On sharp Strichartz inequalities in low dimensions

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Plan

- The Strichartz inequality
- Some History
- Proof of sharp Strichartz
  - The Representation Theorem
  - Classification of maximizers: how to use rotation invariance

The free Schrödinger evolution

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$$i\partial_t u = -\Delta u \qquad \text{on } L^2(\mathbb{R}^d)$$

With initial condition

$$u(0,x) = f(x) \in L^2(\mathbb{R}^d).$$

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Thus

$$u \in L_t^\infty L_x^2 \qquad \text{as a space-time function}$$

Strichartz inequality

Theorem (Strichartz 1977). *For* $p = p(d) = 2 + \frac{4}{d}$,

$$\|u\|_{L^p_{t,x}} = \Big(\int_{\mathbb{R}} dt \int_{\mathbb{R}^d} dx \, |u(t,x)|^p \Big)^{1/p} \leq S_d \|f\|_{L^2_x}$$

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the paraboloid $\tau = k^2$ has positive Gaussian curvature in $\mathbb{R}^{d+1}$. Now apply the Stein-Tomas theorem.

Some history

- Tomas 1975: Fourier restriction theorem for densities on the sphere
- Extended by Strichartz to some non-compact manifolds with non-vanishing Gaussian curvature.
- Much simplified proof by Ginibre and Velo 1985.
- Strichartz inequalities are at the heart of most studies of non-linear Schrödinger equations (from mid 1980 to now).

Some questions

- What is

$$S_d = \sup_{f \neq 0} \frac{\|e^{it\Delta} f\|_{L^p_{t,x}}}{\|f\|_{L^2_x}} = ?$$

- Existence of maximizers: Are there $f_* \in L^2(\mathbb{R}^d)$ with

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- all questions above turn out to be very hard to answer: The Strichartz inequality is invariant under all Galilei transformations (translations and boosts) and scaling, which is a *huge* non-compact group.

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- Milena Stanislavova 2004: for a related functional (the dispersion management functional) all maximizers are smooth.
- Damiano Foschi preprint 2004: $S_1 = 12^{-1/12}$ and $S_2 = 2^{-1/2}$ and Gaussians are (among the) maximizers.

The Representation Theorem

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Theorem (100DM and Vadim Zharnitsky 2006). *Let* $f \in L^2(\mathbb{R}^d)$.

a) *If* $d = 1$ *then*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |e^{it\Delta} f(x)|^6 \, dx dt = \frac{1}{2\sqrt{3}} \langle f \otimes f \otimes f, P_1(f \otimes f \otimes f) \rangle_{L^2(\mathbb{R}^3)}$$

b) *If* $d = 2$ *then*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} |e^{it\Delta} f(x)|^4 \, dx dt = \frac{1}{4} \langle f \otimes f, P_2(f \otimes f) \rangle_{L^2(\mathbb{R}^4)}.$$

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P_2: orthogonal projection onto functions in $L^2(\mathbb{R}^4)$ invariant under rotations fixing the $(1, 0, 1, 0)$ and $(0, 1, 0, 1)$ directions.

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i.e.,

$$S_1 \leq (2\sqrt{3})^{-1/6} = 12^{-1/12}.$$

and similarly,

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- If $f(x) = e^{-ax^2}$, then

$$f \otimes f \otimes f(\eta) = f(\eta_1)f(\eta_2)f(\eta_3) = e^{-a(\eta_1^2+\eta_2^2+\eta_3^2)} = e^{-a\eta^2}$$

is invariant under all rotations of $\mathbb{R}^3$.

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- In particular, $S_1 = 12^{-1/12}$ and analogously $S_2 = 2^{-1/2}$.

Classification of maximizers

Theorem (100DM and Vadim Zharnitsky). *a) If* $f \in L^2(\mathbb{R})$ *is not identically zero and* $f \otimes f \otimes f$ *is invariant under rotations of* $\mathbb{R}^3$ *which keep the* $(1,1,1)$ *direction fixed, then there exist* $A \in \mathbb{C}$, $\mu \in \mathbb{R}$, $\lambda > 0$, *and* $b \in \mathbb{C}$ *such that*

$$f(x) = Ae^{(-\lambda+i\mu)x^2+bx}.$$

b) If $f \in L^2(\mathbb{R}^2)$ *and* $f \otimes f$ *is invariant under rotations of* $\mathbb{R}^4$ *fixing* $(1,0,1,0)$ *and* $(0,1,0,1)$ *directions, then*

$$f(x) = Ae^{(-\lambda+i\mu)|x|^2+b\cdot x}$$

for some $A \in \mathbb{C}$, $\mu \in \mathbb{R}$, $\lambda > 0$, *and* $b \in \mathbb{C}^2$

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That is,

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then

$$F(v) = e^{-av^2}.$$

That is, F is a centered Gaussian.

Proof of the Representation Formula $d = 1$

Have

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$$u(t,x)^3 = \frac{1}{\sqrt{4\pi i t}^3} \Big(\int_{\mathbb{R}} e^{i(x-y)^2/(4t)} f(y)\, dy \Big)^3$$

and

$$|u(t,x)|^6 = \frac{1}{(4\pi|t|)^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-ix(\eta_1+\eta_2+\eta_3-\zeta_1-\zeta_2-\zeta_3)/(2t)} e^{-i(|\eta|^2-|\zeta|^2)/(4t)}$$
$$\overline{f(\eta_1) f(\eta_2) f(\eta_3)} f(\zeta_1) f(\zeta_2) f(\zeta_3)\, d\eta d\zeta$$

Now do the x-integration with the change of variables $x = 2tz$, so $dx = 2|t|dz$, using $\delta(\beta) = \frac{1}{2\pi}\int e^{ix\beta}\,dx$,

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Then do the t-integration with $t = 1/(4\tau)$, so $dt = 4t^2 d\tau$,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |u(t,x)|^6 dx dt = \frac{1}{2\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\delta((1,1,1)(\eta-\zeta))\delta(|\eta|^2-|\zeta|^2)$$

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$$\int \int |u(t,x)|^6 \, dx dt = q(f \otimes f \otimes f, f \otimes f \otimes f)$$

Thus any bound on q yields a bound on S_1^6.

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Claim:

$$A = \frac{1}{2\sqrt{3}} P_1$$

where $P_1 : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^2)$ is the orthogonal projection operator mapping into functions F which are invariant under rotations of $\mathbb{R}^3$ around the $(1,1,1)$ direction.

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Fact 2:

$$2\sqrt{3}A \text{ is the identity on } \text{Range}(P_1)$$

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$$2\sqrt{3}A\widetilde{G}(\eta) = \frac{\sqrt{3}}{\pi}\int_{\mathbb{R}^3} \widetilde{G}((1,1,1)\cdot\zeta, |\zeta|^2)\delta(\tau - (1,1,1)\zeta)\delta(\xi - |\zeta|^2)\, d\zeta$$

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Gaussians and rotations

Gaussians and rotations

Toy problem: Assume that for $f \in L^2(\mathbb{R})$ the function

$$h = f \otimes f : \mathbb{R}^2 \to \mathbb{C}$$

$$(x, y) \mapsto h(x, y) := f(x)f(y)$$

is invariant under rotations of $\mathbb{R}^2$. Then f is a centered Gaussian.

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Thus $(\log(f))' = cx$ and f is a centered Gauss.

Step 2

Let $P_t(z) = \frac{1}{2\pi t} e^{-(z)^2/(2t)}$ for $z \in \mathbb{R}^2$ and $Q_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)}$ for $x \in \mathbb{R}$.

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$$P_t * h = Q_t * f \otimes Q_t * f$$

is also invariant under rotations and $Q_t * f$ is smooth and never vanishes, by assumption.

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In particular, $Q_t * f$ never vanishes!