

Decay estimates for Dispersion Management Solitons

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- Physical relevance.

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 - harmonic analysis meets real space.
 - Sub-harmonic rules!

The Dispersion Management functional

Interested in minimizers (for fixed $\|f\|_{L^2(\mathbb{R})}$) of

$$H(f) := \frac{\bar{d}}{2} \int_{\mathbb{R}} |f'|^2 dx - \frac{1}{4} \mathcal{Q}(f, f, f, f)$$

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with

$$\begin{aligned} \mathcal{Q}(f, f, f, f) &= \int_0^1 dt \int_{\mathbb{R}} dx |T_t f(x)|^4 \\ &= \int_0^1 \int_{\mathbb{R}} \overline{T_t f(x)} T_t f(x) \overline{T_t f(x)} T_t f(x) dx dt \end{aligned}$$

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and

$$T_t = e^{it\partial_x^2}.$$

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Stabilize pulses by **strongly periodically varying** the dispersion d .

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Effective equation for the average amplitude v

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Separation of variables: $v(t, x) = e^{i\omega t} f(x)$ yields

$$-\omega f = -\alpha f'' - Q(f, f, f)$$

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Solutions of this equation are critical points of

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Now widely used commercially: e.g., fast internet.

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 - Kunze (2004): Existence of a minimizer for $\bar{d} = 0$
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- **Conjecture:** No solution exists for negative average dispersion.

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Even for $\bar{d} = 0$, the functional $H(f)$ is bounded below:

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now use

$$a^3 b \leq \frac{1}{2} (\varepsilon^{-1} a^6 + \varepsilon b^2) \quad \text{Cauchy-Schwarz with epsilon}$$

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using one-dim Strichartz and unicity of T_t .

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Decay conjecture

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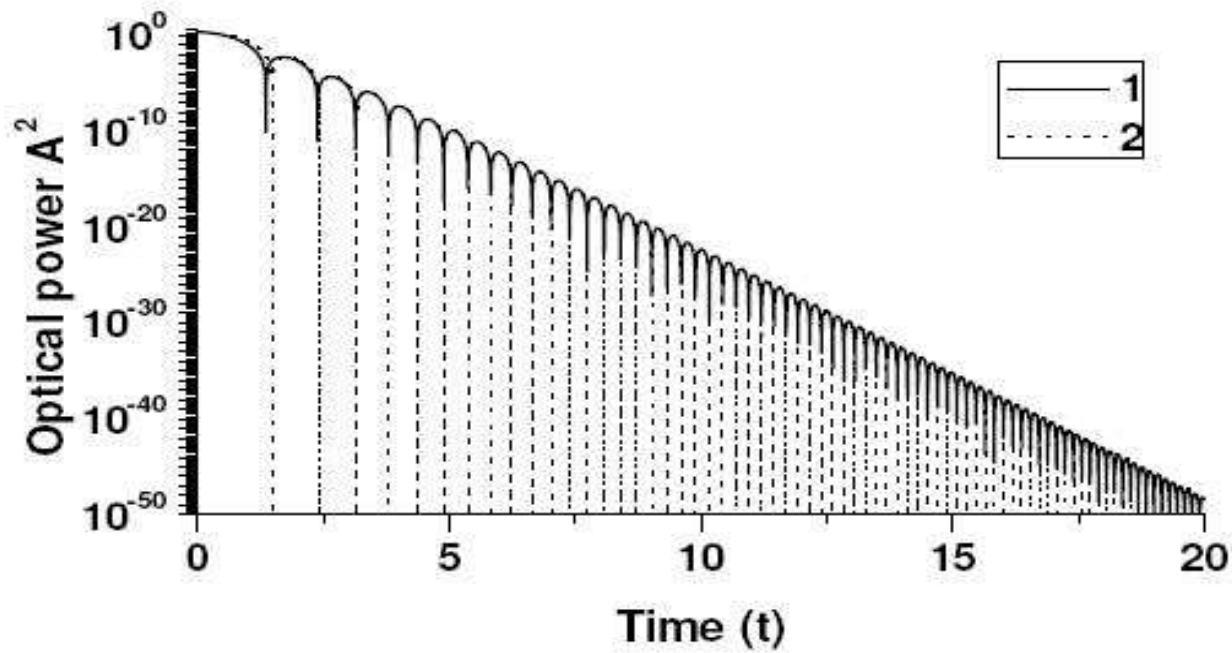
P. Lushnikov (2004) gave convincing arguments that the dispersion managed soliton should look like

$$f(x) \sim \cos(x^2(a_0 + O(x^{-1}))) \exp(-b|x|) \quad \text{as } x \rightarrow \infty.$$

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Regularity

Theorem (100DM and Young-Ran Lee (2007)). *Any weak solution of*

$$f = Q(f, f, f)$$

is a Schwartz function. That is,

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Recall: f is a weak solution if

$$\langle g, f \rangle = \langle g, Q(f, f, f) \rangle = Q(g, f, f, f)$$

for all $g \in L^2(\mathbb{R})$.

Estimates on the tail

Let $f \in L^2(\mathbb{R})$ be a weak solution of $f = Q(f, f, f)$. Set

$$\alpha(s) := \left(\int_{|x| \geq s} |f(x)|^2 dx \right)^{1/2}$$

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Theorem.

$$\alpha(s) \lesssim (\alpha(s/3))^3 + \frac{1}{\sqrt{s}} \alpha(0)^2 \alpha(s/3)$$

$$\hat{\alpha}(s) \lesssim (\hat{\alpha}(s/3))^3 + \frac{1}{\sqrt{s}} \hat{\alpha}(0)^2 \hat{\alpha}(s/3)$$

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If $\alpha \rightarrow 0$ at infinity and $\alpha(s) \lesssim (\alpha(s))^3$. Then α has **compact support!**

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Then

$$\alpha_d(s) \leq C(\alpha_d(s/3))^2\alpha_d(s/3) + C\frac{\alpha_d(s/3)}{\sqrt{s}} + \chi_{[d,\infty)}(s)\alpha(s)$$

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Hence

$$s^\gamma \alpha_d(s) \leq C 3^\gamma (\alpha_d(s/3))^2 (s/3)^\gamma \alpha_d(s/3) + C \frac{3^\gamma (s/3)^\gamma \alpha_d(s/3)}{\sqrt{s}} \\ + (3d)^\gamma \chi_{[d,3d)}(s) \alpha(s)$$

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and

$$\|s^\gamma \alpha_d\|_\infty \leq C 3^\gamma (\alpha(d))^2 \|s^\gamma \alpha_d\|_\infty + C \frac{3^\gamma \|s^\gamma \alpha_d\|_\infty}{\sqrt{s}} \\ + (3d)^\gamma \alpha(0)$$

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Rearrange to

$$\left(1 - C3^\gamma \alpha(d)^2 - \frac{C3^\gamma}{\sqrt{s}}\right) \|s^\gamma \alpha_d\|_\infty \leq (3d)^\gamma \alpha(0)$$

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In particular, $s \mapsto s^\gamma \alpha(s)$ is bounded or

$$\alpha(s) \lesssim s^{-\gamma} \quad \text{for all } \gamma > 0.$$

Corollary. *If $f \in L^2(\mathbb{R})$ solves $f = Q(f, f, f)$ weakly,*

i) $(1 + x^2)^{\gamma/2} f(x) \in L^2(\mathbb{R})$ and

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Homework: *If $(1 + x^2)^{n/2} f(x) \in L^2(\mathbb{R})$ and $D^n f \in L^2(\mathbb{R})$ for all $n \in \mathbb{N}_0$, then $(1 + x^2)^{n/2} D^m f \in L^2(\mathbb{R})$ for all $n, m \in \mathbb{N}_0$, i.e., f is a Schwartz function.*

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Remaining Question: How to get the a-priori bound on the tail distribution?

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It remains to estimate $Q(g, f, f, f)$ uniformly in g .

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The map $f_1, f_2, f_3, f_4 \mapsto \mathcal{Q}(f_1, f_2, f_3, f_4)$ is multi-linear.

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Fact 1: If g is supported in $\{|x| \geq s\}$ and f_1, f_2, f_3 are supported in $\{|x| \leq s/3\}$, then

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So even though \mathcal{Q} is **non-local**, it retains some locality (**quasi-local**).

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$$\begin{aligned} \mathcal{Q}(g, f, f, f) &= \mathcal{Q}(g, f_{<} + f_{>}, f_{<} + f_{>}, f_{<} + f_{>}) \\ &= \mathcal{Q}(g, f_{<}, f_{<}, f_{<}) + \mathcal{Q}(g, f_{>}, f_{>}, f_{>}) \\ &\quad + \mathcal{Q}(g, f, f_{<}, f_{>}) + \mathcal{Q}(g, f_{>}, f, f_{<}) + \mathcal{Q}(g, f_{<}, f_{>}, f) \end{aligned}$$

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$$\begin{aligned} Q(g, f, f, f) &= Q(g, f_{<} + f_{>}, f_{<} + f_{>}, f_{<} + f_{>}) \\ &= Q(g, f_{<}, f_{<}, f_{<}) + Q(g, f_{>}, f_{>}, f_{>}) \\ &\quad + Q(g, f, f_{<}, f_{>}) + Q(g, f_{>}, f, f_{<}) + Q(g, f_{<}, f_{>}, f) \\ &\stackrel{\text{Fact 1}}{=} Q(g, f_{>}, f_{>}, f_{>}) \\ &\quad + Q(g, f, f_{<}, f_{>}) + Q(g, f_{>}, f, f_{<}) + Q(g, f_{<}, f_{>}, f) \end{aligned}$$

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Fact 3:

$$|Q(g, f, f_<, f_>)| \lesssim \frac{\|g\| \|f\| \|f_<\| \|f_>\|}{\sqrt{s}} \leq \frac{\alpha(0)^2 \alpha(s/3)}{\sqrt{s}}.$$

and similar for the other terms.

Multi-linear II

So

$$|Q(g, f, f, f)| = |Q(g, f_{<} + f_{>}, f_{<} + f_{>}, f_{<} + f_{>})|$$

Multi-linear II

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$$\begin{aligned} |\mathcal{Q}(g, f, f, f)| &= |\mathcal{Q}(g, f_{<} + f_{>}, f_{<} + f_{>}, f_{<} + f_{>})| \\ &\leq |\mathcal{Q}(g, f_{>}, f_{>}, f_{>})| \\ &\quad + |\mathcal{Q}(g, f, f_{<}, f_{>})| + |\mathcal{Q}(g, f_{>}, f, f_{<})| + |\mathcal{Q}(g, f_{<}, f_{>}, f)| \end{aligned}$$

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Hence

$$\alpha(s) = \sup(|\langle g, f \rangle| : \|g\| = 1, \text{supp}(g) \subset \{|x| \geq s\})$$

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Why quasi-local?

$$Q(f_1, f_2, f_3, f_4) = \int_0^1 \int_{\mathbb{R}} \overline{T_t f_1(x)} T_t f_2(x) \overline{T_t f_3(x)} T_t f_4(x) dx dt$$

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Now use

$$T_t f(x) = \frac{1}{\sqrt{4\pi it}} \int_{\mathbb{R}} e^{i(x-y)^2/(4t)} f(y) dy$$

multiply out everything like crazy and do the x -integration

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if the supports of f_1, f_2, f_3 and f_4 are such that their product is zero whenever $y_1 - y_2 + y_3 - y_4 = 0$.

$$|\mathcal{Q}(f_1, f_2, f_3, f_4)| = \left| \int_0^1 \int_{\mathbb{R}} \overline{T_t f_1(x)} T_t f_2(x) \overline{T_t f_3(x)} T_t f_4(x) dx dt \right|$$

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&= \prod_{j=1}^4 \left(\mathcal{Q}(f_j, f_j, f_j, f_j) \right)^{1/4} \leq S_1^3 \prod_{j=1}^4 \|f_j\|
\end{aligned}$$

Multi-linear enhancement

Why

$$|Q(g, f, f_<, f_>)| \lesssim \frac{\|g\| \|f\| \|f_<\| \|f_>\|}{\sqrt{\text{dist}(\text{supp}(\widehat{g}), \text{supp}(\widehat{f_<}))}}$$

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By Cauchy-Schwarz

$$|Q(g, f, f_<, f_>)| \leq \|T_t g T_t f_<\|_{L^2([0,1] \times \mathbb{R})} \|T_t f T_t f_>\|_{L^2([0,1] \times \mathbb{R})}$$

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and

$$\|T_t f T_t f_>\|_{L^2([0,1] \times \mathbb{R})} \lesssim \|f\| \|f_>\|$$

Bi-linear Strichartz

Claim:

$$\|T_t g T_t f_{<} \|_{L^2([0,1] \times \mathbb{R})} \lesssim \frac{\|g\| \|f_{<}\|}{\sqrt{\text{dist}(\text{supp}(\widehat{g}), \text{supp}(\widehat{f_{<}}))}}$$

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New insight: The same bound holds if the supports of g and $f_{<}$ are separated!

New Symmetry

$$\|T_t f_1 T_t f_2\|_{L^2(\mathbb{R} \times \mathbb{R}^d, t^{d-2} dt dx)} = 2^{1-d/2} \|T_\tau \check{f}_1 T_\tau \check{f}_2\|_{L^2(\mathbb{R} \times \mathbb{R}^d, d\tau dz)}$$

where \check{f}_j is the inverse Fourier transform of f_j .

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Consequence: All previously known bounds involving conditions on the Fourier transforms of f_1 and f_2 have direct x -space analogues.