphi(n) = (-1)^{|n|}\varphi(n)$. Then

$$UH_0U^{-1} = -H_0 \text{ and } UVU^{-1} = V.$$

for any multiplication operator.

Def: For $H = H_0 + V$ and $\varphi_+, \varphi_- \in l^2(\mathbb{Z}^d)$ set

$$\begin{aligned}\Delta(\varphi_+, \varphi_-) &= \Delta(\varphi_+, \varphi_-, V) \\ &= \langle \varphi_+, (H - 2d)\varphi_+ \rangle + -\langle \varphi_-, (H + 2d)\varphi_- \rangle\end{aligned}$$

Note:

$\Delta(\varphi_+, \varphi_-) > 0$ implies that H has spectrum outside $[-2d, 2d]$.

Lemma 3. *For* $f, g \in \ell^2(\mathbb{Z}^d)$

$$\Delta(f+g, U(f-g)) \geq 2\langle f, (H_0 - 2d)f\rangle - 8d\|g\|^2 + 4\mathrm{Re}\langle f, Vg\rangle \quad (3)$$

Proof.

$$\begin{aligned}\Delta(f+g, U(f-g)) &= \\ &= \langle (f+g), (H_0 - 2d + V)(f+g)\rangle \\ &\quad + \langle (f-g), (H_0 - 2d - V)(f-g)\rangle \\ &= 2\langle f, (H_0 - 2d)f\rangle + 2\underbrace{\langle g, (H_0 - 2d)g\rangle}_{\geq -4d\|g\|^2} + 4\mathrm{Re}\langle f, Vg\rangle\end{aligned}$$

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Remark: An obvious choice is to take $f = \varphi$, $g = tV\varphi$. The potential term on the r.h.s. of (3) is then

$$4t(-2dt + 1)\langle \varphi, V^2\varphi\rangle$$

which is maximal for $t = \frac{1}{4d}$ with $4t(-2dt+1) = \frac{1}{2d}$.

Thus we get

Lemma 4 (Comparison lemma). *For any* $\varphi \in l^2(\mathbb{Z}^d)$,

$$\Delta \geq 2\langle\varphi, (H_0 - 2d + \tfrac{1}{4d}V^2)\varphi\rangle \tag{4}$$

Note: The pair of test functions in this case is given by $\varphi_+ = \varphi(1 + \frac{1}{4d}V)$ and $\varphi_- = U(\varphi(1 - \frac{1}{4d}V))$

Sometimes one would like to cut off large values of V (to make the norm of φ and $\varphi_\pm$ comparable). For this one can use $f = \varphi$ and $g = tFV\varphi$ for some function $0 \leq F \leq 1$. A similar calculation shows that the potential terms in the r.h.s. of (3) are given by

$$\begin{aligned} &- 8dt^2\langle\varphi, VF^2V\varphi\rangle + 4t\langle\varphi, VFV\varphi\rangle \\ &\geq -8dt^2\langle\varphi, VFV\varphi\rangle + 4t\langle\varphi, VFV\varphi\rangle \\ &= 4t(-2dt + 1)\langle\varphi, FV^2\varphi\rangle \end{aligned}$$

since $F^2 \leq F$, and, again, the choice $t = \frac{1}{4d}$ is optimal.

So we also have

Lemma 5. *For any* $\varphi \in l^2(\mathbb{Z}^d)$, $0 \leq F \leq 1$,

$$\Delta \geq 2\langle\varphi, (H_0 - 2d + \tfrac{1}{4d}FV^2)\varphi\rangle. \tag{5}$$

Theorem 6. *Let $V(n) \to 0$ as $n \to \infty$. If $H_0 + \frac{1}{4d}V^2$ has at least one eigenvalue outside $[-2d, 2d]$, then so does $H_0 + V$. Moreover, if $H_0 + \frac{1}{4d}V^2$ has infinitely many eigenvalues outside $[-2d, 2d]$, then so does $H_0 + V$.*

Proof. The first part is just the bound (4) (+ minmax theorem), since the r.h.s. of (4) will be positive for a suitable chosen test function φ.

For the second part one has to play with local compactness type arguments to see that one can find a sequence φ_n such that each φ_n has finite support, $\langle \varphi_n, H_0 + \frac{1}{4d}V^2 \varphi_n \rangle > 2d \,\|\varphi_n\|^2$, and dist(supp($\varphi_l$), supp($\varphi_m$)) ≥ 2 for all $l \neq m$.

This (+ Lemma 4) gives the existence of a sequence $\tilde{\varphi}_n$ such that either

$$\langle \tilde{\varphi}_n, (H_0 + V)\tilde{\varphi}_n \rangle > 2d\|\tilde{\varphi}_n\|^2$$

or

$$\langle \tilde{\varphi}_n, (H_0 + V)\tilde{\varphi}_n \rangle < 2d\|\tilde{\varphi}_n\|^2$$

and (since supp($\tilde{\varphi}_n$) $\subset$ supp(φ_n)) we also have

$$\langle \tilde{\varphi}_l, \tilde{\varphi}_m \rangle = 0 = \langle \tilde{\varphi}_l, (H_0 + V)\tilde{\varphi}_m \rangle \quad \text{for } l \neq m.$$

Thus minmax applies again. ∎

Essential spectrum and compactness:

Proposition 7 (= Theorem 1.b). *Let $d \leq 2$. If $\sigma_{\mathrm{ess}}(H_0 + V) = \sigma_{\mathrm{ess}}$, then $V(n) \to 0$ as $n \to \infty$.*

The key to this result is the fact that in dimension $d \leq 2$ there are sequences φ_n such that $\varphi_n(0) = 1$ and $\langle \varphi_n, (2d - H_0)\varphi_n \rangle \to 0$.

This is not possible for $d \geq 3$. Indeed, let $1 = \varphi(0) = \langle \varphi, \delta_0 \rangle$. Since

$$\begin{aligned} 1 &= \langle \varphi, \delta_0 \rangle \\ &= \langle (2d - H_0)^{1/2}\varphi, (2d - H_0)^{-1/2}\delta_0 \rangle \\ &\leq \left\| (2d - H_0)^{1/2}\varphi \right\| \left\| (2d - H_0)^{-1/2}\delta_0 \right\|, \end{aligned}$$

we see

$$\langle \varphi, (2d - H_0)\varphi \rangle \geq \langle \delta_0, (2d - H_0)^{-1}\delta_0 \rangle^{-1} > 0.$$

So any $\varphi \in l^2(\mathbb{Z}^d)$ with $\varphi(0) = 1$ has a minimal kinetic energy in dimension $d \geq 3$.

Lemma 8. *a) Let* $L_1, L_2 \geq 1$. *There exists* $\varphi_{L_1,L_2} \in l^2(\mathbb{Z})$, *supported in* $[-L_1, L_2]$, *so that*

(i) $\varphi_{L_1,L_2}(0) = 1$

(ii) $\langle \varphi_{L_1,L_2}, (2-H_0)\varphi_{L_1,L_2} \rangle = (L_1+1)^{-1} + (L_2+1)^{-1}$

(iii) *for suitable constants* $c_1 > 0$ *and* $c_2 < \infty$,

$$c_1(L_1 + L_2) \leq \|\varphi_{L_1,L_2}\|^2 \leq c_2(L_1 + L_2)$$

b) Let $L \geq 1$. *There exists* $\varphi_L \in l^2(\mathbb{Z}^2)$ *supported in* $\{(n_1, n_2) \mid |n_1| + |n_2| \leq L\}$ *so that*

(i) $\varphi_L(0) = 1$

(ii) $0 \leq \langle \varphi_L, (4 - H_0)\varphi_L \rangle \leq c[\ln(L + 1)]^{-1}$ *for some* $c > 0$

(iii) $(L^{-1} \ln(L))^2 \|\varphi_L\|^2 \to d > 0$

Proof. Define

$$\varphi_{L_1,L_2}(n) = \begin{cases} 1 - \frac{n}{L_2+1} & 0 \le n \le L_2 + 1 \\ 1 - \frac{|n|}{L_1+1} & 0 \le -n \le L_1 + 1 \\ 0 & n \ge L_2 + 1 \text{ or } n \le -L_1 - 1 \end{cases}$$

and

$$\varphi_L(n_1, n_2) = \begin{cases} \frac{-\ln[(1+|n_1|+|n_2|)/(L+1)]}{\ln(L+1)} & \text{if } |n_1| + |n_2| \le L \\ 0 & \text{if } |n_1| + |n_2| \ge L. \end{cases}$$

These will do the job. ∎

Remark: If $\psi(0) = 1$ and $\text{supp}(\psi) \subset [-L_1, L_2]$,

$$\sum_{j=1}^{L_2+1} \psi(j) - \psi(j-1) = -1$$

so, by Cauchy-Schwarz,

$$1 \le (L_2 + 1) \sum_{j=1}^{L_2+1} |\psi(j) - \psi(j-1)|^2$$

Hence

$$\langle \psi, (2 - H_0)\psi \rangle \ge (L_1 + 1)^{-1} + (L_2 + 1)^{-1}$$

and φ_{L_1,L_2} is an extremal function.

Proof of Proposition 7 ($d = 1$):

Assume $\limsup |V(n)| = a > 0$. Pick L with $2(L+1)^{-1} < \frac{1}{8}\min(a^2, 2a)$. Pick a sequence n_j with $|V(n_j)| \to a$ and $|n_j - n_l| \geq 2(L+2)$ for all $j \neq l$.

Set

$$F(n) = \min\left(1, \frac{2}{|V(n)|}\right) \tag{6}$$

and let $\psi_j(n) = \varphi_{L,L}(n - n_j)$. Then

$$\begin{aligned}
&\langle \psi_j, (H_0 - 2 + \tfrac{1}{4} FV^2)\psi_j \rangle \\
&\quad \geq -2(L+1)^{-1} + \tfrac{1}{4} F(n_j)V(n_j)^2 \\
&\quad \geq -\tfrac{1}{8} \min(a^2, 2a) + \tfrac{1}{4} \min(|V(n_j)|^2, 2|V(n_j)|)
\end{aligned}$$

Thus we have that

$$\liminf \langle \psi_j, (H_0 - 2 + \tfrac{1}{4} FV^2)\psi_j \rangle \geq \tfrac{1}{8} \min(a^2, 2a)$$

As $|FV| \leq 2$, if $\varphi_{\pm,j} = (1 \pm \frac{1}{4} FV)\psi_j$, we have

$$\tfrac{1}{2} \|\psi_j\| \leq \|\varphi_{\pm,j}\| \leq \tfrac{3}{2} \|\psi_j\| \leq C_L \tag{7}$$

where C_L is independent of j.

By Lemma 5, we have a subsequence of j's so that either

$$\liminf \langle \varphi_{+,j_\ell}, (H_0 + V - 2)\varphi_{+,j_\ell} \rangle \geq \tfrac{1}{16} \min(a^2, 2a)$$

or

$$\liminf \langle \varphi_{-,j_\ell}, (-H_0 - V - 2)\varphi_{+,j_\ell} \rangle \geq \tfrac{1}{16} \min(a^2, 2a)$$

Moreover, the φ's are orthogonal. Thus H has essential spectrum in either

$$[2 + \tfrac{1}{16} C_L^{-1} \min(a^2, 2a), \infty)$$

or

$$(-\infty, -2 - \tfrac{1}{16} C_L^{-1} \min(a^2, 2a)]$$

The proof of the two dimensional result is similar, using the second part of Lemma 8. ■

Proposition 9 (= Theorem 1.a). *Let* $d \leq 2$. *If* $\sigma_0(H_0 + V) \subset [-2d, 2d]$, *then* $V = 0$.

Proof. By Proposition 7, $V(n) \to 0$. According to Theorem 6, if $H_0 + V$ has no bound states, neither does $H_0 + \frac{1}{4d}V^2$.

Let φ_L be the function guaranteed by Lemma 8. Then

$$\begin{aligned} 0 &\geq \langle \varphi_L, (H_0 + \frac{1}{4d}V^2 - 2d)\varphi_L \rangle \\ &\geq \frac{1}{4d}V(0)^2 + \langle \varphi_L, (H_0 - 2d)\varphi_L \rangle \end{aligned}$$

Since $\langle \varphi_L, (H_0 - 2d)\varphi_L \rangle \to 0$, we must have $V(0)^2 = 0$. By translation invariance, $V(n) = 0$ for all n.

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Decay and bound states:

In the following we will consider the half-line operator on $l^2(\mathbb{N})$. We will use J_0 for H_0 with Dirichlet b.c. at 0 (say).

Proposition 10. *Suppose* $V(n) \geq 0$ *and that* $J_0 + V$ *has no bound states. Then*

$$|V(n)| \leq n^{-1}$$

Moreover, this bound cannot be improved in that for each n_0*, there exists* V_{n_0} *so that* $V_{n_0}(n_0) = n_0^{-1}$ *and* $J_0 + V_{n_0}$ *has no bound states.*

Proof: Let $n_0 \in \mathbb{N}$. $J_0 + \delta_{n_0}$ has a bound state if and only if $|\lambda| > n_0^{-1}$. Indeed, w.l.o.g. assume $\lambda > 0$. By Sturm oscillation theory, there is a bound state in $(2, \infty)$ iff the solution of

$$u(n+1) + u(n-1) + \lambda\delta_{n_0}(n)u(n) = 2u(n)$$

with $u(0) = 0$ and $u(1) = 1$ has a negative value for some $n \in \mathbb{N}$.

The solution is given by

$$u(n) = \begin{cases} n & n \leq n_0 \\ n_0 + (1 - \lambda n_0)(n - n_0) & n \geq n_0 \end{cases}$$

Thus u gets negative iff $\lambda n_0 > 1$.

If $V(n_0) > n_0^{-1}$, then $V(n) \geq V(n_0)\delta_{n_0}(n)$ for all n (recall that $V \geq 0$). By the minmax theorem and the fact that $J_0 + V(n_0)\delta_{n_0}$ has a bound state, $J_0 + V$ must have a bound state in $(2, \infty)$.

The contrapositive of $V(n_0) > n_0^{-1} \Rightarrow \sigma(J_0 + V) \neq [-2, 2]$ gives the first assertion of the theorem.

The second follows from an explicit example. ■

Theorem 11. *If $J_0 + V$ has no bound states, then*

$$|V(n)| \leq 2n^{-1/2}$$

Moreover, the above bound cannot be improved by more than a factor of 2 in that for each n_0, there exists V_{n_0} so that $J_0 + V_{n_0}$ has no bound states and

$$\lim_{n_0 \to \infty} n_0^{1/2} |V_{n_0}(n_0)| = 1$$

Proof. If $J_0 + V$ has no bound states so does $J_0 + \frac{1}{4}V^2$. (By an easy extension of the comparison Theorem 6 to the half-line case.)

By Proposition 10, we must have $\frac{1}{4}V^2(n) \leq n^{-1}$.

The second claim follows from analysing the potential

$$W_{n_0} = \begin{cases} 1 & n = n_0 \\ -1 & n = n_0 + 1 \\ 0 & n \neq n_0, n_0 + 1 \end{cases}$$

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The case $|V(n)| \sim \beta n^{-\alpha}$:

Theorem 12. *If* $\liminf_{n\to\infty} n|V(n)| > 1$, *then* $J_0 + V$ *has infinitely many eigenvalues outside* $[-2, 2]$.

Remark: We only need that for some sequence $n_k \to \infty$

$$\frac{2}{n_k} \sum_{j=n_k/2}^{n_k} V(j)^2$$

is not too small (that is, the potential may have zeros, but should not be sparse).

If $|V(n)| = \beta n^{-\alpha}$ and $\alpha < 1$, then $J_0 + V$ has infinitely many eigenvalues outside $[-2, 2]$.

If $\alpha = 1$ and $\beta > 1$, then $J_0 + V$ has infinitely many eigenvalues outside $[-2, 2]$.

For $\beta \in [-1, 1]$, the potential $V(n) = \beta(-1)^{n-1}/n$ has no bound states (on $l^2(\mathbb{N})$).

A preparatory lemma:

Lemma 13. *For* $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}_+)$ *let*

$$h_\lambda[\varphi] = \int \Big(|\varphi'(x)|^2 - \lambda \frac{|\varphi(x)|^2}{x^2}\Big)\, dx.$$

If $\lambda > \frac{1}{4}$, *then there exists a function* $\varphi \in \mathcal{C}_0^\infty((0,1))$ *such that* $h[\varphi] < 0$.

More generally, let

$$h_{\lambda,\delta}[\varphi] = \int \Big(|\varphi'(x)|^2 - \lambda \frac{|\varphi(x)|^2}{(x+\delta)^2}\Big)\, dx$$

for $\delta \geq 0$. *If* $\lambda > \frac{1}{4}$, *then there exist* $\delta_0 > 0$ *and* $\varphi \in \mathcal{C}_0^\infty((0,1))$ *such that* $h_{\lambda,\delta}[\varphi] < 0$ *for all* $0 \leq \delta \leq \delta_0$.

Proof: The proof of the first part is just the proof of Hardy's inequality, once the coupling constant is larger than $\frac{1}{4}$, the form is unbounded from below (by scaling).

Also by scaling, once one has a single φ with compact support for which $h[\varphi]$ is negative, there are infinitely many of those with support in $(0,1)$.

The second part follows from the first one by continuity in δ. ■

Proof of Theorem 12: By Lemma 4, it is enough to find a sequence of functions $(\varphi_k)_k$ with disjoint supports, such that

$$A[\varphi_k] = \langle \varphi_k, (2 - H_0)\varphi_k \rangle - \tfrac{1}{4}\langle \varphi_k, V^2 \varphi_k \rangle < 0 \qquad (8)$$

infinitely often.

Put $U_k = 2^{k-l}$, $O_k = 2^{k+l}$, then $D_k = O_k - U_k = (4^l - 1)2^{n-l}$. Choose $\varepsilon > 0$ with

$$r := \limsup (nV(n))^2 - \varepsilon > 1.$$

Furthermore, put $\delta_l = \frac{1}{4^l - 1}$. According to Lemma 13, there exist l and $\varphi \in \mathcal{C}_0^\infty(0,1)$ such that $h_{r/4,\delta_l}[\varphi] < 0$.

With $\varphi_k(n) = \varphi(\frac{n-U_k}{D_k}) = \varphi(\frac{n}{D_k} - \frac{U_k}{D_k}) = \varphi(\frac{n}{D_k} - \delta_l)$ we have

$$\begin{aligned}\langle \varphi_k, (2 - H_0)\varphi_k \rangle &= \sum_n |\varphi_k(n+1) - \varphi_k(n)|^2 \\ &= \sum_n |\varphi(\tfrac{n+1}{D_k}) - \varphi(\tfrac{n}{D_k})|^2 \\ &= D_k^{-1} \sum_n \left| \frac{\varphi(\frac{n+1}{D_k}) - \varphi(\frac{n}{D_k})}{1/D_k} \right|^2 D_k^{-1}\end{aligned}$$

and

$$\langle \varphi_k, V^2 \varphi_k \rangle = \sum_n |\varphi_k(n)|^2 V(n)^2$$
$$= \sum_n |\varphi(\tfrac{n}{D_k})|^2 V(n + U_k)^2$$
$$= \sum_n |\varphi(\tfrac{n}{D_k})|^2 V(D_k(\tfrac{n}{D_k} + \delta_l))^2.$$

Since, $\liminf_{n\to\infty} n|V(n)| > 1$, there exists K_0 such that, for all $k \geq K_0$ and all $n \geq 0$,

$$V(D_k(\tfrac{n}{D_k} + \delta_l))^2 \geq \frac{r}{(D_k(\frac{n}{D_k} + \delta_l))^2} = D_k^{-2} \frac{r}{(\frac{n}{D_k} + \delta_l)^2},$$

in particular,

$$\langle \varphi_k, V^2 \varphi_k \rangle \geq D_k^{-1} \sum_n |\varphi(\tfrac{n}{D_k})|^2 \frac{r}{(\frac{n}{D_k} + \delta_l)^2} D_k^{-1}.$$

for all large enough k.

Since

$$\sum_n \left|\frac{\varphi(\frac{n+1}{D_k}) - \varphi(\frac{n}{D_k})}{1/D_k}\right|^2 D_k^{-1} \to \int |\varphi'(x)|^2\, dx$$

and

$$\sum_n |\varphi(\tfrac{n}{D_k})|^2 \frac{r}{(\frac{n}{D_k} + \delta_l)^2} D_k^{-1} \to r \int \frac{|\varphi(x)|^2}{(x+\delta_l)^2}$$

as $k \to \infty$, we see that

$$\lim_{k\to\infty} D_k A[\varphi_k] = \int \Big(|\varphi'(x)|^2 - \tfrac{r}{4}\frac{|\varphi(x)|^2}{(x+\delta)^2}\Big)\, dx$$

$$= h_{r/4,\delta_l}[\varphi] < 0.$$

To finish the proof, one only has to note that we can choose a subsequence k_j such that $\mathrm{dist}(\mathrm{supp}(\varphi_l), \mathrm{supp}(\varphi_m)) \geq 2$ for all $l \neq m$.

Blowup of suitable eigenvalue moments:

Lemma 14. *Let* $|V| \leq 4d$ *on* supp(φ). *Then there exists* ψ *with* supp$(\psi) =$ supp(φ) *so that*

$$\|\psi\|^{-2}\Big|\langle\psi,(H_0+V)\psi\rangle\Big| - 2d$$
$$\geq \tfrac{1}{4}\Big[\|\varphi\|^{-2}\langle\varphi,(H_0+\tfrac{1}{4d}V^2)\varphi\rangle - 2d\Big]$$

Proof. Put $\psi_\pm = (1 \pm (4d)^{-1})\varphi$. By assumption, $\|\psi_\pm\| \leq 4\,\|\varphi\|$. The claim follows from the basi- Comparison Lemma 4, choosing either ψ_+ or ψ_- for ψ. ∎

To convert this into bounds for eigenvalue moments we use the following elementary

Lemma 15. *Let* A *be a bounded selfadjoint operator. Let* $\{\varphi_j\}_{j=1}^{\infty}$ *be an orthonormal set with*

$$\langle\varphi_j, A\varphi_k\rangle = \alpha_j\delta_{jk}$$

If F *is a nonnegative even function on* $\mathbb{R}$ *that is monotone nondecreasing on* $[0,\infty)$, *then*

$$\mathrm{tr}\Big(F(A)\Big) \geq \sum_j F(\alpha_j)$$

Theorem 16. *Let J be a Jacobi matrix of the form $J_0 + V$ where*

$$|V(n)| \geq Cn^{-\alpha}$$

for some $\alpha < 1$ and $V(n) \to 0$. Then

$$\sum_j \Big(|E_j| - 2\Big)^\gamma = \infty$$

for

$$\gamma < \frac{1-\alpha}{2\alpha}$$

where E_j are eigenvalues of J outside $[-2, 2]$.

Remark: If the constant C is large enough, one gets divergence for $\gamma < \frac{1-\alpha}{2\alpha}$.

An inspection of the proof shows that this result is *independent* of the dimension! (Replacing $|E_j| - 2$ by $|E_j| - 2d$, of course.)

Proof. Fix $p > 0$. Let φ_m be supported near m^{p+1} on an interval $[m^{p+1} - C_1 m^p, m^{p+1} + C_1 m^p]$ (C_1 is picked to arrange that the supports are separated by at least 2). Taking the slopes fixed on each half-interval, we see

$$\langle \varphi_m, (2 - H_0)\varphi_m \rangle \leq \frac{C_2}{m^p} \tag{9}$$

$$\langle \varphi_m, \tfrac{1}{4} V^2 \varphi_m \rangle \geq \frac{C_3 m^p}{m^{2\alpha(p+1)}} \tag{10}$$

and

$$\langle \varphi_m, \varphi_m \rangle \geq C_4 m^p.$$

So long as $\alpha(p+1) < p$ (that is, $p < \frac{\alpha}{1-\alpha}$), (10) beats (9) for large m, and we find

$$\langle \varphi_m, \varphi_m \rangle^{-1} \langle \varphi_m, (H_0 + \tfrac{1}{4} V^2 - 2)\varphi_m \rangle \geq C_5 m^{-2\alpha(p+1)}$$

Note that, as $p \downarrow \frac{\alpha}{1-\alpha}$, $2\alpha(p+1) \downarrow \frac{2\alpha}{1-\alpha}$.

By the last lemma with $F(x) = \mathrm{dist}(x, [-2,2])^\gamma$, we see that we have divergence if $\gamma < \frac{1-\alpha}{2\alpha}$. ■