Finite-Volume Fractional-Moment Criteria for Anderson Localization

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Abstract

A technically convenient signature of localization, exhibited by discrete operators with random potentials, is exponential decay of the fractional moments of the Green function within the appropriate energy ranges. Known implications include: spectral localization, absence of level repulsion, strong form of dynamical localization, and a related condition which plays a significant role in the quantization of the Hall conductance in two-dimensional Fermi gases. We present a family of finite-volume criteria which, under some mild restrictions on the distribution of the potential, cover the regime where the fractional moment decay condition holds. The constructive criteria permit to establish this condition at spectral band edges, provided there are sufficient 'Lifshitz tail estimates' on the density of states. They are also used here to conclude that the fractional moment condition, and thus the other manifestations of localization, are valid throughout the regime covered by the "multiscale analysis". In the converse direction, the analysis rules out fast power-law decay of the Green functions at mobility edges.

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Contents

1. Introduction

1.a Overview

Operators with extensive disorder are known to have spectral regimes (energy ranges) where the spectrum consists of a dense collection of eigenvalues corresponding to exponentially localized eigenfunctions. This phenomenon is of relevance in different contexts; e.g., it plays a role in the conductive properties of metals [1, 2, 3], in the quantization of Hall conductance [4, 5, 6, 7, 8], and in the emerging subject of optical crystals [9].

Most of the mathematical results on localization for operators with random potential in dimensions $d > 1$ have been derived using the *multiscale analysis* introducted by Fröhlich and Spencer [10] (and later evolved through various other works). For discrete systems there is an alternative approach, based on the analysis of the Green function's *fractional moments* [11]. This approach has so far been developped for only a subset of the localization regime, but were it applies it yields somewhat stronger conclusions (through elementary arguments). In this work we present a further extension of that method. In particular, we derive a family of constructive finite-volume criteria for the exponential decay for the fractional moments of Green functions. This decay condition is a technically convenient characterization of localization, for it is known to imply spectral localization, absence of level repulsion, dynamical localization (in a strong exponential sense) and a related condition which plays a significant role in the quantization of the Hall conductance in two-dimensional Fermi gases. The constructive criteria are used to prove that for the discrete random operators described below all these properties hold throughout the regime of localization – if that is defined through either the criteria of the multiscale analysis or those presented here. The constructive criteria also preclude fast power-law decay of the Green functions at mobility edges.

A guiding example for the operators discussed here is the discrete Schrödinger operator, acting in $\ell^2(\mathbb{Z}^d)$:

$$H_\omega \;=\; T + \lambda V_\omega \,, \tag{1.1}$$

with T denoting the off-diagonal part, whose matrix elements are referred to as the *hopping terms*, and V_ω a random multiplication operator – referred to as the *potential.* The symbol ω represents a particular realization of the disorder, in this case the potential variables $\{V_\omega(x)\}$, and λ serves as the disorder strength parameter.

For the discrete Schrödinger operator

$$T_{u,v} \;=\; \begin{cases} 1 & \text{if } |u-v| = 1 \,, \\ 0 & \text{if } |u-v| \neq 1 \,, \end{cases} \tag{1.2}$$

and the random potential is given by a collection of independent identically distributed random variables, $\{V_\omega(x)\}_{x\in\mathbb{Z}^d}$. However, we shall also consider a more general class of operators, allowing the incorporation of magnetic fields, periodic terms, and off-diagonal disorder (see Section 3). We focus on the case of extensive disorder, where the distribution of the random operator H_ω is either translation invariant, or at least gauge equivalent to shifts by multiples of basic periods (i.e. invariant under periodic magnetic shifts).

Our main goal is to present a sequence of finite-volume criteria for localization, which permit to conclude that the following fractional-moment condition is satisfied in some energy interval $[a,b] \in \mathbb{R}$:

$$\mathbb{E}(| < x|\frac{1}{H_\omega - E - i\eta}|y > |^s) \ \le \ A(s)\ e^{-\mu(s)|x-y|} \ , \tag{1.3}$$

for all $E \in [a,b]$, $\eta \in \mathbb{R}$, and suitable $s \in (0,1)$. $\mathbb{E}(\cdot)$ represents here the average over the disorder, *i.e.* the random potential.

Needless to say, the bound (1.3) is of interest mainly in situations where the energy E is within the spectrum, *i.e.* $[H_\omega - E]^{-1}$ is an unbounded operator and the exponential decay occurs only due to the localization of the eigenvalues with energies within the interval $[a,b]$. As in ref. [11], fractional powers are used in order to avoid infinity, however the value of $0 < s < 1$ at which eq. (1.3) is derived is of almost no importance (if eq. (1.3) holds for a particular value of s, then it will hold for all $s < \tau$, where $\tau < 1$ is a number which depends only on the regularity of the probability distribution of $V_\omega(x)$, see Appendix – Lemma B.2).

For the systems considered here, eq. (1.3) is known to imply various other properties, mentioned above, which are commonly associated with localization. More explicitely:

i. *Spectral localization ([11] - using [12]):* The spectrum of H_ω within the interval (a,b) is almost-surely of the pure-point type, and the corresponding eigenfunctions are exponentially localized.

ii. *Dynamical localization ([13], expanded here in Appendix A):* wave packets with energies in the specified range do not spread –

$$\mathbb{E}\left(\sup_{t\in\mathbb{R}} | < x|\ e^{-itH} P_{H\in[a,b]}\ |y > |\right) \ \le \tilde{A} e^{-\tilde{\mu}|x-y|} \tag{1.4}$$

iii. *Exponential decay of the projection kernel ([8])*; the condition expressed in a bound similar to eq. (1.4) for $\mathbb{E}(| < x|\ P_{H\le E}\ |y > |)$ with $E \in [a,b]$. This condition plays an important role in the quantization of Hall conductance, in the ground state of the two dimensional electron gas with Fermi level $E_F \in [a,b]$ [7, 6, 8].

iv. *Absence of level repulsion ([14]).* Minami has shown that eq. (1.3) implies, for operators of the type considered here, that in the range $[a, b]$ the energy gaps have Poisson-type statistics.

The fractional moment condition has already been established for certain regimes: extreme energies, as well as all energies at high enough disorder [11], and also for weak disorder but far enough from the unperturbed spectrum [13]. The results presented below permit to extend it to band edges, provided there are sufficient 'Lifshitz tail estimates' on the density of states (ref. [15, 16, 17, 18, 19]), and to other regimes mapped by a sequence of constructive criteria.

1.b The finite-volume criteria

Our main results admit a number of variations. In this section we present a formulation which is natural for the prototypical example of the discrete random Schrödinger operators, *i.e.* Hamiltonians of the form (1.1) with T the discrete Laplacian (given by (1.2)). In Section 3 we formulate various extensions of the results, including to operators incorporating magnetic fields and to operators with hopping terms of unbounded range.

The results are derived under some mild regularity assumptions on the probability distribution of the variables $\{V_\omega(x)\}_{x\in\mathbb{Z}^d}$ which form the random potential. For simplicity we address ourselves here to the *IID* case: the potential variables are independent with a common probability distribution $\rho(dV)$. The assumption is then that $\rho(dV)$ satisfies the regularity conditions listed below, $R_1(s)$ or $R_2(s)$. However, the independence is not essential. What matters is that the stated regularity condition be satisfied, with a uniform constant, by the conditional distribution of each of the potential variables, conditioned on arbitrary values of the other potentials.

The two regularity conditions mentioned here are:

$R_1(s)$: A probability distribution $\rho(dV)$, on $\mathbb{R}$, is said to be *s-regular*, or to satisfy the condition $R_1(s)$ at some $0 < s \leq 1$, if there exists $C < \infty$ such that

$$\rho(a-\epsilon, a+\epsilon) \leq C\epsilon^s. \tag{1.5}$$

$R_2(s)$: The probability distribution $\rho(dV)$ is said to have the *decoupling property* $R_2(s)$, with some $0 < s \leq 1$, if there exists $C < \infty$ such that for any pair of functions f and g of the form

$$f(V) = \frac{1}{V-a}, \qquad g(V) = \frac{V-b}{V-c}\,, \tag{1.6}$$

with $a, b, c \in \mathbb{C}$, the expectation of the product can be dominated as follows:

$$\mathbb{E}(|f(V)|^s |g(V)|^s) \;\leq\; C\, \mathbb{E}(|f(V)|^s)\; \mathbb{E}(|g(V)|^s)\,, \tag{1.7}$$

The smallest C such that eq. (1.7) holds for all $a, b, c \in \mathbb{C}$ is called here the *decoupling constant* for ρ, and is denoted by $D_s(\rho)$.

A sufficient condition for $R_2(s)$ is that ρ have bounded support and satisfy $R_1(\tau)$ for some $\tau > 4s$ (see Appendix C; related discussion is found in Refs. [11, 8].)

In Appendix B we show that given any τ-regular measure ρ and any $s < \tau$, there is a finite constant C such that for any 2×2 self adjoint matrix $A_{2\times 2}$

$$\int\int \rho(du)\rho(dv) \left| \left[\left(A_{2\times 2} + \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \right)^{-1} \right]_{i,j} \right|^s \leq C , \tag{1.8}$$

where $[\cdot]_{i,j}$ denotes the i,j matrix element with $i,j = 1,2$. Throughout this work, we denote by C_s the smallest value of C at which (1.8) holds. For $\rho(dV)$ which also satisfy $R_2(s)$ we let: $\widetilde{C}_s = C_s \cdot D_s(\rho)^2$.

For $\Lambda \subset \mathbb{Z}^d$ we denote by $H_{\Lambda;\omega}$ the operator obtained from H_ω by "turning off" the hopping terms outside Λ. Thus, the restriction of $H_{\Lambda;\omega}$ to $\ell^2(\Lambda)$ (considered as a subspace of $\ell^2(\mathbb{Z}^d)$), is nothing but H_ω with the Dirichlet boundary conditions on the boundary of Λ.

We also denote by $\Gamma(\Lambda)$ the set of the nearest-neighbor bonds reaching out of Λ (*i.e.* pairs with one site in Λ and the other outside), by Λ^+ the collection of sites within distance 1 from Λ, and by $|\Gamma(\Lambda^+)|$ the number of bonds reaching out of that set. These notions will be generalized in Section 2.a.

Following are our basic results for operators of the form (1.1).

Theorem 1.1 *Let H_ω be a random Schrödinger operator with the probability distribution of the potential $V(x)$ satisfying the regularity condition $R_1(\tau)$ and fix $s < \tau$. If for some $z \in \mathbb{C}$ (possibly real) and some finite region $\Lambda \subset \mathbb{Z}^d$ which contains the origin 0:*

$$b(\Lambda, z) \; := \; \sup_{W\subset\Lambda} \left\{ |\Gamma(\Lambda^+)| \, \frac{C_s}{\lambda^s} \sum_{<u,u'>\in\Gamma(\Lambda)} \mathbb{E}\left(|<0|\frac{1}{H_{W;\omega} - z}|u>|^s \right) \right\} \; < \; 1 \, , \tag{1.9}$$

then there are some $\mu(s) > 0$ and $A(s) < \infty$ — which depend on the energy z only through the bound $b(\Lambda, z)$ — such that for any region $\Omega \subset \mathbb{Z}^d$

$$\mathbb{E}_{\pm i0}\left(|<x|\frac{1}{H_{\Omega;\omega} - z}|y>|^s \right) \; \leq \; A(s)\; e^{-\mu(s)\,|x-y|} \, . \tag{1.10}$$

The subscript of $\mathbb{E}_{\pm i0}$, in (1.10) is to be interpreted as saying that the bound is valid for either of the two limiting expressions:

$$\lim_{\eta \searrow 0} \mathbb{E}\left(| < x| \frac{1}{H_{\Omega;\omega} - E - (+)\, i\eta} |y > |^s \right) . \tag{1.11}$$

The "cutoff" $\pm i\eta$ is needed for an unambiguous interpretation in case z is a real energy (E) within the spectrum of H. For the random operators considered here it is well understood that: i) the expectation may be exchanged with the limit $\eta \searrow 0$, ii) it suffices to verify the uniform bounds (1.10) for finite regions, and iii) the finite volume expectations are continuous in η. In the proofs we shall be dealing with finite systems; the subscript will, therefore, be omitted there.

Let us note that already the special case $\Lambda = \{0\}$ is of interest. It provides the following variant of the single-site criterion of ref. [11] (which is, in fact, a bit simpler since it does not invoke the *decoupling lemma*).

Corollary *For the random Schrödinger operator a sufficient condition for localization (1.3) is that for all $E \in [a, b]$*

$$2d(2d-1)\, \frac{C_s}{\lambda^s} \int \frac{1}{|\lambda V - E|^s}\, \rho(dV) \;<\; 1\, . \tag{1.12}$$

Just as the main result of ref. [11], the above criterion permits to easily conclude localization for the cases of high disorder or extreme energies. However, we may now move beyond that. By testing the hypothesis of Theorem 1.1 in the increasing sequence of volumes $\Lambda = [-L, L]^d$, one may extend the conclusion to increasing regimes in the 'energy $\times$ disorder plane'. In fact, it is easy to see that for each energy at which the strong localization condition (1.10) is satisfied, the hypothesis (1.9) will be met at all sufficiently large L. (This may, however, be far from a practical test, as the necessary computation may be rather difficult for large L).

Observant readers may note that the conclusion of Theorem 1.1 provides not only the localization condition eq. (1.3), but it also rules out *extended boundary states*. The flip side of this observation is that if such states are present in some geometry, *e.g.* the half space, then the hypothesis of Theorem 1.1 will fail to be satisfied even if the operator exhibits localization in the bulk. Therefore, we present also the following result which permits to establish bulk localization regardless of the possible presence of extended boundary states.

Theorem 1.2 *Let H_ω be a random Schrödinger operator with the probability distribution of the potential $V(x)$ satisfying $R_1(\tau)$ and $R_2(s)$, for some $s < \tau$. If for some $z \in \mathbb{C}$ and* *some finite region* $0 \in \Lambda \subset \mathbb{Z}^d$

$$\left(1 + \frac{\widetilde{C}_s}{\lambda^s}\ |\Gamma(\Lambda)|\right)^2 \sum_{<u,u'>\in\Gamma(\Lambda)} \mathbb{E}\left(|<0|\frac{1}{H_{\Lambda;\omega} - z}|u>|^s\right) \ < \ 1, \tag{1.13}$$

then H_ω *satisfies the fractional-moment condition (1.3), and there exist* $\mu(s) > 0, A(s) < \infty$ *so that for any region* $\Omega \subset \mathbb{Z}^d$*,*

$$\mathbb{E}_{\pm i0}\left(|<x|\frac{1}{H_{\Omega;\omega} - z}|y>|^s\right) \ \leq \ A(s)\ e^{-\mu(s)\,\mathrm{dist}_\Omega(x,y)}\ , \tag{1.14}$$

with

$$\mathrm{dist}_\Omega(x,y) = \min\{|x-y|, [\mathrm{dist}(x,\partial\Omega) + \mathrm{dist}(y,\partial\Omega)]\}\ . \tag{1.15}$$

Let us add that, as in Theorem 1.1, $A(s)$ and $\mu(s)$ of (1.14) depend on z only through the value of the LHS in eq. (1.13).

The modified metric, $\mathrm{dist}_\Omega(x,y)$, is a distance function relative to which the entire boundary of Ω is regarded as one point. It permits us to state that there is exponential decay in the bulk without ruling out non-exponential decay along the boundary. We supplement the last result by the following observation.

Theorem 1.3 *Let* H_ω *be a random operator given by eq. (1.1), with the probability distribution of the potential* $V(x)$ *satisfying* $R_1(\tau)$ *and* $R_2(s)$*, for some* $s < \tau$*. If at some energy* E *(or* $z \in \mathbb{C}$*) the localization condition (1.3) is satisfied, with some* $A < \infty$ *and* $\mu > 0$*, then for all large enough (but finite)* L *the condition (1.13) is met for* $\Lambda = [-L, L]^d$*.*

The statement is a bit less immediate than the analogous claim for Theorem 1.1. We shall therefore include the proof below.

It is natural to compare the above criteria for localization with those of the multiscale analysis. The two methods share the basic feature that the analysis requires an initial condition which one may expect to be met in a finite system provided its linear size is of the order of the localization length, or larger. However, for the method presented here if a suitable input is received on some scale, then the analysis can proceed using steps, or blocks, of only that size. An important difference in the results is that the fractional moment condition yields exponential decay for the expectation values, which are important for some of the conclusions listed above. Such bounds have not been derived by methods based on the multiscale analysis, since (at least without further improvement) the bounds the latter yields on the "error terms", i.e., the probabilities of "bad blocks", decay not faster than $\exp[-(\log L/\log L_o)^\alpha]$. This rate is faster than any power of L, but in itself not fast enough to imply exponential bounds for the mean values. However, it should be noted that the extension of the present method to operators in the continuum, for which a number of basic localization results have been established using the multiscale analysis [20, 21, 17], is still unaccomplished. Also not covered are discrete operators with the potential assuming discrete values (e.g., $V_\omega(x) = \pm 1$ [22]).

In Section 4 we discuss various implications of the basic results. In particular it is shown that, for discrete random operators of the type considered here, the fractional moment condition (1.3) is satisfied throughout the regime in which the multiscale analysis applies (see Theorem 4.4). This carries the further implication that the properties listed above hold throughout the entire regime for which localization can be proven by any of the known methods. One of those properties is a strong form of dynamical localization, on which more is said in Appendix A.

2. Proofs of the main results

2.a Some useful notation

The proofs of the above statements will be presented in terms which permit a direct extension to operators with more general hopping terms. We start by generalizing the notation; in particular, the sets Λ^+ and $\Gamma(\Lambda)$ will be made to depend implicitly on the operator T.

In the study of $H_{\Omega;\omega}$ we shall often consider 'depleted' Hamiltonians, $H_{\Omega;\omega}^{(\Gamma)}$, obtained by setting to zero the operator's non-diagonal matrix elements (*hopping terms*) along some collection of ordered pairs of sites (referred to here as *bonds*) $\Gamma \subset \mathbb{Z}^d \times \mathbb{Z}^d$. The difference is the operator $T^{(\Gamma)}$, with

$$T^{(\Gamma)}_{x,y} = \begin{cases} T_{x,y} & \text{if } <x,y>\in \Gamma \text{ or } <y,x>\in\Gamma \\ 0 & \text{if } <x,y>\notin\Gamma \text{ and } <y,x>\notin\Gamma\,, \end{cases} \tag{2.1}$$

so that

$$H_{\Omega;\omega} \;=\; H_{\Omega;\omega}^{(\Gamma)} + T^{(\Gamma)}\,. \tag{2.2}$$

Typically, Γ will be a collection of bonds which forms the 'cut set' of some $W\subset\mathbb{Z}^d$, i.e., the set of bonds with $T_{x,y}\neq 0$ connecting sites in W with sites in its complement. Thus we denote

$$\Gamma(W) \;=\; \left\{ <u,u'> \mid u\in W, u'\in\mathbb{Z}^d\backslash W, \text{ and } T_{u,u'}\neq 0\right\}\,, \tag{2.3}$$

and also

$$W^+ \;=\; W \cup \left\{ u' \in \mathbb{Z}^d |\; T_{u,u'} \neq 0 \text{ for some } u \in W \right\} \;. \tag{2.4}$$

The number of elements (*i.e.* bonds) in Γ is denoted $|\Gamma|$.

In addition, we use the "Green function" notation:

$$G_{\Omega;\omega}(x,y;z) = < x | \frac{1}{H_{\Omega;\omega} - z} | y > \;, \tag{2.5}$$

with $G^{(\Gamma)}_{\Omega;\omega}(x,y;z)$ defined correspondingly. Often, where it is obvious from context that an operator is a random variable, we shall suppress the subscript ω.

In broad terms, the strategy for the proof is to derive a bound on the average Green function, of the form

$$\mathbb{E}\left(|G_\Omega(x,y;z)|^s\right) \;\leq \sum_{<u,u'>\in\Gamma(\Lambda(x))} \gamma_{\Lambda(x)}(<u,u'>) |T_{u,u'}|^s \; \mathbb{E}\left(|G_\Omega^{(\Gamma(\Lambda(x))}(u',y;z)|^s \right) \;, \tag{2.6}$$

for all $y \in \mathbb{Z}^d \backslash \Lambda(x)$, where: $\Lambda(x) = \{x + y \,:\, y \in \Lambda\}$ is a finite neighborhood of x, translate of some fixed region $\Lambda \ni 0$, and $\gamma_{\Lambda(x)}$ is a quantity which is small when the typical values of the finite volume Green function between x and the boundary of $\Lambda(x)$ are small (in a suitable sense).

An inequality of the form (2.6) is particularly useful when

$$\sum_{<u,u'>\in\Gamma(\Lambda(x))} \gamma_{\Lambda(x)}(<u,u'>) \; |T_{u,u'}|^s \;<\; 1 \;, \tag{2.7}$$

since in that case eq. (2.6) is akin to the statement that $\mathbb{E}(|G_\Omega(x,y;z)|^s)$ is a strictly subharmonic function of x, as long as $|x-y| > \text{diam}|\Lambda|$, and thus — if it is also uniformly bounded (which it is) — it decays exponentially.

The first step towards a bound of the form (2.6) is, naturally, the resolvent identity:

$$\begin{aligned} G_{\Omega,\omega} \;&=\; G^{(\Gamma)}_{\Omega,\omega} - G^{(\Gamma)}_{\Omega,\omega} \cdot T^{(\Gamma)} \cdot G_{\Omega,\omega} \\ &=\; G^{(\Gamma)}_{\Omega,\omega} - G_{\Omega,\omega} \cdot T^{(\Gamma)} \cdot G^{(\Gamma)}_{\Omega,\omega} \end{aligned} \tag{2.8}$$

(written here in the operator form). However, one then reaches an obstacle, since the quantity whose mean needs to be estimated is a product of two Green functions which are not independent. For some time now this co-dependence has been the main obstacle on the road to an argument along the lines outlined above, since otherwise the general strategy applied here is well familiar from its various successful applications in the context of the statistical mechanics of homogeneous systems ([23, 24, 25, 26, 27]), and the other auxiliary tools specific to the present context have in essence been available since ref. [11]. The co-dependence problem is solved here through a second application of the resolvent identity (followed by a decoupling argument of a familiar type). In fact, a similar tactic was applied by von Dreifus to the mean correlation functions, in a study of the phase transitions in disordered ferromagnetic models [28] (as we learned from T. Spencer after the completion of the first draft of this work).

The two applications of the resolvent identity, for which the depletion sets Γ_1 and Γ_2 need not coincide, may be combined by starting our argument from the identity:

$$G_\Omega \;=\; G_\Omega^{(\Gamma_1)} - G_\Omega^{(\Gamma_1)} \cdot T^{(\Gamma_1)} \cdot G_\Omega^{(\Gamma_2)} + G_\Omega^{(\Gamma_1)} \cdot T^{(\Gamma_1)} \cdot G_\Omega \cdot T^{(\Gamma_2)} \cdot G_\Omega^{(\Gamma_2)} \; . \tag{2.9}$$

Readers familiar with the current techniques may note that once the middle term G_Ω is replaced by a uniform bound, the remaining expression can be made free from co-dependence by an appropriate choice of Γ_1 and Γ_2. The rest are technicalities, to which we turn next.

2.b Key Lemmas

We shall now present three Lemmas which will be used in the proofs of our main results. The first is a known estimate which provides the afore-mentioned uniform upper bound.

Lemma 2.1 *Let $V(x)$ be a random potential satisfying the regularity condition $R_1(\tau)$. Then for each $s < \tau$, any region Ω, and any random operator of the form (1.1)*

$$\mathbb{E}\left(|G_\Omega(x,y;z)|^s \right) \;\leq\; \frac{C_s}{\lambda^s} \, , \tag{2.10}$$

for all $z \in \mathbb{C}$.

The statement is an immediate consequence of a version of the Wegner estimate which we present in the appendix. (See lemma B.1; also eq. (2.18) below.)

Next is our new bound.

Lemma 2.2 *Let H_ω be a random operator given by eq. (1.1) with the probability distribution of the potential $V(x)$ satisfying the regularity condition $R_1(\tau)$, and let W be a subset of Ω. Then, denoting $\widetilde{\Gamma} = \Gamma(W^+)$ and $\Gamma = \Gamma(W)$, for all $z \in \mathbb{C}$:*

1. The following 'depleted-resolvent bound' holds for any pair of sites $x \in W$, $y \in \Omega \backslash W^+$,

$$\mathbb{E}\left(|G_\Omega(x,y;z)|^s\right) \;\leq\; \gamma(W) \sum_{<v,v'>\in\widetilde{\Gamma}} |T_{v,v'}|^s \; \mathbb{E}\Big(|G_{\Omega\backslash W^+}(v',y;z)|^s\Big) \;, \tag{2.11}$$

with

$$\gamma(W) \;=\; \frac{C_s}{\lambda^s} \sum_{<u,u'>\in\Gamma} |T_{u,u'}|^s \; \mathbb{E}\left(|G_W(x,u;z)|^s\right) \;. \tag{2.12}$$

2. If, furthermore, the probability distribution of the potential satisfies also $R_2(s)$ then the following bound holds for any pair of sites $x \in W$, $y \in \Omega \backslash W$,

$$\mathbb{E}\left(|G_\Omega(x,y;z)|^s\right) \;\leq\; \sum_{<v,v'>\in\Gamma} \gamma_x(<v,v'>) \; |T_{v,v'}|^s \; \mathbb{E}\Big(|G_{\Omega\backslash W}(v',y;z)|^s\Big) \;, \tag{2.13}$$

with

$$\gamma_x(<v',v>) \;=\; \mathbb{E}\left(|G_W(x,v';z)|^s\right) + \frac{\widetilde{C}_s}{\lambda^s} \sum_{<u,u'>\in\Gamma} |T_{u,u'}|^s \mathbb{E}\left(|G_W(x,u;z)|^s\right) \;. \tag{2.14}$$

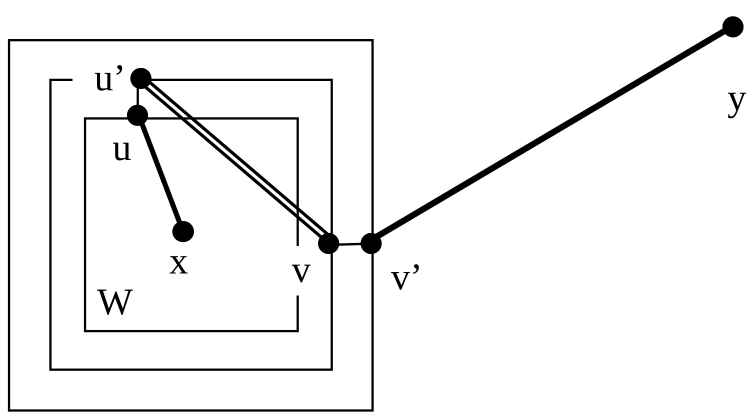

Figure 1: Diagramatic depiction of the bound (2.16) on $G(x,y;z)$, for $x,y \in \mathbb{Z}^d$ and $z \in \mathbb{C}$. The long solid lines are 'depleted Green functions', the two short segments correspond to the hoping terms (T) and the double line is a full Green function. Once the latter is replaced by a uniform upper bound, the expectation value of the product of the remaining terms factorizes.

Proof: Both results follow from the second-order resolvent identity eq. (2.9), which yields:

$$G_\Omega(x,y;z) \;=\; G_\Omega^{(\Gamma_1)}(x,y;z) \;-\; <x|G_\Omega^{(\Gamma_1)}\;T_\Omega^{(\Gamma_1)}\;G_\Omega^{(\Gamma_2)}|y> \\ +<x|G_\Omega^{(\Gamma_1)}\;T_\Omega^{(\Gamma_1)}\;G_\Omega\;T_\Omega^{(\Gamma_2)}\;G_\Omega^{(\Gamma_2)}|y> \;. \quad (2.15)$$

For the proof of the first claim, we take $\Gamma_1 = \Gamma = \Gamma(W)$ and $\Gamma_2 = \widetilde{\Gamma} = \Gamma(W^+)$. Then, the first term of eq. (2.15) is zero because $\Gamma(W)$ decouples x and y and the second term is zero because $\Gamma(W^+)$ decouples W^+ and y. Thus

$$G_\Omega(x,y;z) \;=\; \sum_{\substack{<u,u'>\in\Gamma \\ <v,v'>\in\widetilde{\Gamma}}} T_{u,u'}\;T_{v,v'}\;G_\Omega^{(\Gamma)}(x,u;z)G_\Omega(u',v;z)G_\Omega^{(\widetilde{\Gamma})}(v',y;z)\;. \qquad (2.16)$$

It follows that for any $s \in (0,1)$

$$\mathbb{E}(|G_\Omega(x,y;z)|^s) \\ \leq \sum_{\substack{<u,u'>\in\Gamma \\ <v,v'>\in\widetilde{\Gamma}}} |T_{u,u'}|^s|T_{v,v'}|^s\;\mathbb{E}\Big(|G_\Omega^{(\Gamma)}(x,u;z)G_\Omega(u',v;z)G_\Omega^{(\widetilde{\Gamma})}(v',y;z)|^s\Big)\;. \quad (2.17)$$

(note that for $0<s<1$: $|a+b|^s \leq |a|^s + |b|^s$.)

In estimating the terms on the right hand side of eq. (2.17) let us consider first the conditional expectation of the central factors, $G_\Omega(u',v;z)$. Only these factors depend on the values of the potential at u' and v, and therefore they can be replaced by their conditional expectation $\mathbb{E}\big(|G_\Omega(u',v;z)|^s|\,\{V(q)\}_{q\in\Omega\setminus\{u',v\}}\big)$. As will be proven in the appendix, under the regularity condition $R_1(\tau)$ these are uniformly bounded (Lemma B.1):

$$\mathbb{E}\big(|G_\Omega(u',v;z)|^s|\,\{V(q)\}_{q\in\Omega\setminus\{u',v\}}\big) \;\leq\; \frac{C_s}{\lambda^s}\;. \qquad (2.18)$$

(The proof involves a reduction to a two-dimensional problem via the Krein formula, and a two-dimensional Wegner-type estimate.)

Once the central factor in each expectation on the right hand side of eq. (2.17) is replaced by the above bound, what remains there are two independent random variables which are $|G_\Omega^{(\Gamma)}(x,u;z)|^s = |G_W(x,u;z)|^s$ and $|G_\Omega^{(\widetilde{\Gamma})}(v',y;z)|^s = |G_{\Omega\setminus W^+}(v',y;z)|^s$. The expectation now factorizes, and the resulting expression yields the first claim of the Lemma.

For the second claim, we take $\Gamma_1 = \Gamma_2 = \Gamma = \Gamma(W)$. Once again the first term of eq. (2.15) is zero because $\Gamma(W)$ decouples x and y. However, the second term is non-zero, and we obtain

$$\begin{aligned}
&\mathbb{E}(|G_\Omega(x,y;z)|^s) \\
&\quad\le \sum_{<v,v'>\in\Gamma} |T_{v',v}|^s \mathbb{E}\left(|G_\Omega^{(\Gamma)}(x,v;z)G_\Omega^{(\Gamma)}(v',y;z)|^s\right) \\
&\qquad + \sum_{\substack{<u,u'>\in\Gamma\\ <v,v'>\in\Gamma}} |T_{u,u'}|^s|T_{v,v'}|^s\ \mathbb{E}\left(|G_\Omega^{(\Gamma)}(x,u;z)G_\Omega(u',v;z)G_\Omega^{(\Gamma)}(v',y;z)|^s\right) \ .
\end{aligned} \tag{2.19}$$

At this point we may not use the previous argument, since in the last expectation $V(v)$ affects each of the first two factors and $V(u')$ affects each of the last two factors. However, the dependence of each of these factors on the potentials is of a particularly simple form: they are ratios of two functions (determinants) which are separately linear in each potential variable. Using the decoupling hypotheses, *i.e.* the regularity conditions $R_1(\tau)$ and $R_2(s)$, the expectation may be bounded by the product of expectations. Specifically, we prove in Lemma C.1 that:

$$\begin{aligned}
&\mathbb{E}\left(|G_\Omega^{(\Gamma)}(x,u;z)G_\Omega(u',v;z)G_\Omega^{(\Gamma)}(v',y;z)|^s\right) \\
&\qquad\qquad \le \frac{\widetilde{C}_s}{\lambda^s}\mathbb{E}\left(|G_\Omega^{(\Gamma)}(x,u;z)G_\Omega^{(\Gamma)}(v',y;z)|^s\right) \ .
\end{aligned} \tag{2.20}$$

Once again, we are left with a product of two independent random variables, $|G_\Omega^{(\Gamma)}(x,u;z)|^s = |G_W(x,u;z)|^s$ and $|G_\Omega^{(\Gamma)}(v',y;z)|^s = |G_{\Omega\backslash W}(v',y;z)|^s$. The factorization of the remaining expectation yields the second claim of the Lemma, eq. (2.13). ∎

The above Lemma provides a bound for the Green function in terms of its depleted versions. This suffices for the derivation of the first of our two main Theorems (Thm 1.1). However, this does not suffice for the second Theorem, Thm 1.2, for which we shall use an inequality that is linear in the original function. That "closure" will be attained with the help of the following bound on the depleted resolvent in terms of the full one.

Lemma 2.3 *Let $H_{\Omega,\omega}$ be a random operator in $\ell^2(\Omega)$, $\Omega \subseteq Z^d$, given by eq. (1.1), with the probability distribution of the potential $V(x)$ satisfying the regularity conditions $R_1(\tau)$ and $R_2(s)$ for some $s<\tau$. Let W be a subset of Ω. Then, the following holds for any pair of sites $u,y \in \Omega\backslash W$, and every $z \in \mathbb{C}$*

$$\mathbb{E}\left(|G_{\Omega\backslash W}(u,y;z)|^s\right) \ \le \ \mathbb{E}(|G_\Omega(u,y;z)|^s) + \frac{\widetilde{C}_s}{\lambda^s}\sum_{<v,v'>\in\Gamma}|T_{v',v}|^s\mathbb{E}(|G_\Omega(v,y;z)|^s) \ , \tag{2.21}$$

with $\Gamma = \Gamma(W)$ the 'cut-set' of W.

Proof: Starting from the first order resolvent identity, eq. (2.8), and taking expectation values of its matrix elements, we find:

$$\mathbb{E}\left(|G_\Omega^{(\Gamma)}(u,y;z)|^s\right) \leq \mathbb{E}\left(|G_\Omega(u,y;z)|^s\right) + \sum_{<v,v'>\in\Gamma(W)} |T_{v',v}|^s \mathbb{E}\left(|G_\Omega^{(\Gamma)}(u,v';z)|^s |G_\Omega(v,y;z)|^s\right) , \quad (2.22)$$

where $\Gamma = \Gamma(W)$, and $G^{(\Gamma)} = G_{\Omega\setminus W}$. It suffices, therefore, to show that in the last term the factor $|G_\Omega^{(\Gamma)}(u,v';z)|^s$ may be replaced (for an upper bound) by the constant $\frac{\widetilde{C}_s}{\lambda^s}$. This follows through a decoupling argument which we present in the Appendix — see Lemma C.1. ■

Remark In the applications we shall use Lemmas 2.2 and 2.3 both in the stated form and in the conjugated form, with the arguments of the Green functions reversed. One form of course implies the other (at conjugate energy).

2.c Proofs of the main results

We are now ready to derive the results stated in the Introduction. For simplicity these were stated in the context of the Schrödinger operators, for which T is the discrete Laplacian. The proofs given in this section will be restricted to this case. A more generally applicable treatment is presented in the next section.

Proof of Theorem 1.1: Assume that for some $z \in \mathbb{C}$ and a finite region Λ the smallness condition (1.9) holds. By Lemma 2.2 and translation invariance, we learn that for any region Ω and any $x, y \in \Omega$ with $y \in \mathbb{Z}^d \setminus \Lambda^+(x)$:

$$\mathbb{E}\left(|G_\Omega(x,y;z)|^s\right) \leq b \cdot \frac{1}{|\Gamma(\Lambda^+)|} \sum_{<v,v'>\in\Gamma(\Lambda^+(x))} \mathbb{E}\left(|G_{\Omega\setminus\Lambda^+(x)}(v',y;z)|^s\right) , \quad (2.23)$$

where $b = b(\Lambda, z)$ of eq. (1.9), and $\Lambda(x)$ is the translate of Λ by x.

By Lemma 2.1, each of the terms in the sum is bounded by C_s/λ^s. Since the sum is normalized by the prefactor $1/|\Gamma(\Lambda^+)|$, the inequality (2.23) permits to improve that bound for $\mathbb{E}(|G_\Omega(x,y;z)|^s)$ by the factor b (< 1). Furthermore, the inequality may be iterated a number of times, each iteration resulting in an additional factor of b.

One should take note of the fact that the iterations bring in Green functions corresponding to modified domains. It is for this reason that the initial input assumption was required to hold for modified geometries, *i.e.* not just for Λ but also for all its subsets.

Inequality (2.23) can be iterated as long as the resulting sequences $(x, v', \ldots, v^{(n)})$

do not get closer to y than the distance $L = \sup\{|u| \mid u \in \Lambda^+\}$. Thus:

$$\mathbb{E}(|G_\Omega(x,y;z)|^s) \ \le \ \frac{C_s}{\lambda^s} \cdot b^{\lfloor |x-y|/L \rfloor} \ \le \ \frac{C_s}{\lambda^s b} \, e^{-\mu|x-y|} \ , \tag{2.24}$$

with $\mu = |\ln b|/L$. ■

Next, let us turn to the proof of the second theorem (Thm 1.2). The main change is that we now proceed under the assumption that the smallness condition holds for some region Λ without requiring it to hold also in all subsets. As explained in the introduction, the difference may be meaningful if H_ω has extended boundary states in some geometry.

Proof of Theorem 1.2: Our first goal is to show that under the assumption (1.13) there is $b < 1$ such that for all pairs $\{x, y\}$ with $\Lambda(x) \subset \Omega$ and $y \in \Omega \backslash \Lambda(x)$,

$$\mathbb{E}(|G_\Omega(x,y;z)|^s) \ \le \ b \sum_{u \in \Lambda^+(x)} P^l_x(u) \, \mathbb{E}(|G_\Omega(u,y;z)|^s) \ , \tag{2.25}$$

with non-negative weights satisfying:

$$\sum_{u \in \Lambda^+(x)} P^l_x(u) \ = \ 1 \ . \tag{2.26}$$

We shall use this inequality along with its conjugate:

$$\mathbb{E}(|G_\Omega(x,y;z)|^s) \ \le \ b \sum_{v \in \Lambda^+(y)} P^r_y(v) \, \mathbb{E}(|G_\Omega(x,v;z)|^s) \ , \tag{2.27}$$

where $P^r_y(v)$ satisfy the suitable analog of the normalization condition (2.26).

It is important that – unlike in the inequality (2.23), the functions which appear on the right hand side of (2.25) and (2.27) are computed in the same domain as those on the left hand side.

The first step is by Lemma 2.2, which yields

$$\mathbb{E}(|G_\Omega(x,y;z)|^s) \ \le \sum_{<u,u'> \in \Gamma(\Lambda(x))} \gamma_x(<u,u'>) \, \mathbb{E}\left(|G_{\Omega\backslash\Lambda(x)}(u',y;z)|^s\right) \ , \tag{2.28}$$

whenever $\Lambda(x) \subset \Omega$ and $y \in \mathbb{Z}^d \backslash \Lambda(x)$, with $\gamma_x(<u,u'>)$ specified in eq. (2.14).

Next, we apply Lemma 2.3, eq. (2.21), to bound $\mathbb{E}\left(|G_{\Omega\backslash\Lambda(x)}(u',y;z)|^s\right)$ in terms of a sum of quantities of the form $\mathbb{E}(|G_\Omega(v,y;z)|^s)$ with $v \in \Lambda^+(x)$. The result is initially expressed as a sum over bonds:

$$\mathbb{E}(|G_\Omega(x,y;z)|^s) \leq \sum_{<u,u'>\in\Gamma(\Lambda(x))} \gamma_x(<u,u'>)\, \mathbb{E}(|G_\Omega(u',y;z)|^s) + \frac{\widetilde{C}_s}{\lambda^s}\, \Theta \sum_{<u,u'>\in\Gamma(\Lambda(x))} \mathbb{E}(|G_\Omega(u,y;z)|^s) \ , \quad (2.29)$$

where, using translation invariance,

$$\Theta := \sum_{<u,u'>\in\Gamma(\Lambda)} \gamma_0(<u,u'>) \ .$$

Collecting terms, and pulling out normalizing factors, one may cast the inequality (2.29) in the form (2.25) with

$$b := \sum_{<u,u'>\in\Gamma(\Lambda(x))} \left(\gamma_x(<u,u'>) + \frac{\widetilde{C}_s}{\lambda^s}\Theta\right) = \left(1+\frac{\widetilde{C}_s}{\lambda^s}\,|\Gamma(\Lambda)|\right)\Theta \quad (2.30)$$

$$= \left(1+\frac{\widetilde{C}_s}{\lambda^s}\,|\Gamma(\Lambda)|\right)^2 \sum_{<u,u'>\in\Gamma(\Lambda)} \mathbb{E}(|G_\Lambda(0,u;z)|^s) \ . \quad (2.31)$$

The smallness condition (1.13) is nothing other than the assumption that $b<1$.

The above argument proves eq. (2.25). By the transposition, or time-reflection, symmetry of H ($H^T = H$) also eq. (2.27) holds. (Such symmetry of H is not essential for our analysis: it suffices to assume that the smallness condition eq. (1.13) holds along with its transpose.)

We proceed in the proof by iterating the inequalities (2.25) and (2.27). However an adaptation is needed in the argument which was used in the proof of Theorem 1.1 since the iteration can be carried out only as long as the two points (the arguments of the resolvent) stay at distance $L = \sup\{|u| : u \in \Lambda^+\}$ not only from each other but also from the boundary $\partial\Omega$. The relevant observation is that for every pair of sites $x, y \in \Omega$ there is a pair of integers $\{n, m\}$ such that:

1. $n + m = \mathrm{dist}_\Omega(x,y)$,
2. the ball of radius n centered at x and the ball of radius m centered at y form a pair of disjoint subsets of Ω.

For the desired bound on $\mathbb{E}(|G_\Omega(x,y;z)|^s)$, we shall iterate eq. (2.25) $\lfloor n/L \rfloor$ times from the left, and (2.27) $\lfloor m/L \rfloor$ times from the right. Similar to eq. (2.24), we obtain:

$$\mathbb{E}(|G_\Omega(x,y;z)|^s) \leq \frac{C_s}{\lambda^s b^2}\, e^{-\mu\ \mathrm{dist}_\Omega(x,y)} \ , \quad (2.32)$$

with $\mu = |\ln b|/L$. ∎

The third Theorem stated in the introduction (Thm 1.3) is the claim that the condition which is shown above to be sufficient for exponential localization, in the sense of eq. (1.3), is also a necessary one. We shall now prove this to be the case.

Proof of Theorem 1.3: Suppose that eq. (1.3) holds with some $A < \infty$ and $\mu > 0$. We need to show that also in finite systems the Green function is sufficiently small between an interior point and the boundary. To bound the finite volume function in terms of the infinite volume one, we may use lemma 2.3, by which

$$\sum_{<u,u'>\in\Gamma(\Lambda)} \mathbb{E}\left(|G_\Lambda(0,u;z)|^s\right) \le \sum_{<u,u'>\in\Gamma(\Lambda)} \mathbb{E}\left(|G(0,u;z)|^s\right) + \frac{\widetilde{C}_s}{\lambda^s}\,|\Gamma(\Lambda)| \sum_{<v,v'>\in\Gamma(\Lambda)} |T_{v,v'}|^s\, \mathbb{E}\left(|G(0,v';z)|^s\right) \, , \quad (2.33)$$

for any finite region Λ containing the origin. We need to show that for $\Lambda = [-L,L]^d$ with L large enough

$$\left(1 + \frac{\widetilde{C}_s}{\lambda^s}|\Gamma(\Lambda)|\right)^2 \sum_{<u,u'>\in\Gamma(\Lambda)} \mathbb{E}\left(|G_\Lambda(0,u;z)|^s\right) \; < \; 1 \, . \quad (2.34)$$

After applying eq. (2.33) to the terms on the left side of eq. (2.34) we find that the number of summands involved and their prefactors grow only polynomially in L, whereas under our assumption the relevant factors $\mathbb{E}\left(|G(0,u;z)|^s\right)$ are exponentially small in L. Hence the condition (2.34) is satisfied for L large enough. ∎

3. Generalizations

3.a Formulation of the general results

We shall now turn to some generalizations of the theorems which were presented in Section 1.b for the random Schrödinger operator. The setup may be extended in a number of ways.

1. *Addition of magnetic fields.* The hopping terms $\{T_{x,y}\}$ need not be real. In particular, the present analysis remains valid when one includes in H_ω a constant magnetic field, or a random one with a translation invariant distribution.

A magnetic field is incorporated in $T_{x,y}$ through a factor $\exp(-iA_{x,y})$, with $A_{x,y}$ an anti-symmetric function of the bonds. (It represents the integral of the 'vector potential' $\times(-e/\hbar)$ along the bond $< x, y >$.) Except for the trivial case, with such a factor T is no longer shift invariant. However, in the case of a constant magnetic field, T will still be invariant under appropriate"magnetic shifts", which consist of ordinary shifts followed by gauge transformations.

Translation-invariance plays a role in our discussion. However, since gauge transformations do not affect the absolute values of the resolvent, it suffices for us to assume that H_ω is *stochastically invariant under magnetic shifts* – in the sense of Definition 3.1.

2. *Extended hopping terms.* The discrete Laplacian may be replaced by an operator with hopping terms of unlimited range. For exponential localization we shall however require $\{T_{x,y}\}$ to decay exponentially in $|x-y|$.

3. *Off-diagonal disorder.* $\{T_{x,y}\}$ may also be made random. It is convenient however to assume exponentially decaying uniform bounds. The regularity conditions on the potential will now be assumed for the conditional distribution of $V(x)$ at specified off-diagonal disorder.

4. *Periodicity.* H_ω may also include a periodic potential, i.e., eq. (1.1) may be modified to:

$$H_\omega \ = \ T_{x,y;\omega} + U_{per}(x) + \lambda V_\omega(x) \tag{3.1}$$

   This may be further generalized by requiring periodicity only of the probability distribution of H.

5. *More general lattices.*

In the previous discussion, the underlying sets $\mathbb{Z}^d$ may be replaced by other graphs, with suitable symmetry groups. The graph structure is relevant if the hopping terms are limited to graph edges. However, since we consider also operators with hoping terms of unlimited range, let us formulate the result for operators on $\ell^2(\mathcal{T})$ where the underlying set is of the form $\mathcal{T} = \mathcal{G} \times S$, with $\mathcal{G}$ a countable group and S a finite set. We let $\mathrm{dist}(x, y)$ denote a metric on $\mathcal{T}$ which is invariant under the natural action of $\mathcal{G}$ on that set.

For example, this setup allows for $\mathcal{T}$ to be a Bethe lattice, or a more general Cayley lattice. (Instructive discussion of some statistical mechanical models in such settings may be found in refs.[29]). The set S is included here in order to leave room for periodic structures. We denote by $\mathcal{C}$ the "periodicity cell", which is $\{\imath\} \times S$ where $\imath$ is the identity in $\mathcal{G}$.

Some of the relevant concepts are summarized in the following definition.

Definition 3.1 *With $\mathcal{T} = \mathcal{G} \times S$ as above, let H_ω be a random operator on $\ell^2(\mathcal{T})$ (i.e., one with some specified probability distribution), whose off-diagonal part is denoted by T_ω and the diagonal part is referred to as the potential (for consistency, we denote it as λV_ω).*

1. *We say that H_ω is <u>stochastically invariant under magnetic shifts</u> if for each $\kappa \in \mathcal{G}$ and almost every ω there is a unitary map of the form*

$$(U_{\kappa,\omega}\psi)(x) \;=\; e^{i\phi_{\kappa,\omega}(x)}\psi(\kappa x)\ , \tag{3.2}$$

(with some function $\phi_{\kappa,\omega}(\cdot)$) under which

$$U^*_{\kappa,\omega}\ H_\omega\ U_{\kappa,\omega} \;\stackrel{\mathcal{D}}{=}\; H_\omega\ , \tag{3.3}$$

where $\stackrel{\mathcal{D}}{=}$ means equality of the probability distributions.

2. *The operator is said to have <u>tempered off-diagonal matrix elements</u>, at a specified value of $s < 1$, if there is a kernel $\tau_{x,y}$, and some $m > 0$, such that $T_{x,y;\omega} \;\le\; \tau_{x,y}$, almost surely, and*

$$\sup_{x\in\mathcal{T}} \sum_{y\in\mathcal{T}} \tau^s_{x,y}\ e^{+\,m\,\mathrm{dist}(x,y)} \;<\; \infty\ . \tag{3.4}$$

3. *We say that the potential has an <u>s-regular distribution</u> if for some $\tau > s$ the conditional distributions of $\{V_\omega(x)\}$, <u>at specified values of</u> the hopping terms variables $\{T_{u,v;\omega}\}$, are independent and satisfy the regularity conditions $R_1(\tau)$ and $R_2(s)$ with uniform constants.*

Following is the generalization of Theorem 1.1.

Theorem 3.1 *Let H_ω be a random operator on $\ell^2(\mathcal{T})$ ($\mathcal{T} = \mathcal{G} \times S$, as above) with an s-regular distribution for the potential $V_\omega(\cdot)$, and with tempered off-diagonal matrix elements ($T_{x,y;\omega}$), which is stochastically invariant under magnetic shifts. Assume that for some $z \in \mathbb{C}$ and a finite region $\Lambda \subset \mathcal{T}$, which contains the periodicity cell $\mathcal{C}$, the following is satisfied for all subsets $W \subset \Lambda$*

$$\left(1 + \frac{\widetilde{C}_s}{\lambda^s}\, \Xi_s(\Lambda)\right) \sup_{x \in \mathcal{C}} \sum_{<u,u'> \in \Lambda \times (\mathcal{T} \setminus \Lambda)} \tau(u-u')^s \; \mathbb{E}\left(| < x| \frac{1}{H_{W;\omega} - z} |u> |^s \right) \; < \; 1 \, , \tag{3.5}$$

where

$$\tau(v) = \sup_{u \in \mathcal{T}} \text{ess sup}_\omega |T_{u,u+v;\omega}| \, , \qquad \Xi_s(\Lambda) \; = \sum_{<u,u'> \in \Lambda \times (\mathcal{T} \setminus \Lambda)} \tau(u-u')^s \, . \tag{3.6}$$

Then there exist $\mu > 0$, $A < \infty$, such that for all $\Omega \subset \mathcal{T}$, and all $y \in \Omega$,

$$\sum_{x \in \Omega} \mathbb{E}_{\pm i0} \left(| < x| \frac{1}{H_{\Omega;\omega} - z} |y> |^s \right) \; e^{+\mu \, \text{dist}(x,y)} \; \leq \; A \tag{3.7}$$

Remarks:

1. For graphs which grow at an exponential rate, such as the Bethe lattice, exponentially decaying functions need not be summable. The conclusion, eq. (3.7), was therefore formulated in the stronger form, which implies both exponential decay, and almost sure summability. In particular, it is useful to recall that for $s/2 < 1$:

$$\mathbb{E}\left(\left[\sum_y |G(x,y)|^2 \right]^{s/2} \right) \; \leq \; \mathbb{E}\left(\sum_y |G(x,y)|^s \right) \, . \tag{3.8}$$

2. One may note that in the more general theorem we do make use of the "decoupling Lemma", which was not used in Theorem 1.1.

3. Translation invariance played a limited role here: the analysis extends readily to random operators with non-translation invariant distributions, provided only that the required bounds are satisfied uniformly for all translates of Λ, and the distribution of the potential is uniformly s-regular. To demonstrate the required change we cast the next statement in that form.

As we discussed in the preceding sections, condition (3.5) may fail due to the existence of extended states at some surfaces. The following generalization of Theorem 1.2 provides criteria for localization in the bulk which are less affected by such surface states.

Theorem 3.2 *Let H_ω be a random operator on $\ell^2(\mathcal{T})$ ($\mathcal{T} = \mathcal{G} \times S$, as above) with an s-regular distribution for the potential $V_\omega(\cdot)$, and with tempered off-diagonal matrix elements ($\{T_{x,y;\omega}\}$). Assume that for some $z \in \mathbb{C}$ and a finite region Λ, $\mathcal{C} \subset \Lambda \subset \mathcal{T}$,*

$$\left(1 + \frac{\widetilde{C}_s}{\lambda^s}\, \Xi_s(\Lambda)\right)^2 \sup_{x\in\mathcal{T}} \sum_{\substack{u\in\Lambda(x)\\ u'\in\mathcal{T}\setminus\Lambda(x)}} \tau^s_{u,u'}\ \mathbb{E}\left(|<x|\frac{1}{H_{\Lambda;\omega} - z[\bar{z}]}|u>|^s\right) \ < \ 1 \ , \tag{3.9}$$

where $\Lambda(x)$ is the unique translate of Λ, by an element of $\mathcal{G}$, which contains x, and $z[\bar{z}]$ means that the bound is satisfied for both z and $\bar{z}$. Then the condition (3.7) holds for the full operator H_ω (i.e., with $\Omega = \mathcal{T}$), and there exist $B < \infty$, $\tilde{\mu} > 0$ with which for arbitrary $\Omega \subset \mathcal{T}$:

$$\mathbb{E}_{\pm i0}\left(|<x|\frac{1}{H_{\Omega;\omega} - z}|y>|^s\right) \ \leq \ B\ e^{-\,\tilde{\mu}\,\mathrm{dist}_\Omega(x,y)}\ . \tag{3.10}$$

The modified distance $\mathrm{dist}_\Omega(x,y)$ is defined by the natural extension of eq. (1.15).

3.b Derivation of the general results

The derivation of Theorems 3.1 and 3.2 follows very closely the proofs of Section 2. The main difference is in the second portion of the argument where we encounter a more general "sub-harmonicity" relation.

The first part of the proof rests on Lemmas 2.2 and 2.3 which are easily seen to extend to the setup described in Theorem 3.2. (The hopping terms $T_{x,y}$ appearing in section 2.b are replaced with the uniform upper-bound $\tau_{x,y}$.) We thus obtain the following extension of the resolvent bounds.

Lemma 3.3 *Let H_ω be a random operator with the properties listed in Theorem 3.2, and let Λ be a finite subset of $\mathcal{T}$, containing the periodicity cell $\mathcal{C}$, for which the condition (3.5) is satisfied. Then the following bound is valid for any $x \in \Lambda, y \in \mathcal{T}\setminus\Lambda$,*

$$\sup_{\Omega\subset\mathcal{T}} \mathbb{E}\left(|G_\Omega(x,y;z)|^s\right) \ \leq \ b\ \sum_{u\in\mathcal{T}} p_\Lambda(x,u)\ \sup_{\Omega\subset\mathcal{T}} \mathbb{E}\left(|G_\Omega(u,y;z)|^s\right)\ , \tag{3.11}$$

with some $b < 1$ and a "sub-probability kernel" $p_\Lambda(x,u)$, satisfying

$$\sum_u p_\Lambda(x,u) \ \leq \ 1\ ,\ \ \textit{and}\ \sum_x p_\Lambda(x,u)\ \leq\ 1\ , \tag{3.12}$$

which is tempered in the sense that for some $m > 0$

$$\sup_x \sum_u e^{m\,\mathrm{dist}(x,u)} p_\Lambda(x,u) \ < \ \infty \ , \ \textit{and} \ \sup_u \sum_x e^{m\,\mathrm{dist}(x,u)} p_\Lambda(x,u) \ < \ \infty \ . \tag{3.13}$$

Furthermore, assuming (3.9) instead of (3.5), the following bound is valid for any $x \in \Lambda, y \in \mathcal{T}\backslash\Lambda$, and $\Omega \supset \Lambda$

$$\mathbb{E}\left(|G_\Omega(x,y;z)|^s\right) \ \leq \ \tilde{b} \ \sum_{u \in \mathcal{T}} \widetilde{p}_\Lambda(x,u) \ \mathbb{E}\left(|G_\Omega(u,y;z)|^s\right) \ , \tag{3.14}$$

with some $\widetilde{b} < 1$ and $\widetilde{p}_\Lambda(x,u)$ which satisfies the same conditions as $p_\Lambda(x,u)$.

The bounds presented in the above lemma may be read as stating that the resolvent $\mathbb{E}(|G(x,y;z)|^s)$ is sub-harmonic (we use this term here in the sense of "sub-mean") with respect to a tempered probability kernel whenever x, y are sufficiently far apart. Theorems 3.2 and 3.1 follow from these bounds via a general principle which applies to such sub-harmonic functions. We expect this principle to be well known, but for completeness we include a proof here.

Proposition 3.4 *Let $(\mathcal{T}, \mathrm{dist})$ be a countable metric space, $\Lambda \subset \mathcal{T}$ a finite subset, and $g : \mathcal{T} \to \mathbb{R}$ a bounded and non-negative function, which for all $x \in \mathcal{T}\backslash\Lambda$ satisfies:*

$$g(x) \ \leq \ b \sum_u p(x,u) g(u) \ , \tag{3.15}$$

with a kernel on $\mathcal{T} \times \mathcal{T}$ satisfying

$$\sup_x \sum_u p(x,u) \ \leq \ 1 \ , \qquad \sup_u \sum_x p(x,u) \ \leq \ 1 \ , \tag{3.16}$$

which is tempered in the sense of eq. (3.13). Then $g(x)$ is exponentially summable, i.e., for some $\mu > 0$:

$$\sum_y e^{\mu\,\mathrm{dist}(y,\Lambda)} \ g(y) \ < \ \infty \ . \tag{3.17}$$

Proof: One may read the claim as saying that the function $g(\cdot)$ lies in the space $\ell^{1;\mu}(\mathcal{T})$ of functions for which the following norm is finite:

$$\|f\|_{1,\mu} \ := \ \sum_{x \in \mathcal{T}} e^{\mu\,\mathrm{dist}(x,\Lambda)} |f(x)| \ . \tag{3.18}$$

We shall deduce this claim after arriving first at a bound formulated within the larger space of bounded functions $\ell^\infty(\mathcal{T})$.

Let P be the linear operator with the kernel $p(x,y)$. Within $\ell^\infty(\mathcal{T})$ the operator acts as a contraction, since its norm there is

$$\|P\|_{\infty,\infty} \;=\; \sup_x \sum_u p(x,u) \;\leq\; 1 \tag{3.19}$$

(using (3.16)). It is convenient to paraphrase the assumption on $g(\cdot)$ in the following form, which holds for all $x \in \mathcal{T}$:

$$g(x) \;\leq\; \|g\|_\infty \cdot I_\Lambda(x) \;+\; b\,[P \cdot g](x)\ , \tag{3.20}$$

with I_Λ the "indicator function" of Λ. Iterating this relation N times, one obtains a bound in the form of a finite geometric series with a "remainder" which is uniformly bounded by $(b\,\|P\|_{\infty,\infty})^N \cdot \|g\|_\infty$. As $N \to \infty$ the reminder vanishes, since $(b\,\|P\|_{\infty,\infty}) < 1$, and one is left with a bound in the form of a convergent series:

$$g(x) \;\leq\; \|g\|_\infty \sum_{n=0}^{\infty} b^n\,[P^n \cdot I_\Lambda](x)\ . \tag{3.21}$$

We now note that for a finite region Λ, the function I_Λ lies in the "weighted-ℓ^1 space" $\ell^{1;\mu}$. The norm of P as an operator within $\ell^{1;\mu}$ is easily seen to obey:

$$\|P\|_{1,\mu;1,\mu} \;\leq\; \sup_u \sum_x e^{\mu\,\mathrm{dist}(x,u)} p(x,u)\ . \tag{3.22}$$

The expression on the right hand side is convex in μ, and by the temperedness assumption (the analog of eq. (3.13)) it is finite for small enough $\mu > 0$. Since convexity implies continuity, using (3.16) we conclude that there is some $\mu > 0$ for which

$$b\,\|P\|_{1,\mu;1,\mu} \;<\; 1\ . \tag{3.23}$$

With this choice of μ we conclude:

$$\sum_x e^{\mu\,\mathrm{dist}(x,\Lambda)}\, g(x) \;\equiv\; \|g\|_{1,\mu} \;\leq\; \frac{\|g\|_\infty\ |\Lambda|}{1 - b\,\|P\|_{1,\mu;1,\mu}} \;<\; \infty\ . \tag{3.24}$$

■

Theorems 3.1 and 3.2 now follow by a combination of the proposition just shown with Lemma 3.3.

Proof of Theorem 3.1 : To establish the claimed bound (3.7) fix $y \in \mathcal{T}$, and let $g(x) = \sup_\Omega \mathbb{E}(|G_\Omega(x,y;z)|^s)$. We note that for each $x \in \mathcal{T}$ there is a unique element of the symmetry group, $h_x \in \mathcal{G}$, such that $h_x x \in \Lambda$. Starting from the kernel $p_\Lambda(h_x x, h_x u)$ which appears in Lemma 3.3, let us define a shift-invariant kernel $p(x,y)$ by:

$$p(x,u) \;=\; p_\Lambda(h_x x, h_x u)\;. \tag{3.25}$$

Due to the shift invariance of the distribution of H_ω, eq. (3.11) implies that the function $g(x)$ is sub-harmonic, in the sense of (3.15), with respect to the kernel $p(x,u)$, which satisfies (3.16) and is tempered . Thus, a direct application of Proposition 3.4 yields now the claimed bound (3.7). ■

Proof of Theorem 3.2 : The situation to be discussed now is different from that encountered in the last proof in that now for each Ω the basic sub-harmonicity bound can be assumed only for points which are not too close to the boundary $\partial\Omega$. The claim made for the special case $\Omega = \mathcal{T}$ is covered by the above analysis. However, the second claim, i.e., eq. (3.10), requires a somewhat different argument.

The argument we shall use shadows the proof of Proposition 3.4, replacing there the weighted-ℓ^1 estimate by its weighted-ℓ^∞ version. The starting observation is that $\mathbb{E}(|G_\Omega(x,y;z)|^s)$ has the sub-mean property with respect to averages over either x or y – provided the point is at distance at least diam(Ω) from the other and from the boundary $\partial\Omega$. (In allowing the averaging procedure to occur from either side, we rely on the fact that the smallness condition holds for both the kernel $G(x,y;z)$ and its conjugate, or equivalently the fact that the smallness condition is assumed to hold for both z and $\bar{z}$.)

To cast the situation in terms reminiscent of the proof of Proposition 3.4, let us consider the function $g(<x,y>) = \mathbb{E}(|G_\Omega(x,y;z)|^s)$ as defined over the space of pairs, $\Omega \times \Omega$, equipped with the distance function

$$\mathrm{dist}_\Omega(<x_1,y_1>,<x_2,y_2>) \;=\; \mathrm{dist}_\Omega(x_1,x_2) + \mathrm{dist}_\Omega(y_1,y_2)\;. \tag{3.26}$$

For $<x,y>$ not in the set $W := \{<u,v> \;|\; \mathrm{dist}_\Omega(u,v) \le 2L, \text{ with } L = \mathrm{diam}(\Lambda)\}$, we have the basic sub-mean estimate:

$$g(<x,y>) \;\le\; b \sum_{<u,v>} \widetilde{p}(<x,y>,<u,v>)\; g(<u,v>)\;, \tag{3.27}$$

with

$$\widetilde{p}(<x,y>,<u,v>) := \begin{cases} p(x,u)\,\delta_{y,v} & \text{if } \mathrm{dist}_\Omega(x,y) > 2L \text{ and } \mathrm{dist}(x,\partial\Omega) > L \;, \\ \delta_{x,u}\,p(y,v) & \text{if } \mathrm{dist}_\Omega(x,y) > 2L \text{ and } \mathrm{dist}(x,\partial\Omega) \le L \;, \\ \delta_{x,u}\,\delta_{y,v} & \text{if } \mathrm{dist}_\Omega(x,y) \le 2L \;, \end{cases} \tag{3.28}$$

where $p(x,y)$ is given by eq. (3.25).

By repeating the arguments seen there we find that $g(< x, y >)$ obeys the analog of eq. (3.21) — formulated within the space $\ell^\infty(\Omega \times \Omega)$, with the set Λ replaced by W, and the operator P replaced by $\widetilde{P}$ defined by the kernel $\widetilde{p}(< x, y >, < u, v >)$. Unlike in the previous case, we have no fixed bound on the size of the set W. Thus we shall not use here the weighted-ℓ^1 estimate. However, we may reuse the argument applying it to weighted-ℓ^∞ norm of $g(\cdot)$, which is defined as:

$$\|g\|_{\infty;\mu} \; = \; \sup_{<x,y>} e^{\mu \operatorname{dist}(x,y)} \, |g(< x, y >)| \tag{3.29}$$

The conclusion is that there is some $\mu > 0$ at which $\|g\|_{\infty;\mu} < \infty$. Equivalently:

$$\mathbb{E}\left(|G_\Omega(x, y; z)|^s\right) \; \leq \; \text{Const.} e^{-\mu \operatorname{dist}_\Omega(x,y)} \quad , \tag{3.30}$$

as claimed in Theorem 3.2. ∎

4. Some Implications

We shall now present a number of implications of the finite volume criteria for localization, focusing on the finite dimensional lattices $\mathbb{Z}^d$. The statements will bear some resemblance to results derived using the multiscale approach, however the conclusions drawn here go beyond the latter by yielding results on the exponential decay of the mean values. The significance of that was described in the introduction.

4.a Fast power decay ⇒ exponential decay

An interesting and useful implication (as is seen below) is that fast enough power law implies exponential decay. In this sense, random Schrödinger operators join other statistical mechanical models in which such principles have been previously recognized. The list includes the general Dobrushin-Shlosman results [24] and the more specific two-point function bounds in: percolation (Hammersley[23] and Aizenman-Newman [27]), Ising ferromagnets (Simon [25] and Lieb [26]), certain $O(N)$ models (Aizenman-Simon [30]), and time-evolution models (Aizenman-Holley [31], Maes-Shlosman [32].)

Theorem 4.1 *Let H_ω be a random operator on $\ell^2(\mathbb{Z}^d)$ with an s-regular distribution for the potential $(V_\omega(x))$ and tempered off-diagonal matrix elements $(T_{x,y;\omega})$. There are $L_0, B_1, B_2 < \infty$, which depend only on the temperedness bound (3.4), such that if for some $E \in \mathbb{R}$ and some finite $L \geq L_0$, either*

$$L^{3(d-1)} \quad \sup_{L/2 \leq \|x-y\| \leq L} \mathbb{E}\left(| < x| \frac{1}{H_{\Lambda_L(x),\omega} - E} |y > |^s \right) \; \leq \; B_1 \; , \tag{4.1}$$

or

$$L^{4(d-1)} \sup_{L/2\le\|x-y\|\le L} \mathbb{E}\left(|<x|\frac{1}{H_\omega - E - i0}|y>|^s \right) \;\le\; B_2 \;, \tag{4.2}$$

where $\Lambda_L(x) = [-L, L]^d + x$ and $\|y\| \equiv \max_j |y_j|$, then the exponential *localization (1.3) holds for all energies in some open interval (a, b) containing E.*

Proof: By Theorem 3.2, to establish exponential decay at the energy E it suffices to show that for each $x \in \mathbb{Z}^d$

$$\left(1 + \frac{\widetilde{C}_s}{\lambda^s}\, \Xi_s(\Lambda_L)\right)^2 \sum_{\substack{u\in\Lambda_L(x)\\ u'\in\mathbb{Z}^d\setminus\Lambda_L(x)}} \tau^s_{u,u'}\, \mathbb{E}\left(|G_{\Lambda_L(x)}(x,u;E)|^s\right) \;<\; 1 \;. \tag{4.3}$$

Because the off diagonal elements are tempered we have the following bounds

$$\tau^s_{u,u'} \;\le\; \text{Const.}\; e^{-m|u-u'|} \;, \qquad \Xi_s(\Lambda_L) \;\le\; \text{Const.}\; L^{d-1} \;, \tag{4.4}$$

for some $m > 0$, and all $L > 1$. Under the assumption eq. (4.1):

$$\begin{aligned} \sum_{\substack{u\in\Lambda_L(x)\\ u'\in\mathbb{Z}^d\setminus\Lambda_L(x)}} \tau^s_{u,u'}\, \mathbb{E}\left(|G_{\Lambda_L(x)}(x,u;E)|^s\right) \;\le & \\ \le\; \frac{\widetilde{C}_s}{\lambda^s}\; \text{Const.}\; (L/2)^d\, e^{-mL}/2 \;+ & \\ +\; \text{Const.} \sup_{L/2\le\|x-y\|\le L} \mathbb{E}\left(|<x|\frac{1}{H_{\Lambda_L(x),\omega} - E}|y>|^s \right) L^{d-1} \;. & \end{aligned} \tag{4.5}$$

For this bound the sum was split according to $\|u - u'\| <$ (or $\ge$)$L/2$, and in the first case we used the uniform upper bound $\mathbb{E}(|G(x,u;E)|^s) \le \widetilde{C}_s/\lambda^s$.

It is now easy to see that with an appropriate choice of L_0 and B_1 condition (4.1) implies the claimed bound (4.3) – for the given energy E. The extension to an interval of energies around E then follows from the continuity of the fractional moments of finite volume Green functions.

To show the sufficiency of the second condition, we first use Lemma 2.3 to bound finite volume Green functions in terms of the corresponding infinite volume funtions

$$\mathbb{E}\left(|G_{\Lambda_L(x)}(x,y;E)|^s\right) \le\; \mathbb{E}\left(|G(x,y;E)|^s\right) + \frac{\widetilde{C}_s}{\lambda^s} \sum_{\substack{u\in\Lambda_L(x)\\ u'\in\mathbb{Z}^d\setminus\Lambda_L(x)}} \tau^s_{u',u}\, \mathbb{E}\left(|G(x,u';E)|^s\right) \;. \tag{4.6}$$

Splitting the sum as in eq. (4.5), we get

$$\sup_{L/2 \le \|x-y\| \le L} \mathbb{E}\left(|G_{\Lambda_L(x)}(x,y;E)|^s\right) \le$$

$$\le \left[\frac{\widetilde{C}_s}{\lambda^s}\right]^2 \text{Const. } (L/2)^d \, e^{-mL}/2 \; +$$

$$+ \; \left(1 + \text{Const. } L^{d-1}\right) \times L^{d-1} \sup_{L/2 \le \|x-y\| \le L} \mathbb{E}\left(|G(x,y;E)|^s\right) \quad (4.7)$$

The combination of eq. (4.7) with (4.5), yields the claim - for the given energy. Again, the existence of an open interval of energies in which the condition is met is implied by the continuity of the finite-volume expectation values. ■

4.b Lower bounds for $G_\omega(x,y;E_{\text{edge}} + i0)$ at mobility edges

Boundary points of the continuous spectrum are often referred to as *mobility edges*. (In an ergodic setting the location of such points does not depend on the realization ω [33].) The proof of the occurance of continuous spectrum for random stochastically shift-invariant operators on $\mathbb{Z}^d$ is still an open problem (one may add that we are glossing here over some fine distinctions in the dynamical behaviour [34]). However it is intersting to note that Theorem 4.1 directly yields the following pair of lower bounds on the decay rate of the Green function at mobility edges, E_{edge}, for stochastically shift invariant random operators with regular probability distribution of the potential:

$$\sup_{L/2 \le \|y\| \le L} \mathbb{E}\left(| < 0| \frac{1}{H_{[-L,L]^d,\omega} - E_{\text{edge}}} |y > |^s \right) \ge B_1 \, L^{-3(d-1)} \,, \qquad (4.8)$$

$$\sup_{L/2 \le \|y\| \le L} \mathbb{E}\left(| < 0| \frac{1}{H_\omega - E_{\text{edge}} - i0} |y > |^s \right) \ge B_2 \, L^{-4(d-1)} \,, \qquad (4.9)$$

with $\|y\| \equiv \max_j |y_j|$. We do not expect the power laws provided here to be optimal. As mentioned above, vaguely similar bounds are known for the critical two-point functions in certain statistical mechanical models (percolation, Ising spin systems, and some $O(N)$ spin models).

4.c Extending off the real axis

For various applications, such as the decay of the projection kernel (see [8] Sect. 5), it is useful to have bounds on the resolvent at $z = E + i\eta$ which are uniform in η. The following result shows that in order to establish such uniform bounds it is sufficient to verify our criteria for real energies in some neighborhood of E.

Theorem 4.2 *Let H_ω be a random operator on $\ell^2(\mathbb{Z}^d)$ with an s-regular distribution for the potential $(V_\omega(x))$ and tempered off-diagonal matrix elements $(T_{x,y;\omega})$. Suppose that for some $E \in \mathbb{R}$, and $\Delta E > 0$, the following bound holds uniformly for $\xi \in [E-\Delta E, E+\Delta E]$:*

$$\mathbb{E}\left(|<x|\frac{1}{H_\omega - \xi - i0}|y>|^s \right) \;\leq\; A\, e^{-\mu|x-y|} \,. \tag{4.10}$$

Then for all $\eta \in \mathbb{R}$:

$$\mathbb{E}\left(|<x|\frac{1}{H_\omega - E - i\eta}|y>|^s \right) \;\leq\; \widetilde{A}\;\; e^{-\tilde{\mu}|x-y|} \,, \tag{4.11}$$

with some $\widetilde{A} < \infty$ and $\tilde{\mu} > 0$ – which depend on ΔE and the bound (4.10).

Remarks:

1. This result is not needed in situations covered by the single site version of the criterion provided by Theorem 1.1, since if eq. (1.12) is satisfied at some $E \in \mathbb{R}$ then it automatically holds uniformly along the entire line $E + i\mathbb{R}$. We do not see a monotonicity argument for such a deduction in case of other finite-volumes.

2. One way to derive the statement is by using the fact that exponential decay may be tested in finite volumes: if a finite volume criterion holds for some E then continuity allows one to extend it to all $E + i\eta$ with η sufficiently small. The Combes-Thomas estimate [35] can then be used to cover the rest of the line $E + i\mathbb{R}$. However, by this approach one gets only a weaker decay rate for energies off the real axis. It is tempting to think that some contour integration argument could be found to significantly improve on that. The proof given below is a step in that direction (though it still leaves one with the feeling that a more efficient argument should be possible).

Proof: Assume that the condition (4.10) is satisfied for all $\xi \in [E - \Delta E, E + \Delta E]$. We shall show that this implies that for any power α

$$\mathbb{E}\left(|<x|\frac{1}{H_\omega - \xi - i\eta}|y>|^s \right) \;\leq\; \frac{A_\alpha}{|x-y|^\alpha} \,, \tag{4.12}$$

with the constant $A_\alpha < \infty$ uniform in η. The stated conclusion then follows by an application of Theorem 4.1 (and the uniform bounds seen in its proof).

We shall deal separately with large and small $|\eta|$, splitting the two regimes at $\Delta E \times \pi/\alpha$. The case $|\eta| \geq \Delta E \times \pi/\alpha$ is covered by the general bound of Combes-Thomas [35], which states that:

$$|G(x, y; E + \mathrm{i}\eta)| \leq (2/\eta)\mathrm{e}^{-m|x-y|} \tag{4.13}$$

for any $m \geq 0$ such that

$$\sum_{x \in \mathbb{Z}^d} \tau(x) \, (e^{m|x|} - 1) \; \leq \; \eta/2 \, . \tag{4.14}$$

To estimate the resolvent for $|\eta| \leq \Delta E \times \pi/\alpha$, we shall use the fact that the function

$$f_L(\zeta) \; = \; \mathbb{E}\left(|G_{[-L,L]^d}(x,y;\zeta)|^s \right) \tag{4.15}$$

is subharmonic in the upper half plane, and continuous at the boundary. The subharmonicity is a general consequence of the analyticity of the resolvent in ζ, and the continuity is implied through the continuity of the distribution of the potential. L serves as a convenient cutoff, which may be removed after the bounds are derived (since $H_{[-L,L]^d,\omega} \underset{L\to\infty}{\longrightarrow} H_\omega$ in the strong resolvent sense).

Let $D \subset \mathbb{C}$ be the triangular region in the upper half plane in the form of an equilateral triangle based on the real interval $[E - \Delta E, E + \Delta E]$ with the side angles equal to θ – determined by the condition

$$\alpha \; = \; \frac{2\pi}{\theta} - 1 \, . \tag{4.16}$$

The Poisson-kernel representation of harmonic functions yields, for $E + i\eta \in D$,

$$f_L(E + i\eta) \; \leq \; \int_{\partial D} f_L(\zeta) \; P^D_{E+i\eta}(d\zeta) \tag{4.17}$$

where $P^D_{E+i\eta}(d\zeta)$ is a certain probability measure on ∂D. We now rely on the fact that this probability measure satisfies

$$P^D_{E+i\eta}(d\zeta) \; \leq \; \text{Const.} \; d(\eta^{2\pi/\theta}) \, / \Delta E^{2\pi/\theta} \, . \tag{4.18}$$

(This is easily understood upon the unfolding of D by the map $z \mapsto z^{2\pi/\theta}$ applied from either of the base corners of D, i.e., from $\zeta = E \pm \Delta E$, and a comparison with the Poisson kernel in the upper half plane.)

For $\zeta \in \partial D \cap \mathbb{R}$ the integrand satisfies the exponential bound (4.10). Along the rest of the boundary of D we use the Combes-Thomas bound (4.13). Putting it all together we get

$$f_L(E + i\eta) \; \leq \; A \, e^{-\mu|x-y|} \; + \; \text{Const.} \int_0^{\Delta E \, \theta} \frac{2}{\eta} \, e^{-\text{Const.} \, |x-y| \, \eta} \; d(\eta^{2\pi/\theta}) \; / \Delta E^{2\pi/\theta} \, . \tag{4.19}$$

The claimed eq. (4.12) follows by simple integration, and the relation (4.16). ■

4.d Relation with the multiscale analysis and density of states estimates

Using the above results we shall now show that the fractional moment localization condition is satisfied throughout the regime for which localization can be shown via the multiscale analysis, and also in regimes over which one has suitable bounds (e.g., via Lifshitz tail estimates) on the density of states of the operators restricted to finite regions $\Lambda_L = [-L, L]^d$. The following result is useful for the latter case.

Theorem 4.3 *Let H_ω be a random operator on $\ell^2(\mathbb{Z}^d)$ with tempered off-diagonal matrix elements $(T_{x,y;\omega})$ and a distribution of the potential which is s-regular for all s small enough, which is stochastically invariant under magnetic shifts. Then, given $\beta \in (0,1)$, $C_1 > 0$, and $\xi > 3(d-1)$, there exist $L_0 > 0$ and $C_2 > 0$ such that if for some $L \geq L_0$*

$$\text{Prob}\left[\text{dist}\left(\sigma(H_{\Lambda_L;\omega}), E\right) \leq C_1 L^{-\beta}\right] \ < \ C_2 L^{-\xi} \ , \tag{4.20}$$

at some energy E, then the exponential localization condition (1.3) holds in some open interval containing E.

The condition (4.20) is similar to the one used in the multiscale analysis, although there one can also find a sufficient diagnostic with arbitrary $\xi > 0$. It may therefore not be initially clear that the methods of this paper may be used throughout the regime in which the multiscale analysis applies. However, the proof of Theorem 4.3 is easily adapted to prove the following result which implies fractional moment localization via the *conclusions* of the multiscale analysis.

Theorem 4.4 *Let H_ω be a random operator with tempered off-diagonal matrix elements $(T_{x,y;\omega})$ and a distribution of the potential which is s-regular for all s small enough, which is stochastically invariant under magnetic shifts. If for some $E \in \mathbb{R}$ there exist $A < \infty$, $\mu > 0$, and $\xi > 3(d-1)$ such that*

$$\lim_{L\to\infty} L^{\xi}\, \text{Prob}\left[|G_{\Lambda_L;\omega}(0,x)| \ > \ Ae^{-\mu|x|} \text{ for some x} \in \Lambda_\text{L}\right] \ = \ 0 \ , \tag{4.21}$$

then the exponential localization condition (1.3) holds in some open interval containing E.

Remarks:

1. When the multiscale analysis applies, it allows one to conclude that there are $A < \infty$ and $\mu > 0$ such that the probabilities appearing on the left side of eq. (4.21) decay faster than *any* power of L as $L \to \infty$. Thus, the conclusions of the multiscale analysis imply that exponential localization in the stronger sense discussed in our work applies throughout the regime which may be reached by this prior method.

2. It is of interest to combine the criterion presented above with Lifshitz tail estimates on the density of states at the bottom of the spectrum, E_0, and at band edges. Using Lifshitz tail estimates, it is possible to show that [36]:

$$\text{Prob}\left[\inf \sigma(H_{\Lambda_L;\omega}) \le E_0 + \Delta E\right] \;\le\; \text{Const. } L^d e^{-\Delta E^{-d/2}} \, . \tag{4.22}$$

Theorem 4.3 then implies fractional moment localization in a neighborhood of E_0; we need only choose $\Delta E \propto L^{-\beta}$ with $\beta \in (0,1)$ for large enough L. Previous results in this vein may be found in [21, 16, 17, 18].

Proof of Theorems 4.3 and 4.4: We first prove Theorem 4.3 and then indicate how the proof can be modified to show Theorem 4.4.

Fix an energy $E \in \mathbb{R}$. For $L > 0$, define

$$p_L(\delta) \;:=\; \text{Prob}\left[\text{dist}\left(\sigma(H_{\Lambda_L;\omega}), E\right) \le \delta\right] \, , \tag{4.23}$$

and let

$$\delta_L \;:=\; C_1 L^{-\beta} \, . \tag{4.24}$$

We will show that for suitable $s \in (0,1)$, $L_0 > 0$ and $C_2 > 0$, if

$$p_L(\delta_L) \;<\; C_2 L^{-\xi} \tag{4.25}$$

then the input condition (4.1) of Theorem 4.1:

$$L^{3(d-1)} \sup_{L/2 \le \|y\| \le L} \mathbb{E}\left(|<0|\frac{1}{H_{\Lambda_L,\omega} - \widetilde{E}}|y>|^s \right) \;\le\; B_1 \, , \tag{4.26}$$

is satisfied for all energies $\widetilde{E} \in [E - \frac{1}{2}\delta_L, E + \frac{1}{2}\delta_L]$. Exponential localization in the corresponding interval (and strip, with $\eta \ne 0$) follows then by Theorems 4.1 (and Theorem 4.2).

First we must show how to estimate $\mathbb{E}\left(|G_{\Lambda_L;\omega}(0,u;\widetilde{E})|^s\right)$ in terms of $p_L(\delta)$. This is achieved by considering separately the contributions from the "good set":

$$\Omega_G \;=\; \{\omega \mid \text{dist}\left(\sigma(H_{\Lambda_L;\omega}), E\right) > \delta\} \, , \tag{4.27}$$

and its complement, the "bad set": $\Omega_B \;=\; \Omega_G^c$.

On the "good set", $\omega \in \Omega_G$, the energy $\widetilde{E}$ is at a small yet significant distance ($\Delta E \ge \frac{1}{2}\delta$) from the spectrum of $H_{\Lambda_L;\omega}$. In this situation, we use the Combes-Thomas [35] bound, by which:

$$|G_{\Lambda_L;\omega}(0,u;\widetilde{E})| \;\le\; \frac{2}{\Delta E} e^{-\frac{1}{2}\Delta E |u|} \, . \tag{4.28}$$

The above estimate does not apply on the "bad set". However, using the Hölder inequality, we find that the net contribution to the expectation is small because $\mathrm{Prob}(\Omega_B) = p_L(\delta)$ is small. The two estimates are combined in the following bound:

$$\begin{aligned}
&\mathbb{E}\left(|G_{\Lambda_L;\omega}(0,u;\widetilde{E})|^s \right) \\
&\quad = \mathbb{E}\left(|G_{\Lambda_L;\omega}(0,u;\widetilde{E})|^s \; I[\omega \in \Omega_G] \right) \; + \; \mathbb{E}\left(|G_{\Lambda_L;\omega}(0,u;\widetilde{E})|^s \; I[\omega \in \Omega_B] \right) \\
&\quad \le 4^s \delta^{-s} e^{-s\,|u|\,\delta\,/4} \; + \; \mathbb{E}\left(|G_{\Lambda_L;\omega}(0,u;\widetilde{E})|^t \right)^{\frac{s}{t}} \; \mathbb{E}\left(I[\omega \in \Omega_B] \right)^{1-\frac{s}{t}} \\
&\quad \le 4^s \delta^{-s} e^{-s\,|u|\,\delta\,/4} \; + \; C_t^{\frac{s}{t}} / \lambda^s \, p_L(\delta)^{1-\frac{s}{t}} \; ,
\end{aligned} \tag{4.29}$$

where t is any number greater than s for which the distribution of the potential is still t-regular (i.e., $C_t < \infty$).

The required bound, eq. (4.26), is satisfied once one chooses s small enough so that $\xi \ge \frac{t}{t-s} 3(d-1)$, and L_0 large enough so that for $L > L_0$

$$4^s C_1^{-s} L^{3(d-1)-s\beta} e^{-s\,C_1\,L^{1-\beta}\,/4} \; \le \; B_1/2 \; . \tag{4.30}$$

Finally let us remark on how this argument can be adapted to prove Theorem 4.4. We simply define the good and bad sets differently:

$$\Omega_G \; = \; \{ \omega \mid |G_{\Lambda_L;\omega}(0,x)| \; \le \; A e^{-\mu |x|} \text{ for all } \mathrm{x} \in \Lambda_\mathrm{L} \} \; , \tag{4.31}$$

and $\Omega_B \; = \; \Omega_G^c$, and then proceed as in the proof of Theorem 4.3 using Hölder's inequality to estimate the contributions from Ω_B. It is easy to see that for large L, the condition (4.21) implies that the input for Theorem 4.1 is satisfied. ■

Thus, we have seen here that the fractional moment localization condition holds throughout the regime for which loclization can be established by any available methods. This is meaningful since that condition carries a number of physically significant implications.

A. Dynamical Localization

Among the implications of the fractional moment condition is dynamical localization, expressed through uniform exponential decay of the average time evolution kernels:

$$\mathbb{E}\left(\sup_{t\in\mathbb{R}} |<x|P_{H_\omega\in F}\; e^{itH}|y>|\right) \;\leq\; A\; e^{-\mu|x-y|}\;, \tag{A.1}$$

where $P_{H_\omega\in F}$ indicates the spectral projection of H_ω onto a set $F\subset\mathbb{R}$ in which the fractional moment condition is known to hold. A derivation of this implication, under some auxiliary assumptions on the distribution of the potential, was given in ref. [13]. For completeness we offer here a streamlined version of that argument, which also extends the result in that we now allow F to be an unbounded set (in particular the full real line).

The inequality expressed in eq. (A.1) is not special to the time evolution operators $f_t(E) = e^{itE}$; it follows, rather, from a similar bound on the average total mass of the spectral measures, $\mu_\omega^{x,y}$, associated to *pairs* of sites x, y. The measures are defined by the spectral representation:

$$\int f(E)\mu_\omega^{x,y}(dE) \;:=\; <x|f(H_\omega)|y>\;, \tag{A.2}$$

for bounded Borel functions f. In the following discussion we denote by $|\mu_\omega^{x,y}|$ the *absolute value* (sometimes called the *total variation*) of $\mu_\omega^{x,y}$.

Theorem A.1 *Let H_ω be a random operator on $\ell^2(\mathbb{Z}^d)$ with tempered off-diagonal matrix elements and a potential V_ω which satisfies:*

1. *For some $\delta\in(0,1)$, the δ-moments of V_ω, $\mathbb{E}\left(|V_\omega(x)|^\delta\right)$, are uniformly bounded.*
2. *For each $x\in\mathbb{Z}^d$ the conditional distribution of $v = V_\omega(x)$ at specified values of all other matrix elements has a density $\rho_\omega^x(v)$, and the functions ρ_ω^x are uniformly bounded.*

Suppose there is an energy domain $F\subset\mathbb{R}$ on which H_ω satisfies a uniform fractional moment bound, i.e., there exist $A<\infty$ and $\mu>0$ such that, for some $s\in(0,1)$,

$$\mathbb{E}\left(|<x|\frac{1}{H_{\Lambda;\omega}-E}|y>|^s\right) \;\leq\; A\; e^{-\mu|x,y|}\;, \tag{A.3}$$

for any finite region $\Lambda\subset\mathbb{Z}^d$, any pair of sites $x,y\in\Lambda$, and every $E\in F$. Then there exist $A'<\infty$ and $\mu'>0$ such that for any pair of sites $x,y\in\mathbb{Z}^d$,

$$\mathbb{E}\left(|\mu_\omega^{x,y}|(F)\right) \;\leq\; A'\; e^{-\mu'|x-y|}\;, \tag{A.4}$$

where $\mu_\omega^{x,y}$ is the spectral measure associated to the pair x,y and H_ω.

Remarks:

1. Recall that for any regular Borel measure μ, $|\mu|(F) = \sup | \int_F f(E)\mu(dE)|$ where the supremum ranges over Borel measurable (or even just continuous) functions f which are point-wise bounded by 1. Thus eq. (A.4) implies that

$$\mathbb{E}\left(\sup_t | < x|f_t(H_\omega) P_{H_\omega \in F}|y > | \right) \;\leq\; C \; A' e^{-\mu'|x-y|} \,, \tag{A.5}$$

for any uniformly bounded family of Borel functions $\{f_t\}$. In particular, we may take $f_t(E) = e^{itE}$ for $t \in \mathbb{R}$ to obtain dynamical localization (A.1) as promised.

2: The requirement that the conditional densities, ρ^x_ω, be uniformly bounded is overly strong. By the arguments presented in ref. [13], the result extends to potentials for which there is some $q > 0$ such that $\int (\rho^x_\omega(v))^{1+q} dv$ are uniformly bounded.

3: Since this work extends now the *exponential* dynamical localization to the regime covered by the multiscale analysis, let us mention that prior results covering this regime include the proof of localization in terms of *power-law* bounds for the time evolution kernel [37, 38]. (The analysis there is more general since it applies also to models for which the fractional moment method has not been developed, e.g., continuum operators).

Proof of Theorem A.1: It is convenient to derive the result through the analysis of the finite volume operators obtained by restricting H_ω to finite regions, $\Lambda_n \subset \mathbb{Z}^d$. It is generally understood that for each $x, y \in \mathbb{Z}^d$ and each increasing sequence of finite regions Λ_n which contain $\{x, y\}$ and whose union is $\mathbb{Z}^d$, the associated spectral measures, $\mu^{x,y}_{\Lambda_n;\omega}$, converge in the vague topology to $\mu^{x,y}_\omega$. Thus, by the lemma of Fatou, for any $F \subset \mathbb{R}$: $\mathbb{E}(|\mu^{x,y}_\omega|(F)) \leq \underline{\lim}_{n\to\infty} \mathbb{E}(|\mu^{x,y}_{\Lambda_n;\omega}|(F))$.

The upshot is that it suffices to prove the following statement regarding finite volume operators.

Under the assumptions of Theorem A.1 there exist $C,\ r > 0$ (which depend only on the regularity assumptions for H_ω) such that for any finite region $\Lambda \subset \mathbb{Z}^d$, any $x, y \in \Lambda$, any $F \subset \mathbb{R}$, and any $s \in (0,1)$:

$$\mathbb{E}\left(\left|\mu^{x,y}_{\Lambda;\omega}\right|(F) \right) \;\leq\; C \; \left[\sup_{E\in F} \mathbb{E}\left(| < x| \frac{1}{H_{\Lambda,\omega} - E} |y > |^s \right) \right]^r \,. \tag{A.6}$$

Following is a summary of the proof of this assertion.

Let us fix a finite region $\Lambda \subset \mathbb{Z}^d$ and a pair of sites $x, y \in \Lambda$. For simplicity of notation, we will suppress the region Λ and denote the restricted operator by H_ω and the associated spectral measure by $\mu^{x,y}_\omega$

Since $\ell^2(\Lambda)$ is finite dimensional, $\mu^{x,y}_\omega$ is a weighted sum of Dirac measures supported on the eigenvalues of H_ω. Integrals with respect to this measure are discrete sums. The argument of ref.[13] makes an essential use of the following representation of this measure.

Let $v = V_\omega(x)$, and let $\hat{v}$ be any other value in $\mathbb{R}$. Denote $\widehat{\Gamma}(E) := -1/ < x|\frac{1}{\hat{H}_\omega - E}|x >$, with $\hat{H}_\omega$ the operator with the potential at x changed to $\hat{v}$. Then,

$$\mu_\omega^{x,y}(dE) \;=\; -(v-\hat{v}) \;\; < x|\frac{1}{\hat{H}_\omega - E}|y > \;\; \delta(v - \hat{v} - \hat{\Gamma}(E)) \; dE \; . \tag{A.7}$$

In what follows, we will take $\hat{v} = \hat{v}_\omega$ to be a random variable independent of v_ω and identically distributed. In this case eq. (A.7) holds almost surely.

A special case of eq. (A.7) is the formula (which was the basis for the important "Kotani-argument"[39, 12]) for the spectral measure at x

$$\mu_\omega^{x,x}(dE) \;=\; \delta(v - \hat{v} - \hat{\Gamma}(E)) \; dE \; . \tag{A.8}$$

The above is a probability measure. Another normalizing condition is:

$$|v - \hat{v}|^2 \int |< x|\frac{1}{\hat{H}_\omega - E}|y >|^2 \; \delta(v - \hat{v} - \hat{\Gamma}(E)) \; dE \;\leq\; 1 \; , \tag{A.9}$$

(which typically holds as equality).

The reason for eq. (A.9) is that by the general structure of the spectral measures, $\mu_\omega^{x,y}(dE) = \Psi_\omega(E)\mu_\omega^{x,x}(dE)$, with $\Psi_\omega(E)$ satisfying $\int |\Psi_\omega(E)|^2 \mu_\omega^{x,x}(dE) \;= < y|\, P_\omega\, |y > \;\leq\; 1$, where P_ω is the projection onto the cyclic subspace for H_ω which contains $|x >$.

Let us first present the necessary estimates for the case that $F \subset \mathbb{R}$ is of finite Lebesgue measure. Using the bound eq. (A.9), and the Hölder inequality,

$$\begin{aligned} &\mathbb{E}(|\mu_\omega^{x,y}|\,(F)) \\ &\quad\leq\; \left[\mathbb{E}\left(|v-\hat{v}|^\alpha \;\int_F |< x|\frac{1}{\hat{H}_\omega - E}|y >|^\alpha \; \delta(v - \hat{v} - \hat{\Gamma}(E)) \; dE\right)\right]^{1/(2-\alpha)} \quad , \end{aligned} \tag{A.10}$$

where α (< 1) is a small number to be specified later. By a further application of the Hölder inequality, followed by the Jensen inequality we obtain

$$\begin{aligned} \mathbb{E}\left(\left|\mu_{\Lambda;\omega}^{x,y}\right|(F)\right)^{2-\alpha} \;\leq\; & \left[2\mathbb{E}(|v|^\delta)\right]^{\alpha/\delta} \\ & \times \left[\mathbb{E}\left(\int_F |< x|\frac{1}{\hat{H}_\omega - E}|y >|^s \; \delta(v - \hat{v} - \hat{\Gamma}(E)) \; dE\right)\right]^{\alpha/s} \quad , \end{aligned} \tag{A.11}$$

where α is fixed by the equation $\alpha/s + \alpha/\delta = 1$. Finally we evaluate:

$$\begin{aligned} \mathbb{E}\left(\int_F |<x|\frac{1}{\hat{H}_\omega - E}|y>|^s \; \delta(v - \hat{v} - \hat{\Gamma}(E)) \; dE\right) \\ = \int_F \mathbb{E}\left(|<x|\frac{1}{\hat{H}_\omega - E}|y>|^s \rho_\omega^x(\hat{v} + \hat{\Gamma}(E))\right) \; dE \\ \leq \kappa \; \int_F \mathbb{E}\left(|<x|\frac{1}{\hat{H}_\omega - E}|y>|^s\right) \; dE \; , \end{aligned} \tag{A.12}$$

where κ is a uniform upper bound for ρ_ω^x. These estimates can be combined to provide a bound of the form eq. (A.6) for F a finite interval, which was the case considered in ref. [13]. We shall now improve the argument, to obtain a statement which covers the case that the localized spectral regime is unbounded.

Since we do not wish our final estimate to depend on the Lebesgue measure of F, we seek a way of introducing an integrable weight $h(E)$, so that the final bound involves the integral of $h(E)dE$ in place of dE. This may be accomplished with the following inequality:

$$|\mu_\omega^{x,y}|\,(F) \;\leq\; \left(<x|\; |g(H)|^{2p} \;|x>\right)^{\frac{1}{2p}} \; \left(\int_F |g(E)|^{-p'} \,|\mu_\omega^{x,y}|\,(dE)\right)^{\frac{1}{p'}} \tag{A.13}$$

where $1/p + 1/p' = 1$ and g is any continuous function which is bounded and bounded away from zero. To prove eq. (A.13), write $|\mu_\omega^{x,y}|\,(F) \;=\; \int_F g(E)/g(E)\,|\mu_\omega^{x,y}|\,(dE)$, and apply the Hölder inequality followed by

$$\left|\int |g(E)|^p \,|\mu_\omega^{x,y}|\,(dE)\right| \;\leq\; \left(<x|\; |g(H)|^{2p} \;|x>\right)^{1/2} \; . \tag{A.14}$$

It is convenient to choose $g(E)^{2p} = (1 + E^2)$, since $<x|(1 + H_\omega^2)|x> = B + V_\omega(x)^2$ where B_ω is a bounded random variable which depends only on the off-diagonal part of H_ω. Upon taking expectations followed by a further application of the Hölder inequality this leads to

$$\begin{aligned} \mathbb{E}\left(|\mu_\omega^{x,y}|\,(F)\right) \;\leq\; \left[\mathbb{E}\left(\left(B_\omega + V_\omega(x)^2\right)^{\frac{q}{2p}}\right)\right]^{1/q} \\ \times \left[\mathbb{E}\left(\left(\int_F \frac{1}{(1+E^2)^{\frac{p'}{2p}}} \,|\mu_\omega^{x,y}|\,(dE)\right)^{\frac{q'}{p'}}\right)\right]^{1/q'} \; , \end{aligned} \tag{A.15}$$

where $1/q + 1/q' = 1$. We estimate the two factors on the right hand side of this inequality separately.

The first factor can be controlled by choosing $q = p\,\delta$ so that

$$\mathbb{E}\left(\left(B_\omega + V_\omega(x)^2\right)^{\frac{q}{2p}} \right) \;\le\; \|B_\omega\|_\infty^{\delta/2} + \mathbb{E}\left(|V_\omega(x)|^\delta\right) \;. \tag{A.16}$$

The exponents p, p', q, q' are all specified once we choose $p > 1/\delta$. Specifically, $q = \delta p$, $q' = p(p-1/\delta)^{-1}$, and $p' = p(p-1)^{-1}$. Note that $p' < q'$.

To estimate the second factor, we note that $|\mu_\omega^{x,y}|$ is a sub-probability measure and $q'/p' > 1$, so by the Jensen inequality,

$$\mathbb{E}\left(\left(\int_F \frac{1}{(1+E^2)^{\frac{p'}{2p}}} \, |\mu_\omega^{x,y}|\,(dE) \right)^{\frac{q'}{p'}} \right) \;\le\; \mathbb{E}\left(\int_F \frac{1}{(1+E^2)^{\frac{q'}{2p}}} \, |\mu_\omega^{x,y}|\,(dE) \right) \;. \tag{A.17}$$

Estimating the right hand side with the argument outlined above for F with finite Lebesgue measure, we find that

$$\mathbb{E}\left(\int_F \frac{1}{(1+E^2)^{\frac{q'}{2p}}} \, |\mu_\omega^{x,y}|\,(dE) \right) \;\le\; \left[2\mathbb{E}(|v|^\delta)\right]^{\alpha/\delta} \times \left[\kappa \int_F \mathbb{E}\left(|<x|\frac{1}{\hat{H}_\omega - E}|y>|^s \right) \frac{dE}{(1+E^2)^{q'/2p}} \right]^{\alpha/s} \;, \tag{A.18}$$

which is uniformly bounded provided we choose p such that $q'/p > 1$. This is possible since $q'/p = (p-1/\delta)^{-1}$ which can be made as large as we like.

Thus, for any finite volume $\mathbb{E}\left(\left|\mu_{\Lambda;\omega}^{x,y}\right|(F)\right)$ can be bounded by a constant multiple of $\sup_{E\in F} \mathbb{E}\left(|<x|\frac{1}{\hat{H}_{\Lambda;\omega}-E}|y>|^s \right)$ raised to a certain power. Which multiple and which power depend only on the δ-moments of the potential and the uniform bound on the conditional distributions ρ_ω^x. By the vague convergence argument outlined at the start of the proof, this proves the theorem. ■

B. A fractional moment bound

The regularity conditions $R_1(\tau)$ and $R_2(s)$ have been used to give *a priori* estimates of certain fractional moments. Such fractional moment bounds are properties of the general class of operators with diagonal disorder. Hence, throughout this appendix, we consider random operators H_ω on $\ell^2(\mathcal{T})$ of the form

$$H_\omega \;=\; T_0 + \lambda V_\omega \;, \tag{B.1}$$

where T_0 is an arbitrary bounded self–adjoint operator and V_ω is a random potential for which $V_\omega(x)$ are independent random variables ($\mathcal{T}$ is any countable set).

Lemma B.1 *Let H_ω be a random operator given by eq. (B.1) such that for each x the probability distribution of the potential $V_\omega(x)$ satisfies $R_1(\tau)$ for some fixed $\tau > 0$ with constants uniform in x. Then there exists $\kappa_\tau < \infty$ such that for any finite subset Λ of $\mathcal{T}$, any $x, y \in \Lambda$, any $z \in \mathbb{C}$, and any $s \in (0, \tau)$*

$$\mathbb{E}\left(| < x| \frac{1}{H_{\Lambda;\omega} - z} |y > |^s \middle| \{V(u)\}_{u \in \Lambda \setminus \{x,y\}} \right) \leq \frac{\tau}{\tau - s} \frac{(4\kappa_\tau)^{s/\tau}}{\lambda^s} . \tag{B.2}$$

Proof: Let us first consider $z = E \in \mathbb{R}$. For such energies eq. (B.2) is a consequence of a Wegner type estimate on the 2-dimensional subspace spanned by $|x>, |y>$. The key is to determine the correct expression for the dependence of $< x|\frac{1}{H_{\Lambda;\omega}-E}|y >$ on $V_\omega(x)$ and $V_\omega(y)$. Such an expression is given by the 'Krein formula':

$$< x| \frac{1}{H_{\Lambda;\omega} - E} |y > = < 1| \left([A]^{-1} + \lambda \begin{bmatrix} V_\omega(x) & 0 \\ 0 & V_\omega(y) \end{bmatrix} \right)^{-1} |2 > , \tag{B.3}$$

where $[A]$ is a 2×2 matrix whose entries do not depend on $V_\omega(x)$ or $V_\omega(y)$. In fact,

$$[A] = \begin{bmatrix} < x|\frac{1}{\widehat{H}_{\Lambda;\omega}-E}|x > & < x|\frac{1}{\widehat{H}_{\Lambda;\omega}-E}|y > \\ < y|\frac{1}{\widehat{H}_{\Lambda;\omega}-E}|x > & < y|\frac{1}{\widehat{H}_{\Lambda;\omega}-E}|y > \end{bmatrix} , \tag{B.4}$$

where $\widehat{H}_{\Lambda;\omega}$ denotes the operator obtained from $H_{\Lambda;\omega}$ by setting $V_\omega(x)$ and $V_\omega(y)$ equal to zero.

The regularity condition $R_1(\tau)$ implies a Wegner type estimate:

$$\text{Prob}\left(\left\| \left([A]^{-1} + \lambda \begin{bmatrix} V_\omega(x) & 0 \\ 0 & V_\omega(y) \end{bmatrix} \right)^{-1} \right\| > t \quad \middle| \{V_\omega(u)\}_{u \neq x,y} \right) \leq \frac{4\kappa_\tau}{(\lambda t)^\tau} , \tag{B.5}$$

where κ_τ is any finite number such that for every $v \in \mathcal{T}$, $a \in \mathbb{R}$, and $\epsilon > 0$

$$\text{Prob}\left(V_\omega(v) \in (a - \epsilon, a + \epsilon)\right) \leq \kappa_\tau \epsilon^\tau . \tag{B.6}$$

The desired bound (B.2) follows easily from eq. (B.5). (The factor, 4, on the right hand side of (B.5) arises as the square of the "volume" of the region $\{x, y\}$. In the case $x = y$, we could replace this factor by 1.)

Although the Krein formula (B.3) is true when E is replaced by any $z \in \mathbb{C}$, the resulting matrix $[A]$ may not be normal if $z \notin \mathbb{R}$. (The resolvent, $\frac{1}{H-z}$, *is* normal. However, given an orthogonal projection, P, the operator $P\frac{1}{H-E}P$ may not be normal!) Yet, the Wegner-like estimate (B.5) holds only when $[A]$ is a normal matrix. At first, this seems to be an obstacle to the extension of (B.2) to all values of z. However, once the inequality is known for real values of z, it follows for all $z \in \mathbb{C}$ from analytic properties of the resolvent. Specifically, the function

$$\phi(z) \ = \ | < x| \frac{1}{H_{\Lambda;\omega} - z} |y > |^s \tag{B.7}$$

is *sub-harmonic* in the upper and lower half planes and decays as $z \to \infty$. Hence, $\phi(z)$ is dominated by the convolution of its boundary values with a Poisson kernel:

$$\phi(E + i\eta) \ \leq \ \int \phi(\widetilde{E}) \frac{|\eta|}{(E - \widetilde{E})^2 + \eta^2} \frac{d\widetilde{E}}{\pi} \ . \tag{B.8}$$

By Fubini's theorem and eq. (B.2) for $\widetilde{E} \in \mathbb{R}$, (B.2) is seen to hold for all $z \in \mathbb{C}$. ■

The "all for one" principle mentioned previously is actually a simple consequence of Lemma B.1.

Lemma B.2 *Let H_ω be a random operator as described in Lemma B.1, and suppose that there is a distance function* dist *on $\mathcal{T}$ such that for some $s < \tau$ and some $z \in \mathbb{C}$*

$$\mathbb{E}\left(| < x| \frac{1}{H_\omega - z} |y > |^s \right) \ \leq \ A(s) \ e^{-\mu(s) \, \mathrm{dist}(x,y)} \ , \tag{B.9}$$

for every $x, y \in \mathcal{T}$. Then, in fact, (B.9) holds, with modified constants $A(r)$ and $\mu(r)$, when s is replaced by any $r < \tau$.

Proof: Note that given $r, s > 0$ with $r < s < \tau$

$$\begin{aligned}
\mathbb{E}\left(| < x| \frac{1}{H_{\Lambda;\omega} - E} |y > |^r \right)^{\frac{s}{r}} \ &\leq \ \mathbb{E}\left(| < x| \frac{1}{H_{\Lambda;\omega} - E} |y > |^s \right) \\
&\leq \ \mathbb{E}\left(| < x| \frac{1}{H_{\Lambda;\omega} - E} |y > |^r \right)^{\frac{t-s}{t-r}} \mathbb{E}\left(| < x| \frac{1}{H_{\Lambda;\omega} - E} |y > |^t \right)^{\frac{s-r}{t-r}} \\
&\leq \ \left(\frac{(4\kappa_\tau)^{t/\tau}}{\lambda^t} \right)^{\frac{s-r}{t-r}} \mathbb{E}\left(| < x| \frac{1}{H_{\Lambda;\omega} - E} |y > |^r \right)^{\frac{t-s}{t-r}} \ ,
\end{aligned} \tag{B.10}$$

where t is any number with $s < t < \tau$. ■

C. Decoupling inequalities

C.a Decoupling inequalities for Green Functions

The condition $R_2(s)$ plays a crucial role in several of the arguments presented in this paper. It has been used to bound expectations of products of Green functions in terms of products of expectations. In this section we demonstrate the validity of the necessary bounds. The main result is the following:

Lemma C.1 *Let H_ω be a random operator given by eq. (B.1), with an s regular distribution of the potential $V_\omega(x)$. Then*

1. *For any $\Omega_1, \Omega_2 \subset \mathcal{T}$, any $x, y \in \Omega_1$, and any $u, v \in \Omega_2$,*

$$\mathbb{E}\left(|G_{\Omega_1}(x,y;z)|^s |G_{\Omega_2}(u,v;z)|^s \right) \;\le\; \frac{\widetilde{C}_s}{\lambda^s}\, \mathbb{E}\left(|G_{\Omega_1}(x,y;z)|^s \right) \;. \tag{C.1}$$

2. *For any $\Omega_1 \cap \Omega_2 = \emptyset$, $x, u \in \Omega_1$, $v, y \in \Omega_2$, and $\Omega_3 \subset \Gamma$,*

$$\begin{aligned} &\mathbb{E}\left(|G_{\Omega_1}(x,u;z)|^s |G_{\Omega_3}(u,v;z)|^s |G_{\Omega_2}(v,y;z)|^s \right) \\ &\qquad \le\; \frac{\widetilde{C}_s}{\lambda^s}\, \mathbb{E}\left(|G_{\Omega_1}(x,u;z)|^s \right) \mathbb{E}\left(|G_{\Omega_2}(v,y;z)|^s \right) \;. \end{aligned} \tag{C.2}$$

Lemma C.1 is a consequence of the conditional expectation bound (B.2), the Krein formula (B.3), and the following:

Lemma C.2 *Let V_1, V_2 be independent real valued random variables which satisfy $R_2(s)$ for some $s > 0$. Then there exists $D_s^{(2)} > 0$ such that*

$$\mathbb{E}\left(|F(V_1,V_2)|^s |F(V_1,V_2)|^s \right) \;\le\; D_s^{(2)}\, \mathbb{E}\left(|F(V_1,V_2)|^s \right)\; \mathbb{E}\left(|G(V_1,V_2)|^s \right) \;, \tag{C.3}$$

where F and G are arbitrary functions of the form

$$F(V_1,V_2) \;=\; \frac{1}{L_1(V_1,V_2)} \tag{C.4}$$

$$G(V_1,V_2) \;=\; \frac{L_2(V_1,V_2)}{L_3(V_1,V_2)} \;, \tag{C.5}$$

with $\{L_i\}$ functions which are linear in each variable separately. In fact, we may take $D_s^{(2)} = D_{s;1} D_{s;2}$, where, for $j = 1, 2$, $D_{s;j}$ is the decoupling constant for V_j.

Proof: Let $f(V)$ and $g(V)$ be two functions of the appropriate form for the decoupling lemma. Then, with $j = 1, 2$

$$\mathbb{E}\left(|f(V_j)|^s |g(V_j)|^s\right) \;\le\; D_{s;1}\, \mathbb{E}\left(|f(\widetilde{V}_j)|^s |g(V_j)|^s\right) , \tag{C.6}$$

where $\widetilde{V}_j$ indicates an independent variable distributed identically to V_j.

Now, if F and G are functions of 2 variables of the given form, then at fixed values of V_2, they satisfy the 1 variable decoupling lemma, so

$$\mathbb{E}\left(|F(V_1,V_2)|^s |G(V_1,V_2)|^s\right) \;\le\; D_{s;1}\, \mathbb{E}\left(|F(\widetilde{V}_1,V_2)|^s |G(V_1,V_2)|^s\right) . \tag{C.7}$$

For fixed values of $\widetilde{V}_1$ and V_1, $F(\widetilde{V}_1, V_2)$ and $G(V_1, V_2)$ (as functions of V_2) are again of the correct form to apply the 1 variable decoupling lemma. Thus,

$$\begin{aligned}\mathbb{E}\left(|F(V_1,V_2)|^s |G(V_1,V_2)|^s\right) \;&\le\; D_{s;1} D_{s;2}\, \mathbb{E}\left(|F(\widetilde{V}_1,\widetilde{V}_2)|^s |G(V_1,V_2)|^s\right) \\ &=\; D_{s;1} D_{s;2}\, \mathbb{E}\left(|F(V_1,V_2)|^s\right)\, \mathbb{E}\left(|G(V_1,V_2)|^s\right) . \end{aligned} \tag{C.8}$$

■

C.b A condition for the validity of $R_2(s)$

Decoupling lemmas have been discussed already in references [11, 13, 8]. Though these contain results similar to those required here, they do not provide the exact condition used in this work. Hence, we briefly present an elementary condition under which $R_2(s)$ is satisfied. The following discussion is by no means exhaustive. Rather, we simply wish to show that the condition $R_2(s)$ is not devoid of meaningful examples.

Lemma C.3 *Let ρ be a measure with bounded support which satisfies $R_1(\tau)$. Then for any $s < \frac{\tau}{4}$, ρ satisfies $R_2(s)$.*

Proof: For each $s > 0$, we define

$$\phi_s(z) \;=\; \int \frac{1}{|V-z|^s} \rho(dV) , \tag{C.9}$$

$$\psi_s(z,w) \;=\; \int \frac{|V-z|^s}{|V-w|^s} \rho(dV) , \tag{C.10}$$

$$\gamma_s(z,w,\zeta), \;=\; \int \frac{|V-z|^s}{|V-w|^s |V-\zeta|^s} \rho(dV) . \tag{C.11}$$

Property $R_2(s)$ amounts to the statement that

$$\sup_{z,w,\zeta\in\mathbb{C}} \frac{\gamma_s(z,w,\zeta)}{\phi_s(\zeta)\psi_s(z,w)} \;<\; \infty\,. \tag{C.12}$$

In fact, if we let

$$F_s(z) \;=\; \frac{\sqrt{\phi_{2s}(z)}}{\phi_s(z)}\,, \tag{C.13}$$

$$G_s(z,w) \;=\; \frac{\sqrt{\psi_{2s}(z,w)}}{\psi_s(z,w)}\,, \tag{C.14}$$

then by the Cauchy-Schwartz inequality, it suffices to show that F_s and G_s are uniformly bounded. However this is elementary since F_s and G_s are continuous functions which are easily shown to have finite limits at infinity. ■

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