ANALYTICITY OF EXTREMALS TO THE AIRY STRICHARTZ INEQUALITY

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Abstract. We prove that there exists an extremal function to the Airy Strichartz inequality

$$\|e^{-t\partial_x^3} f\|_{L^8_{t,x}(\mathbb{R}\times\mathbb{R})} \leq C\|f\|_{L^2(\mathbb{R})},$$

by using the linear profile decomposition. Furthermore we show that, if f is an extremal, then f is exponentially decaying in the Fourier space and so f can be extended to be an entire function on the complex domain.

1. Introduction

It is well known that the (generalized) Korteweg-de Vries equations (KdV or gKdV) are good approximations to describe the evolution of waves on shall water surface [9, 22, 23]:

$$\partial_t u + \partial_x^3 u \pm 3\partial_x(u^p) = 0 \tag{1}$$

for $p \geq 2$. The linear form is the Airy equation

$$\partial_t u + \partial_x^3 u = 0. \tag{2}$$

In general, for an initial data $u(0) = f(x)$ the solution $e^{-t\partial_x^3} f$ to the Airy solution can be expressed as

$$e^{-t\partial_x^3} f(x) := (2\pi)^{-1/2} \int_{\mathbb{R}} e^{ixk+itk^3} \widehat{f}(k) dk. \tag{3}$$

The linear Strichartz inequality for (2) asserts that

$$\|D^\alpha e^{-t\partial_x^3} f\|_{L^q_t L^r_x} \lesssim \|f\|_2, \tag{4}$$

for $-\alpha + \frac{3}{q} + \frac{1}{r} = \frac{1}{2}$ and $-1/2 < \alpha \leq 1/q$, see [14, Theorem 2.1]. When $\alpha = 1/q$, the inequality above is called "endpoints" while "nonendpoints" for $\alpha < 1/q$. It plays an important role in establishing local or global wellposedness theory for the Cauchy problem of (1), see for instance [27]. In this paper, we study the the following symmetry Strichartz inequality

$$\|e^{-t\partial_x^3} f\|_{L^8_{t,x}(\mathbb{R}\times\mathbb{R})} \leq C\|f\|_{L^2(\mathbb{R})}, \tag{5}$$

and consider "extremizers" for (5): the existence of extremals and characterizations of the extremals. Some results on the existence of extremals for general non-endpoints Strichartz inequality is also established, see Remark 3.1.

Date: June 27, 2010.
2000 *Mathematics Subject Classification.* 35Q53; 42A38.

To begin with, we denote the optimal constant for (5) by $\mathcal{A}$:

$$\mathcal{A} := \sup\{\|e^{-t\partial_x^3} f\|_{L^8_{t,x}} : \|f\|_2 = 1\}. \tag{6}$$

Definition 1.1. A function $f \in L^2$ is said to be an extremal for (5) if $f \neq 0$ a. e., and

$$\|e^{-t\partial_x^3} f\|_{L^8_{t,x}(\mathbb{R}\times\mathbb{R})} = \mathcal{A}\|f\|_{L^2(\mathbb{R})}. \tag{7}$$

The first result is the following theorem.

Theorem 1.2. *There exists an extremal function $f \in L^2$ for the Airy Strichartz inequality* (5)*.*

This theorem is proven in Section 3. The proof makes use of the linear profile decomposition for the Airy evolution operator $e^{-t\partial_x^3}$ acting on a bounded sequence of $\{f_n\} \in L^2$, which we develop in Section 2 based on the previous result in [24]. In [25], the profile decomposition appears to be a strong tool to study "extremalizers" problem, where Shao proved the existence of extremals to the Strichartz inequality for the Schrödinger equation in higher dimensions. The profile decomposition can be viewed as a manifestation of the idea of "concentration-compactness" and in our paper it follows in a general framework developed by Lions in his seminal papers [17, 18, 19, 20].

Note that Theorem 1.2 is different from that in [24] where Shao obtained a dichotomy result on the existence of extremals to the Strichartz inequality $\|e^{-t\partial_x^3} D^{1/6} f\|_{L^6_{t,x}} \leq C\|f\|_{L^2}$, which is the symmetric "endpoint" Strichartz inequality; in other words, for this Strichartz inequality, either an extremal exists or a sequence of modulated Gaussians approximates to the extremizer. The method in [24] is based on a refined linear profile decomposition, where the presence of highly oscillatory terms in the profile decomposition gives rise to the dichotomy.

Extremals to Strichartz inequality for the Schrödinger equations and the wave equations have been studied intensively. For the Strichartz inequality for the Schrödinger equation, Kunze [16] proved the existence of extremal to the one dimensional Strichartz inequality by establishing that any nonnegative extremizing sequence converges strongly an extremal in L^2. In the lower dimensional case, the existence of extremals was shown independently by Foschi [11], and Hundertmark, Zharnitsky [13]; they actually proved more: the extemizers are actually Gaussians, which are unique up to the natural symmetries of the inequality. Later works along the line in [13] devoted to the Schrödinger Strichartz with different emphases include [3] and [6]. We remark that all the previous known methods seem not to be adapted directly to finding the explicit forms of "extremals" to (5) to the best of our knowledge. For extremals to the Strichartz inequality for the wave equation, see [11, 4].

Very recently, Christ, Shao [7, 8] studied "extremals" to an adjoint Fourier restriction inequality for the sphere, namely the Tomas-Stein inequality $L^2(S^2) \to L^4_x(\mathbb{R}^3)$ for two dimensional sphere S^2. Although the Strichartz inequality can be viewed as an adjoint Fourier restriction inequality for the paraboloids, the situation for the sphere is different from the paraboloid case due to the nonlocal property and the lack of scaling symmetry of the former operator. However, among other things, they were able to show that there exists an extremal to it by proving that any extremising sequence of nonnegative functions in $L^2(S^2)$ has a strongly convergent subsequence; The main ingredient is to rule out the obstacle that, for the extremising sequence $\{f_\nu\}$, f_ν^2 may converge to a pair of Dirac masses on the sphere.

We turn to the chararacterization of the extremals to (5) from studying the corresponding generalized Euler-Lagrange equation:

$$\mathcal{A}^8\|f\|_2^6 f(x) = \int e^{t\partial_x^3}\big[|e^{-t\partial_x^3}f|^6 e^{-t\partial_x^3}f\big]dt, \text{ a.e. } x\in\mathbb{R}, \tag{8}$$

where $\mathcal{A}$ is the optimal constant defined in (6). The Euler-Lagrange equation (8) can be established by a standard variational argument.

Theorem 1.3. *There exists* $\mu_0 > 0$ *such that for any extremal* f *to the Airy Strichartz inequality* (5),

$$e^{\mu_0|k|^3}\widehat{f}(k)\in L^2; \tag{9}$$

moreover, f *can be extended to be an entire function on the complex plane.*

The proof of this theorem is based on a crucial bootstrap argument, which is refined bilinear Strichartz inequality for Airy evolution operator $e^{-t\partial_x^3}f$, and a weighted Strichartz inequality (5). The argument is similar to Hundertmark, Lee [12] and Erdogan, Hundertmark, and Lee [10]. In [12], Hundertmark, Lee showed that solutions to the dispersion managed non-linear Schrödinger equation in the case of zero residual dispersion are fast decaying not only in the Fourier space but also in the spatial space. Note that the result is stronger than Theorem 1.3. It is essentially due to that, the linear Schrödinger operator $e^{it\Delta}$ enjoys an identity

$$e^{it\Delta}f(x) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{ix\xi+it|\xi|^2}\hat{f}(\xi)d\xi = Ct^{-d/2}\int_{\mathbb{R}^d} e^{\frac{|x-y|^2}{4it}}f(y)dy, \text{ for some } C>0, \tag{10}$$

which enables one to obtain the decay in the spatial space from that on the Fourier side.

The organization of the paper is as follows. In Section 2, we establish the linear profile decomposition. In Section 3, we show the existence of extremals the Airy Strichartz inequality $L^2\to L^8_{t,x}$. In Section 4, we show that any solution to the generalized Euler-Lagrange equation, which include the extremizer as a special case, can be extended to be analytic on the complex plane. It is proven by assuming an important bootstrap lemma, which we establish in Section 5.

2. The linear profile decomposition

Recall from the introduction, we will use the linear profile decomposition for the Airy evolution operator $e^{-t\partial_x^3}$ for L^2 initial data to prove the existence of extremals for (5). Roughly speaking, the linear profile decomposition is to investigate the general structure of solutions $\{e^{-t\partial_x^3}f_n\}$ for bounded $\{f_n\}\in L^2$, and aims to compensate for the loss of compactness of solution operator caused by the symmetries of the equation; for such sequence $\{e^{-t\partial_x^3}f_n\}$, it is expected to be written as a superposition of concentrating waves, "profiles" plus an negligible reminder term; the interaction of the profiles is small, see the precise statements in Theorem

2.3, Theorem 2.4. The profile decomposition for the nonlinear wave and Schrödinger equations, and the gKdV equations have been developed in [1, 2, 5, 15, 21, 24]. To prepare for the linear profile decomposition theorem for the Airy evolution operator in the Strichartz norm $\|u\|_{L^8_{t,x}}$ needed in this paper, we recall two definitions from [24].

Definition 2.1. For any phase $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, position $x_0 \in \mathbb{R}$ and scaling parameter $h_0 > 0$, we define the unitary transform $g_{\theta,x_0,h_0} : L^2 \to L^2$ by the formula

$$[g_{\theta,x_0,h_0} f](x) := \frac{1}{h_0^{1/2}} e^{i\theta} f(\frac{x-x_0}{h_0}).$$

We let G be the collection of such transformations. It is easy to see that G is a group which preserves the L^2 norm.

Definition 2.2. For $j \neq k$, two sequences $\Gamma^j_n := (h^j_n, \xi^j_n, x^j_n, t^j_n)_{n\geq 1}$ and $\Gamma^k_n := (h^k_n, \xi^k_n, x^k_n, t^k_n)_{n\geq 1}$ in $(0,\infty) \times \mathbb{R}^3$ are orthogonal if there holds,

$$\text{either } \limsup_{n\to\infty} \left(\frac{h^j_n}{h^k_n} + \frac{h^k_n}{h^j_n} + h^j_n |\xi^j_n - \xi^k_n| \right) = \infty, \tag{11}$$

$$\text{or } (h^j_n, \xi^j_n) = (h^k_n, \xi^k_n) \text{ and} \tag{12}$$

$$\limsup_{n\to\infty} \left(\frac{|t^k_n - t^j_n|}{(h^j_n)^3} + \frac{3|(t^k_n - t^j_n)\xi^j_n|}{(h^j_n)^2} + \frac{|x^j_n - x^k_n + 3(t^j_n - t^k_n)(\xi^j_n)^2|}{h^j_n} \right) = \infty.$$

The profile decomposition for $e^{-t\partial_x^3} f$ in the Strichartz norm $\|\cdot\|_{L^8_{t,x}}$ will follow from that in the Strichartz norm $\|D^{1/6} \cdot \|_{L^6_{t,x}}$ via the Sobolev embedding. Here D^α with $\alpha \in \mathbb{R}$ denotes the fractional derivative operator defined in terms of the Fourier multiplier, $\widehat{D^\alpha f} = |\xi|^\alpha \hat{f}$. We state the following linear profile decomposition in the Strichartz norm $\|D^{1/6} \cdot \|_{L^6_{t,x}}$ from [24].

Theorem 2.3. *Let $(u_n)_{n\geq 1}$ be a sequence of complex-valued functions satisfying $\|u_n\|_2 \leq 1$. Then up to a subsequence, there exists a sequence of L^2 functions $(\phi^j)_{j\geq 1} : \mathbb{R} \to \mathbb{C}$ and a family of pairwise orthogonal sequences $\Gamma^j_n = (h^j_n, \xi^j_n, x^j_n, t^j_n) \in (0,\infty) \times \mathbb{R}^3$ such that, for any $l \geq 1$, there exists an L^2 function $w^l_n : \mathbb{R} \to \mathbb{C}$ satisfying*

$$u_n = \sum_{\substack{1\leq j\leq l, \xi^j_n \equiv 0 \\ or\ |h^j_n \xi^j_n| \to \infty}} e^{t^j_n \partial_x^3} g^j_n [e^{i(\cdot) h^j_n \xi^j_n} \phi^j] + w^l_n, \tag{13}$$

where $g^j_n := g_{0,x^j_n,h^j_n} \in G$ and

$$\limsup_{l\to\infty} \limsup_{n\to\infty} \|D^{1/6} e^{-t\partial_x^3} w^l_n\|_{L^6_{t,x}} = 0. \tag{14}$$

Moreover, for every $l \geq 1$,

$$\limsup_{n\to\infty} \left(\|u_n\|_2^2 - \left(\sum_{j=1}^{l} \|\phi^j\|_2^2 + \|w^l_n\|_2^2 \right) \right) = 0. \tag{15}$$

As a consequence of this theorem, we can develop a linear profile decomposition in the Airy-Strichartz norm $\|\cdot\|_{L^8_{t,x}}$, where the highly oscillatory terms $e^{ixh_n^j\xi_n^j}\phi^j(x)$ with $|h_n^j\xi_n^j| \to \infty$ disappear.

Theorem 2.4. *Let $(u_n)_{n\geq 1}$ be a sequence of complex-valued functions satisfying $\|u_n\|_2 \leq 1$. Then up to a subsequence, there exists a sequence of L^2 functions $(\phi^j)_{j\geq 1} : \mathbb{R} \to \mathbb{C}$ and a family of parameters $\Gamma_n^j = (h_n^j, x_n^j, t_n^j) \in (0,\infty)\times\mathbb{R}^2$ such that, for any $l \geq 1$, there exists an L^2 function $w_n^l : \mathbb{R} \to \mathbb{C}$ satisfying*

$$u_n = \sum_{1\leq j\leq l} e^{t_n^j\partial_x^3} g_n^j(\phi^j) + w_n^l,$$

where $g_n^j := g_{0,x_n^j,h_n^j} \in G$ and

$$\limsup_{l\to\infty}\limsup_{n\to\infty} \|e^{-t\partial_x^3} w_n^l\|_{L^8_{t,x}} = 0,$$

and for $j \neq k$,

$$\limsup_{n\to\infty}\left(\frac{h_n^j}{h_n^k} + \frac{h_n^k}{h_n^j} + \frac{|t_n^j - t_n^k|}{(h_n^j)^3} + \frac{|x_n^j - x_n^k|}{h_n^j}\right) = \infty. \tag{16}$$

Moreover, we have two orthogonality results: for every $l \geq 1$,

$$\limsup_{n\to\infty}\left(\|u_n\|_2^2 - \left(\sum_{j=1}^{l}\|\phi^j\|_2^2 + \|w_n^l\|_2^2\right)\right) = 0.$$

$$\limsup_{n\to\infty}\left(\|\sum_{1\leq j\leq l} e^{-(t-t_n^j)\partial_x^3} g_n^j(\phi^j)\|_{L^8_{t,x}}^8 - \sum_{1\leq j\leq l}\|e^{-t\partial_x^3}\phi^j\|_{L^8_{t,x}}^8\right) = 0.$$

Proof. This argument consists of three steps. We first see that the error term w_n^l still converges to zero in this new Strichartz norm $\|\cdot\|_{L^8_{t,x}}$. Indeed, by the Sobolev embedding,

$$\|e^{-t\partial_x^3}u_0\|_{L^8_{t,x}} \leq C\|D^{1/6}e^{-t\partial_x^3}u_0\|_{L^6_{t,x}};$$

so an application of (14) yields that

$$\limsup_{l\to\infty}\limsup_{n\to\infty}\|e^{-t\partial_x^3}w_n^l\|_{L^8_{t,x}} = 0.$$

Secondly we claim that, for $1 \leq j \leq l$, when $\lim_{n\to\infty} h_n^j\xi_n^j = \infty$,

$$\lim_{n\to\infty}\|e^{-(t-t_n^j)\partial_x^3} g_n^j[e^{i(\cdot)h_n^j\xi_n^j}\phi^j]\|_{L^8_{t,x}} = 0. \tag{17}$$

It shows that the highly oscillatory terms can be reorganized into the error term. To show (17), by using the symmetries, we reduce to prove

$$\lim_{N\to\infty}\|e^{-t\partial_x^3}[e^{i(\cdot)N}\phi]\|_{L^8_{t,x}} = 0. \tag{18}$$

We may assume $\phi \in \mathcal{S}$, the set of Schwartz functions, and that ϕ has the compact Fourier support $(-1,1)$.

$$e^{-t\partial_x^3}[e^{i(\cdot)N}\phi](x) = e^{ixN+itN^3}\int e^{i(x+3tN^2)\xi+i3Nt\xi^2+it\xi^3}\widehat{\phi}(\xi)d\xi.$$

Setting $x' := x + 3tN^2$ and $t' := 3Nt$, we have,

$$\lim_{N\to\infty} \|e^{-t\partial_x^3}[e^{i(\cdot)N}\phi]\|_{L^8_{t,x}} = cN^{-1/8}\| \int e^{ix'\xi + it'\xi^2 + i\frac{t'}{3N}\xi^3}\widehat{\phi} d\xi\|_{L^8_{t',x'}}$$

for some $c > 0$. Then the dominated convergence theorem yields

$$\lim_{N\to\infty} \| \int e^{ix'\xi + it'\xi^2 + i\frac{t'}{3N}\xi^3}\widehat{\phi} d\xi\|_{L^8_{t',x'}} = \|e^{-it\partial_x^2}\phi^j\|_{L^8_{t,x}}.$$

Here $e^{-it\partial_x^2}$ denotes the Schrödinger evolution operator defined via

$$e^{-it\partial_x^2} f(x) := \int e^{ix\xi + it|\xi|^2}\widehat{f}(\xi)d\xi.$$

Indeed,

$$\int e^{ix'\xi + it'\xi^2 + i\frac{\xi^3}{3N}}\widehat{\phi}(\xi)d\xi \to e^{-it'\partial_x^2}\phi^j(x'),\ a.e.,$$

and by using [26, Corollary, p.334] or integration by parts,

$$\left| \int e^{ix'\xi + it'\xi^2 + i\frac{t'\xi^3}{3N}}\widehat{\phi}(\xi)d\xi \right| \leq C_{\phi^j} B(t', x')$$

for n large enough but still uniform in n. Here

$$B(t', x') = \begin{cases} (1 + |t'|)^{-1/2} \leq C\big[(1 + |x'|)(1 + |t'|)\big]^{-1/4}, & \text{for } |x'| \leq 6|t'|, \\ (1 + |x'|)^{-1} \leq C\big[(1 + |x'|)(1 + |t'|)\big]^{-1/2}, & \text{for } |x'| > 6|t'|. \end{cases}$$

It is easy to observe that $B \in L^8_{t',x'}$. Then (18) follows immediately.

Finally we claim that, for $j \neq k$,

$$\lim_{n\to\infty} \|e^{-(t-t_n^j)\partial_x^3} g_n^j(\phi^j) e^{-(t-t_n^k)\partial_x^3} g_n^k(\phi^k)\|_{L^4_{t,x}} = 0.$$

This is a consequence of the orthogonality condition (16), whose proof is a special case of Lemma 2.6 below. The remaining conclusions in Theorem 2.4 follow from Theorem 2.3 accordingly. □

Remark 2.5. A linear profile decomposition for all non-endpoint Airy Strichartz inequalities can be established by using the first two observations in the previous lemma and Lemma 2.6. The statement is similar to Theorem 2.4 and so we omit the details.

Lemma 2.6. *When* $-\alpha + \frac{3}{q} + \frac{1}{r} = \frac{1}{2}, -1/2 < \alpha < \frac{1}{2}$. *Then for* $j \neq k$,

$$\lim_{n\to\infty} \|e^{-(t-t_n^j)\partial_x^3} D^\alpha g_n^j(\phi^j) e^{-(t-t_n^k)\partial_x^3} D^\alpha g_n^k(\phi^k)\|_{L_t^{q/2} L_x^{r/2}} = 0 \tag{19}$$

provided that $\{(h_n^j, x_n^j, t_n^j)\}$ *and* $\{(h_n^k, x_n^k, t_n^k)\}$ *satisfies the orthogonality condition in* (16).

Proof. We will prove (19) by studying (16) case by case.

Case I. Assume $\limsup_{n\to\infty} \frac{h_n^j}{h_n^k} + \frac{h_n^k}{h_n^j} = \infty$. For any $R > 0$, we define

$$\Omega_n^j(R) := \{(t,x) : \frac{|x - x_n^j|}{h_n^j} + \frac{|t - t_n^j|}{(h_n^j)^3} \le R\},$$
$$\Omega_n^k(R) := \{(t,x) : \frac{|x - x_n^k|}{h_n^k} + \frac{|t - t_n^k|}{(h_n^k)^3} \le R\},$$
$$\left(\Omega_n^j\right)^c := \mathbb{R}^2 \setminus \Omega_n^j(R),\ \left(\Omega_n^k\right)^c := \mathbb{R}^2 \setminus \Omega_n^k(R).$$

By using Hölder's inequality and the Strichartz inequality followed by a change of variables, we have

$$\begin{aligned}
&\|e^{-(t-t_n^j)\partial_x^3} D^\alpha g_n^j(\phi^j) e^{-(t-t_n^k)\partial_x^3} D^\alpha g_n^k(\phi^k)\|_{L_t^{q/2}L_x^{r/2}\left((\Omega_n^j)^c\right)} \\
&\le C\|e^{-(t-t_n^j)\partial_x^3} D^\alpha g_n^j(\phi^j)\|_{L_t^q L_x^r\left((\Omega_n^j)^c\right)} \|e^{-(t-t_n^k)\partial_x^3} D^\alpha g_n^k(\phi^k)\|_{L_t^q L_x^r} \\
&\le C(h_n^j)^{-1/2-\alpha}\|e^{-\frac{t-t_n^j}{(h_n^j)^3}\partial_x^3}(D^\alpha\phi^j)(\frac{x-x_n^j}{h_n^j})\|_{L_t^q L_x^r\left((\Omega_n^j)^c\right)}\|\phi^k\|_2 \\
&\le C\|\phi^k\|_2\|e^{-t\partial_x^3}D^\alpha(\phi^j)\|_{L_t^q L_x^r(\{|x|+|t|\ge R\})}.
\end{aligned}$$

The latter integral converges to zero when R goes to infinity from the dominated convergence theorem. So we can choose a sufficiently large $R > 0$ such that

$$\|e^{-(t-t_n^j)\partial_x^3} D^\alpha g_n^j(\phi^j) e^{-(t-t_n^k)\partial_x^3} D^\alpha g_n^k(\phi^k)\|_{L_t^{q/2}L_x^{r/2}\left((\Omega_n^j)^c\right)}$$

as small as we want. Likewise for $\|e^{-(t-t_n^j)\partial_x^3} D^\alpha g_n^j(\phi^j) e^{-(t-t_n^k)\partial_x^3} D^\alpha g_n^k(\phi^k)\|_{L_t^{q/2}L_x^{r/2}\left((\Omega_n^k)^c\right)}$. So fixing a large R, we may restrict our attention onto $\Omega_n^j \cap \Omega_n^k$. We aim to show that the integral on $\Omega_n^j \cap \Omega_n^k$ converges to zero when n goes to infinity. Indeed, by using trivial $L_{t,x}^\infty$ bounds on $e^{-(t-t_n^j)\partial_x^3} D^\alpha g_n^j(\phi^j)$ and $e^{-(t-t_n^k)\partial_x^3} D^\alpha g_n^k(\phi^k)$, we see that

$$\begin{aligned}
&\|e^{-(t-t_n^j)\partial_x^3} D^\alpha g_n^j(\phi^j) e^{-(t-t_n^k)\partial_x^3} D^\alpha g_n^k(\phi^k)\|_{L_t^{q/2}L_x^{r/2}(\Omega_n^j\cap\Omega_n^k)} \\
&\qquad \le C(h_n^j h_n^k)^{-1/2-\alpha}\min\{(h_n^j)^{6/q+2/r},(h_n^k)^{6/q+2/r}\} \\
&\qquad \le C\min\{(\frac{h_n^j}{h_n^k})^{1/2+\alpha},(\frac{h_n^k}{h_n^j})^{1/2+\alpha}\} \to 0
\end{aligned}$$

as n goes to infinity. Note that $C > 0$ depending on R, $\|\widehat{\phi^j}\|_{L^1}$, and $\|\widehat{\phi^k}\|_{L^1}$. Thus (19) is obtained, which completes the proof of **Case I.**

Case II. Now we may assume that $h_n^j = h_n^k$ for all n, we are left with the case where

$$\limsup_{n\to\infty} \frac{|x_n^j - x_n^k|}{h_n^j} + \frac{|t_n^j - t_n^k|}{(h_n^j)^3} = \infty.$$

We change variables $x' = \frac{x - x_n^k}{h_n^k}$ and $t' = \frac{t - t_n^k}{(h_n^k)^3}$ and see that we need to show that

$$\|e^{-(t'+\frac{t_n^k - t_n^j}{(h_n^j)^3})}(D^\alpha\phi^j)(x' + \frac{x_n^k - x_n^j}{h_n^j})e^{-t'\partial_x^3}(D^\alpha\phi^k)(x')\|_{L_{t'}^{q/2}L_{x'}^{r/2}} \to 0$$

as $n \to \infty$. We define

$$\Omega^k(R) := \{(t,x) : |t'| + |x'| \le R\},$$
$$\Omega_n^j(R) := \{(t,x) : \left| x' + \frac{x_n^k - x_n^j}{h_n^j} \right| + \left| t' + \frac{t_n^j - t_n^k}{(h_n^j)^3} \right| \le R\}.$$

As proving **Case I**, we may reduce to the domain $\Omega^k \cap \Omega_n^j$. While for this case, we observe that, for any fixed large $R > 0$,

$$|\Omega^k \cap \Omega_n^j| \to 0, \text{ as } n \to \infty.$$

This, together with the $L^\infty_{t,x}$ bounds, proves **Case II**. Therefore the proof of Lemma 2.6 is complete. □

Remark 2.7. With this lemma 2.6, we have the following orthogonality result: for (α, q, r) defined as in Lemma 2.6 and $l \ge 1$,

$$\limsup_{n\to\infty} \|D^\alpha \sum_{j=1}^{l} e^{-(t-t_n^j)\partial_x^3} g_n^j \phi^j\|_{L_t^q L_x^r}^q \le \sum_{j=1}^{l} \limsup_{n\to\infty} \|D^\alpha e^{-(t-t_n^j)\partial_x^3} g_n^j \phi^j\|_{L_t^q L_x^r}^q$$

for $q \le r$; while for $r \le q$,

$$\limsup_{n\to\infty} \|D^\alpha \sum_{j=1}^{l} e^{-(t-t_n^j)\partial_x^3} g_n^j \phi^j\|_{L_t^q L_x^r}^r \le \sum_{j=1}^{l} \limsup_{n\to\infty} \|D^\alpha e^{-(t-t_n^j)\partial_x^3} g_n^j \phi^j\|_{L_t^q L_x^r}^r.$$

See [25] for a similar proof.

3. Existence of extremals

In this section we apply the linear profile decomposition Theorem 2.4 to prove the existence of extremals for (5).

Proof. Choose an extremising sequence $(f_n)_{n\ge1}$ such that

$$\|f_n\|_2 = 1, \lim_{n\to\infty} \|e^{-t\partial_x^3} f_n\|_{L^8_{t,x}} = \mathcal{A}.$$

By applying the linear profile decomposition in Theorem 2.4, up to a subsequence,

$$\begin{aligned}\mathcal{A}^8 &= \lim_{n\to\infty} \|e^{-t\partial_x^3} f_n\|_{L^8_{t,x}}^8 = \lim_{l\to\infty}\lim_{n\to\infty} \|\sum_{j=1}^{l} e^{-(t-t_n^j)\partial_x^3} g_n^j(\phi^j)\|_{L^8_{t,x}}^8 \\ &= \sum_{j=1}^{\infty} \|e^{-t\partial_x^3}\phi^j\|_{L^8_{t,x}}^8 \le \mathcal{A}^8 \sum_{j=1}^{\infty} \|\phi^j\|_2^{2\times4} \le \mathcal{A}^8 \left(\sum_{j=1}^{\infty} \|\phi^j\|_2^2\right)^4 \\ &\le \mathcal{A}^8 \|f_n\|_2^8 = \mathcal{A}^8.\end{aligned}$$

Thus the equal signs at the beginning and at the end force all the signs in this chain to be equal. Hence, we have

$$1 = \big(\sum_{j=1}^{\infty} \|\phi^j\|_2^{2\times4}\big)^{1/4} = \sum_{j=1}^{\infty} \|\phi^j\|_2^2, \quad \sup_j \|\phi^j\|_2 \le 1,$$

which yields that there is exactly one j remaining. Without loss of generality, we may assume that

$$\phi^j = 0, \text{ for } j \geq 2.$$

Thus ϕ^1 is an extremiser as desired. □

Remark 3.1. Combining this argument with the orthogonality in Remark 2.7, the existence of extremals for any non-endpoint Strichartz inequality can be obtained similarly. We omit the details here.

4. Analyticity for extremals

In this section, we establish that any extremal f to (5) enjoys an exponential decay in the Fourier space, Theorem 1.3, from which the property of analyticity of extremals follows easily. We begin with a crucial bilinear Airy Strichartz estimate.

Lemma 4.1 (Bilinear Airy estimates). *Suppose* $\mathrm{Supp}\widehat{f_1} \subset \{\xi : N_1 \leq |\xi| \leq 2N_1\}$ *and* $\mathrm{Supp}\widehat{f_2} \subset \{\xi : N_2 \leq |\xi| \leq 2N_2\}$, *and* $N_1 \ll N_2$. *Then*

$$\|e^{-t\partial_x^3} f_1 e^{-t\partial_x^3} f_2\|_{L^4_{t,x}} \leq C\Big(\frac{N_1}{N_2}\Big)^{1/4} \|f_1\|_2 \|f_2\|_2.$$

where the constant $C > 0$ *is independent of* N_1 *and* N_2.

Proof. We observe that

$$\|e^{-t\partial_x^3} f_1 e^{-t\partial_x^3} f_2\|_{L^4_{t,x}} = \Big\| \int e^{ix(\xi_1+\xi_2)+it(\xi_1^3+\xi_2^3)} \widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2) d\xi_1 d\xi_2 \Big\|_{L^4_{t,x}}. \tag{20}$$

We restrict the region to $\{(\xi_1, \xi_2) : \xi_1, \xi_2 \geq 0\}$ and change variables $a := \xi_1 + \xi_2$ and $b := \xi_1^3 + \xi_2^3$; then we see that the Jacobian $J \sim N_2^2$ since $N_1 \ll N_2$. We apply the Hausdorff-Young inequality and changes of variables to see that (20) is bounded by

$$\begin{aligned} &\lesssim \Big(\iint J^{-1/3} |\widehat{f_1}\widehat{f_2}|^{4/3} d\xi_1 d\xi_2 \Big)^{3/4} \\ &\lesssim |J|^{-1/4} \|f_1\|_2 N_1^{1/4} \|f_2\|_2 N_2^{1/4} \\ &\lesssim \Big(\frac{N_1}{N_2}\Big)^{1/4} \|f_1\|_2 \|f_2\|_2. \end{aligned}$$

□

By using dyadic decompositions, we have

Corollary 4.2. *If* $\mathrm{Supp}\widehat{f_1} \subset \{|\xi_1| \leq s\}$ *and* $\mathrm{Supp}\widehat{f_2} \subset \{|\xi_2| \geq Ls\}$ *for some* $s > 1$ *and* $L \gg 1$, *then*

$$\|e^{-t\partial_x^3} f_1 e^{-t\partial_x^3} f_2\|_{L^4_{t,x}} \leq CL^{-1/4} \|f_1\|_2 \|f_2\|_2. \tag{21}$$

where the constant $C > 0$ *is independent of* L.

We define an 8-linear form,

$$Q(f_1, \cdots, f_8) := \iint e^{-t\partial_x^3} f_1 \overline{e^{-t\partial_x^3} f_2} \times \cdots \times e^{-t\partial_x^3} f_7 \overline{e^{-t\partial_x^3} f_8} dt dx. \tag{22}$$

where $f_i \in L^2$, $1 \le i \le 8$. By the Airy Strichartz inequality (5),

$$\|Q\|_{L^\infty} \lesssim \Pi_{1\le i\le 8} \|f_i\|_2^8. \tag{23}$$

Inspired by the Euler-Lagrange equation (8), we define the notion of weak solutions.

Definition 4.3. $f \in L^2$ is said to be a weak solution to the Euler-Lagrange equation (8) if it satisfies the following integral equation

$$\omega\langle g, f\rangle = Q(g, f, \cdots, f), \quad \forall\, g \in L^2. \tag{24}$$

for some $\omega > 0$. Here $\langle \cdot, \cdot\rangle$ is the inner product in L^2 defined by $\langle f, g\rangle = \int_{\mathbb{R}} f\overline{g} dx$.

Remark 4.4. In view of the Euler-Lagrange equation (8), we see that, any extremal f to the Airy Strichartz inequality (5) is actually a weak solution, as f satisfies

$$\omega\langle g, f\rangle = Q(g, f, \cdots, f), \text{ with } \omega = \mathcal{A}^8 \|f\|_2^6. \tag{25}$$

Now we list some additional notations and observations that are used in the following sections: Set

$$a(\eta) := \eta_1^3 - \eta_2^3 + \eta_3^3 - \eta_4^3 + \eta_5^3 - \eta_6^3 + \eta_7^3 - \eta_8^3, \tag{26}$$

$$b(\eta) := \eta_1 - \eta_2 + \eta_3 - \eta_4 + \eta_5 - \eta_6 + \eta_7 + \eta_8, \tag{27}$$

$$M(h_1, \cdots, h_8) := \int_{\mathbb{R}^8} \delta\big(a(\eta)\big)\delta\big(b(\eta)\big)\Pi_{j=1}^8 |h_j(\eta_j)| d\eta, \tag{28}$$

where δ denotes the Dirac mass. Then we rewrite Q as

$$Q(f_1, \cdots, f_8) = (2\pi)^{-3} \int_{\mathbb{R}^8} \delta\big(a(\eta)\big)\delta\big(b(\eta)\big)\widehat{f}(\eta_1)\overline{\widehat{f}}(\eta_2)\widehat{f}(\eta_3)\overline{\widehat{f}}(\eta_4)\widehat{f}(\eta_5)\overline{\widehat{f}}(\eta_6)\widehat{f}(\eta_7)\overline{\widehat{f}}(\eta_8) d\eta. \tag{29}$$

Then it is not hard to see that

$$Q(f_1, \cdots, f_8) \le (2\pi)^{-3} M(|\widehat{f_1}|, \cdots, |\widehat{f_8}|), \tag{30}$$

$$M(h_1, \cdots, h_8) = (2\pi)^3 Q(|h_1|^\vee, \cdots, |h_8|^\vee), \tag{31}$$

where $f^\vee(x) := (2\pi)^{-1/2} \int_{\mathbb{R}} e^{ix\xi} \widehat{f}(\xi) d\xi$.

Now we define a weighted version of M, for any function $F : \mathbb{R} \to \mathbb{R}$,

$$M_F(h_1, \cdots, h_8) := \int_{\mathbb{R}^8} \delta\big(a(\eta)\big)\delta\big(b(\eta)\big) e^{F(\eta_1) - \sum_{l=2}^8 F(\eta_l)} \Pi_{j=1}^8 |h_j(\eta_j)| d\eta. \tag{32}$$

Then

$$M(e^F h_1, e^{-F} h_2, e^{-F} h_3, e^{-F} h_4, e^{-F} h_5, e^{-F} h_6, e^{-F} h_7, e^{-F} h_8) = M_F(h_1, \cdots, h_8). \tag{33}$$

We define, for $\mu > 0$, $\varepsilon > 0$,

$$F = F_{\mu,\varepsilon}(k) := \frac{\mu |k|^3}{1 + \varepsilon |k|^3}. \tag{34}$$

Proposition 4.5. *For F defined as above, we have*

$$M_F(h_1, \cdots, h_8) \le M(h_1, \cdots, h_8). \tag{35}$$

Proof. We see that the claim (35) reduces to proving

$$F(\eta_1) \le \sum_{l=2}^{8} F(\eta_l), \text{ when } a(\eta) = b(\eta) = 0.$$

Since $a(\eta) = 0$ implies $\eta_1^3 = \sum_{l=2}^8 (-1)^l \eta_l^3$,

$$\begin{aligned} F(\eta_1) = \mu \frac{|\eta_1|^3}{1+\varepsilon|\eta_1|^3} &= \mu \frac{|\sum_{l=2}^8 (-1)^l \eta_l^3|}{1+\varepsilon|\sum_{l=2}^8 (-1)^l \eta_l^3|} \le \mu \frac{\sum_{l=2}^8 |\eta_l^3|}{1+\varepsilon \sum_{l=2}^8 |\eta_l^3|} \\ &= \sum_{l=2}^{8} \frac{\mu \sum_{l=2}^8 |\eta_l|^3}{1+\varepsilon\sum_{l=2}^8 |\eta_l|^3} = \sum_{l=2}^{8} F(\eta_l), \end{aligned} \tag{36}$$

where we have used the fact that $t \mapsto \frac{t}{1+\varepsilon t}$ is increasing on $[0, \infty)$. ∎

Then we can easily deduce

Corollary 4.6. *For F defined as above, then*

$$M_F(h_1, \cdots, h_8) \lesssim \Pi_{j=1}^8 \|h_j\|_2, \tag{37}$$

and

$$M_F(h_1, \cdots, h_8) \lesssim L^{-1/4} \Pi_{j=1}^8 \|h_j\|_2 \tag{38}$$

provided that there exists at least one h_j supported on $[-s, s]$ and another h_k supported on $[-Ls, Ls]^c$ where $L \gg 1$ and $s \ge 1$.

The following proposition is the key to the proof of Theorem 1.3. Let $F_{\mu,\varepsilon}$ be defined as above for some $\varepsilon > 0, \mu > 0$. Let $s > 1$, we set

$$\widehat{f}_> := \widehat{f} 1_{[-s^2, s^2]^c}, \text{ and } \|\widehat{f}\|_{\mu,s,\varepsilon} := \|e^{F_{\mu,\varepsilon}} \widehat{f}_>\|_2, \tag{39}$$

where 1_Ω denotes a suitable smooth bump function adapted to the set Ω.

Proposition 4.7. *If f is a weak solution to the Euler-Lagrange equation* (8) *as defined in* (24) *with $\|f\|_2 = 1$. Then for $\mu = s^{-6}$ with $s \gg 1$, there exists a constant $C > 0$ such that*

$$\omega \|\widehat{f}\|_{s^{-6},s,\varepsilon} \le o_1(1) \|\widehat{f}\|_{s^{-6},s,\varepsilon} + C \sum_{l=2}^{7} \|\widehat{f}\|^l_{s^{-6},s,\varepsilon,} + o_2(1), \tag{40}$$

where $o_i(1) \to 0$ uniformly in $\varepsilon > 0$ as $s \to \infty$, $i = 1, 2$; the constant $C > 0$ is independent of ε and s.

Let us postpone the proof of this proposition to the next section and finish the proof of Theorem 1.3.

Proof of Theorem 1.3. We set

$$G(v) := \frac{\omega}{2} v - C\sum_{l=2}^{7} v^l, \text{ for } v \geq 0.$$

Invoking (40), if choosing s large enough such that $o_1(1) \leq \omega/2$, we obtain

$$G\big(\|\widehat{f}\|_{s^{-6},\,s,\,\varepsilon}\big) \leq o_2(1). \tag{41}$$

We observe that the graph of G is concave in $[0,\infty)$ and intersects at x-axis only at two points: $v=0$ and $v=x_0$ for some $x_0>0$. Let $v_0, v_1 > 0$ such that $G(v_0)=G(v_1)=G_{\max}/2$, where $G_{\max} = \max\{G(v):\ v\geq 0\}$. Again we take s to be large enough such that $o_2(1) \leq G_{\max}/2$. Then we have a dichotomy,

$$\text{either } \|\widehat{f}\|_{s^{-6},s,1} \leq v_0, \text{ or } \|\widehat{f}\|_{s^{-6},s,1} \geq v_1. \tag{42}$$

However the second choice is impossible if s is chosen to be large, because by definition

$$F_{s^{-6},1}(k) = \frac{s^{-6}|k|^3}{1+|k|^3} \leq s^{-6} \leq 1,$$

which yields

$$\|\widehat{f}\|_{s^{-6},s,1} = \|e^{F_{s^{-6},1}}\widehat{f}_>\|_2 \leq C\|\widehat{f}1_{[-s^2,s^2]^c}\|_2 \to 0, \text{ as } s\to\infty.$$

Now we fix a large $s>0$ and consider the function $\varepsilon \mapsto \|\widehat{f}\|_{s^{-6},s,\varepsilon}$, which is continuous by the dominated convergence theorem for each $\varepsilon > 0$. Again by (40),

$$G\big(\|\widehat{f}\|_{s^{-6},s,\varepsilon}\big) \leq G_{\max}/2 \tag{43}$$

for all $\varepsilon>0$. Hence by continuity, we must have that $\|\widehat{f}\|_{s^{-6},\,s,\,\varepsilon}$ is in the same connected component of $G^{-1}\big([0, G_{\max}/2]\big) = [0,v_0]\cup[v_1,\infty)$. On the other hand, since we already know that $\|\widehat{f}\|_{s^{-6},s,1} \in [0,v_0]$, we deduce that

$$\|\widehat{f}\|_{s^{-6},\,s,\,\varepsilon} \in [0,v_0],\ \forall\,\varepsilon>0. \tag{44}$$

This implies that, by the monotone convergence theorem,

$$\|\widehat{f}\|_{s^{-6},\,s,\,0} = \lim_{\varepsilon\to 0}\|\widehat{f}\|_{s^{-6},\,s,\,\varepsilon} \leq v_0. \tag{45}$$

In other words,

$$k \mapsto e^{s^{-6}|k|^3}\widehat{f}1_{[-s^2,s^2]^c} \in L^2. \tag{46}$$

Therefore it yields that

$$k \mapsto e^{s^{-6}|k|^3}\widehat{f} \in L^2. \tag{47}$$

Let $\mu_0 = s^{-6}$ for this $s>0$. Then the exponential decay in Theorem 1.3 is established.

We are left with proving that f is an entire function on the complex plane $\mathbb{C}$, which is however a local property. Indeed, by the Cauchy-Schwarz inequality, for any $\mu\in\mathbb{R}$, we have

$$e^{\mu|k|}\widehat{f}(k) = e^{\mu|k|-\mu_0|k|^3}e^{\mu_0|k|^3}\widehat{f}(k) \in L^1(\mathbb{R}), \tag{48}$$

Then for any $z\in\mathbb{C}$, we can always choose $\mu>|z|$ such that

$$f(z) = (2\pi)^{-1/2}\int e^{izk}\widehat{f}(k)dk = (2\pi)^{-1/2}\int e^{izk-\mu|k|}e^{\mu|k|}\widehat{f}(k)dk. \tag{49}$$

Since the first factor $e^{izk-\mu|k|}$ is bounded and the second factor is in L^1 by (48), f is an entire function. $\square$

It remains to prove Proposition 4.7, which we carry out in the next section.

5. The bootstrap argument

In this section, we prove Proposition 4.7, for which we only have the definition of weak solutions in (24) and the definition of Q at our disposal. We write

$$\begin{aligned} f_> &:= f 1_{[-s^2,s^2]^c}(P), \text{ where } P := -i\partial_x, \\ h &:= e^{F(P)} f,\ h_> := e^{F(P)} f_>. \end{aligned} \tag{50}$$

for the same $F = F_{\mu,\varepsilon}$ defined as above. In other words,

$$\widehat{f_>} = \widehat{f} 1_{[-s^2,s^2]^c},\ \widehat{h}(k) = e^{F(k)}\widehat{f}(k),\ \widehat{h}_>(k) := e^{F(k)}\widehat{f}_>(k). \tag{51}$$

Proof of Proposition 4.7. We use $g = e^{2F(P)} f_>$ in (24),

(52)
$$\begin{aligned} \omega\|e^{F(k)}\widehat{f}_>\|_2^2 &= \omega\langle e^{F(k)}\widehat{f}_>(k), e^{F(k)}\widehat{f}_>(k)\rangle = \omega\langle e^{2F(k)}\widehat{f}_>, \widehat{f}\rangle = \omega\langle e^{2F(P)} f_>, f\rangle \\ &= Q(e^{2F(P)} f_>, f, f, f, f, f, f, f) = Q(e^{F(P)} h_>, f, f, f, f, f, f, f) \\ &= Q(e^{F(P)} h_>, e^{-F(P)}h, e^{-F(P)}h, e^{-F(P)}h, e^{-F(P)}h, e^{-F(P)}h, e^{-F(P)}h, e^{-F(P)}h) \\ &=: Q_F. \end{aligned}$$

Then

$$|Q_F| \le CM_F(h_>, h, h, h, h, h, h, h) \le CM(h_>, h, h, h, h, h, h, h), \tag{53}$$

where the last inequality follows from Proposition 4.5. Continuing (53), we split h and use the operator M is sublinear in each component,

(54)
$$\begin{aligned} M(h_>, h, h, h, h, h, h, h) &\le M(h_>, h_<, h_<, h_<, h_<, h_<, h_<, h_<) + \\ &\quad + \sum_{\substack{j_2,\cdots,j_8\in\{>,<\},\\ \text{at least one } j_l = >}} M(h_>, h_{j_2}, h_{j_3}, h_{j_4}, h_{j_5}, h_{j_6}, h_{j_7}, h_{j_8}) =: A + B. \end{aligned}$$

We split further $h_< = h_\ll + h_\sim$, where the low frequency part $\widehat{h}_\ll := \widehat{h} 1_{[-s,s]}$ and the median frequency part $\widehat{h}_\sim := \widehat{h} 1_{[-s^2,s^2]\setminus[-s,s]}$.

We estimate A by using the bilinear Airy Strichartz estimate in Lemma 4.1:

$$\begin{aligned} A =& M(h_>, h_<, h_<, h_<, h_<, h_<, h_<, h_<) \\ &\le M(h_>, h_\ll, h_<, h_<, h_<, h_<, h_<, h_<) + M(h_>, h_\sim, h_<, h_<, h_<, h_<, h_<, h_<) \\ &\le s^{-1/4}\|h_>\|_2\|h_\ll\|_2\|h_<\|_2^6 + \|h_>\|_2\|h_\sim\|_2\|h_<\|_2^6 \\ &= \|h_>\|_2\big(s^{-1/4}\|h_\ll\|_2 + \|h_\sim\|_2\big)\|h_<\|_2^6. \end{aligned} \tag{55}$$

Recalling that $\|f\|_2 = 1$, then

$$\begin{aligned} \|h_<\|_2 &= \|e^{F_{\mu,\varepsilon}} \widehat{f}_<\|_2 \le \|e^{\mu|k|^3} \widehat{f}_<\|_2 \le e^{\mu s^6} \|f\|_2, \\ \|h_\ll\|_2 &= \|e^{F_{\mu,\varepsilon}} \widehat{f}_\ll\|_2 \le e^{\mu s^3} \|f\|_2, \\ \|h_\sim\|_2 &= \|e^{F_{\mu,\varepsilon}} \widehat{f}_\sim\|_2 \le e^{\mu s^6} \|f_\sim\|_2, \end{aligned} \tag{56}$$

we obtain

$$A \le C\|h_>\|_2 \big(s^{-1/4} e^{\mu s^3 - \mu s^6} + \|f_\sim\|_2\big) e^{7\mu s^6}. \tag{57}$$

Now we turn to estimate B.

$$\begin{aligned} B \le & \sum_{\substack{j_2,\cdots,j_8 \in \{>,<\}, \\ \text{exactly one } j_l = >}} M(h_>, h_{j_2}, h_{j_3}, h_{j_4}, h_{j_5}, h_{j_6}, h_{j_7}, h_{j_8}) + \\ & + \sum_{\substack{j_2,\cdots,j_8 \in \{>,<\}, \\ \text{two and more } j_l = >}} M(h_>, h_{j_2}, h_{j_3}, h_{j_4}, h_{j_5}, h_{j_6}, h_{j_7}, h_{j_8}) =: B_1 + B_2. \end{aligned} \tag{58}$$

For B_2,

$$B_2 \lesssim \|h_>\|_2 \Pi_{l=2}^8 \|h_{j_l}\|_2 \lesssim \|h_>\|_2 \left(\sum_{l=2}^7 \|h_<\|_2^{7-l} \|h_>\|_2^l \right) \lesssim \|h_>\|_2 \, e^{5\mu s^6} \sum_{l=2}^7 \|h_>\|_2^l \tag{59}$$

where we have used that $\|h_<\|_2 \lesssim e^{\mu s^6} \|f_<\|_2 \lesssim e^{\mu s^6}$.

For B_1, we split one of the $h_<$ into $h_< = h_\ll + h_\sim$ and then use the sublinearity of M,

$$\begin{aligned} B_1 &\lesssim \|h_>\|_2 \left(s^{-1/4} \|h_\ll\|_2 + \|h_\sim\|_2 \right) \|h_<\|_2^5 \|h_>\|_2 \\ &\lesssim \|h_>\|_2^2 \left(s^{-1/4} e^{\mu s^3 - \mu s^6} + \|f_\sim\|_2 \right) e^{\mu s^6} \|h_<\|_2^5 \\ &\lesssim \|h_>\|_2^2 \left(s^{-1/4} e^{\mu s^3 - \mu s^6} + \|f_\sim\|_2 \right) e^{6\mu s^6}. \end{aligned} \tag{60}$$

Thus we conclude that

$$B \le B_1 + B_2 \lesssim \|h_>\|_2^2 \left(s^{-1/4} e^{\mu s^3 - \mu s^6} + \|f_\sim\|_2 \right) e^{6\mu s^6} + e^{5\mu s^6} \|h_>\|_2 \sum_{l=2}^7 \|h_>\|_2^l. \tag{61}$$

Therefore from (52), (53), (54), (57) and (61), we have

$$\begin{aligned} \omega \|\widehat{h}_>\|_2^2 \lesssim & \|h_>\|_2 \big(s^{-1/4} e^{\mu s^3 - \mu s^6} + \|f_\sim\|_2\big) e^{7\mu s^6} + \\ & + \|h_>\|_2^2 \left(s^{-1/4} e^{\mu s^3 - \mu s^6} + \|f_\sim\|_2 \right) e^{6\mu s^6} + e^{5\mu s^6} \|h_>\|_2 \sum_{l=2}^7 \|h_>\|_2^l \end{aligned} \tag{62}$$

Canceling one $\|\widehat{h}_>\|_2$ on both sides, we see that

$$\begin{aligned} \omega \|\widehat{h}_>\|_2 \lesssim & \big(s^{-1/4} e^{\mu s^3 - \mu s^6} + \|f_\sim\|_2\big) e^{7\mu s^6} + \\ & \|h_>\|_2 \left(s^{-1/4} e^{\mu s^3 - \mu s^6} + \|f_\sim\|_2 \right) e^{6\mu s^6} + e^{5\mu s^6} \sum_{l=2}^7 \|h_>\|_2^l \end{aligned} \tag{63}$$

Since $\|f_\sim\|_2 = \|\widehat{f}1_{[-s^2,s^2]\setminus[-s,s]}\|_2 \le \|\widehat{f}1_{[-s,s]^c}\|_2 = o(1)$ as $s \to \infty$, and $e^{6\mu s^6} = e^6$ if taking $\mu = s^{-6}$, we conclude that

$$\omega\|h_>\|_2 \le o_1(1)\|h_>\|_2 + C\sum_{l=2}^{7}\|h_>\|_2^l + o_2(1). \tag{64}$$

Therefore the proof of Proposition 4.7 is complete. □

Acknowledgements. The research was carried out when the second author visited the math department at the University of Illinois, Urbana Champaign, and he was deeply grateful to its hospitality. During the early preparation of this work, the second author was supported by the National Science Foundation under agreement DMS-0635607. Any opinions, findings, and conclusions or recommendations expressed in this paper are those of the authors and do not necessarily reflect the views of the National Science Foundation.

References

[1] H. Bahouri and P. Gérard. High frequency approximation of solutions to critical nonlinear wave equations. *Amer. J. Math.*, 121(1):131–175, 1999.

[2] P. Bégout and A. Vargas. Mass concentration phenomena for the L^2-critical nonlinear Schrödinger equation. *Trans. Amer. Math. Soc.*, 359(11):5257–5282, 2007.

[3] J. Bennett, N. Bez, A. Carbery, and D. Hundertmark. Heat-flow monotonicity of Strichartz norms. *Analysis and PDE, Vol. 2 (2009), No. 2, 147–158.*

[4] A. Bulut. Maximizers for the strichartz inequalities for the wave equation. *ArXiv:0905.1678.*

[5] R. Carles and S. Keraani. On the role of quadratic oscillations in nonlinear Schrödinger equations. II. The L^2-critical case. *Trans. Amer. Math. Soc.*, 359(1):33–62 (electronic), 2007.

[6] E. Carneiro. A sharp inequality for the Strichartz norm. *Int. Math. Res. Not. IMRN*, (16):3127–3145, 2009.

[7] M. Christ and S. Shao. Existence of extremals for a Fourier restriction inequality. *Preprint.*

[8] M. Christ and S. Shao. On the extremisers of an adjoint Fourier restriction inequality. *Preprint.*

[9] W. Craig. An existence theory for water waves and the Boussinesq and Korteweg-de Vries scaling limits. *Comm. Partial Differential Equations*, 10(8):787–1003, 1985.

[10] B. Erdogan, D. Hundertmark, and Y. R. Lee. Exponential decay of dispersion management solitons. *ArXiv: 0806.1373.*

[11] D. Foschi. Maximizers for the Strichartz inequality. *J. Eur. Math. Soc. (JEMS)*, 9(4):739–774, 2007.

[12] D. Hundertmark and Y. R. Lee. Decay estimates and smoothness for solutions of the dispersion managed non-linear Schrödinger equation. *Comm. Math. Phys.*, 286(3):851–873, 2009.

[13] D. Hundertmark and V. Zharnitsky. On sharp Strichartz inequalities in low dimensions. *Int. Math. Res. Not.*, pages Art. ID 34080, 18, 2006.

[14] C. Kenig, G. Ponce, and L. Vega. Oscillatory integrals and regularity of dispersive equations. *Indiana Univ. Math. J.*, 40(1):33–69, 1991.

[15] S. Keraani. On the defect of compactness for the Strichartz estimates of the Schrödinger equations. *J. Differential Equations*, 175(2):353–392, 2001.

[16] M. Kunze. On the existence of a maximizer for the Strichartz inequality. *Comm. Math. Phys.*, 243(1):137–162, 2003.

[17] P.-L. Lions. The concentration-compactness principle in the calculus of variations. The locally compact case. I. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 1(2):109–145, 1984.

[18] P.-L. Lions. The concentration-compactness principle in the calculus of variations. The locally compact case. II. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 1(4):223–283, 1984.

[19] P.-L. Lions. The concentration-compactness principle in the calculus of variations. The limit case. I. *Rev. Mat. Iberoamericana*, 1(1):145–201, 1985.

[20] P.-L. Lions. The concentration-compactness principle in the calculus of variations. The limit case. II. *Rev. Mat. Iberoamericana*, 1(2):45–121, 1985.
[21] F. Merle and L. Vega. Compactness at blow-up time for L^2 solutions of the critical nonlinear Schrödinger equation in 2D. *Internat. Math. Res. Notices*, (8):399–425, 1998.
[22] G. Schneider and C. E. Wayne. The long-wave limit for the water wave problem. I. The case of zero surface tension. *Comm. Pure Appl. Math.*, 53(12):1475–1535, 2000.
[23] G. Schneider and C. E. Wayne. The rigorous approximation of long-wavelength capillary-gravity waves. *Arch. Ration. Mech. Anal.*, 162(3):247–285, 2002.
[24] S. Shao. The linear profile decomposition for the Airy equation and the existence of maximizers for the Airy Strichartz inequality. *Analysis and PDE, Vol. 2 (2009), No. 1, 83-117.*
[25] S. Shao. Maximizers for the Strichartz inequalities and the Sobolev-Strichartz inequalities for the Schrödinger equation. *Electronic Journal of Differential Equations, Vol. 2009(2009), No. 03, pp. 1-13.*
[26] E. M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.
[27] T. Tao. *Nonlinear dispersive equations*, volume 106 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2006. Local and global analysis.

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