

SUPER-EXPONENTIAL DECAY OF DIFFRACTION MANAGED SOLITONS

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ABSTRACT. This is the second part of a series of papers where we develop rigorous decay estimates for breather solutions of an averaged version of the non-linear Schrödinger equation. In this part we study the diffraction managed discrete non-linear Schrödinger equation, an equation which describes coupled waveguide arrays with periodic diffraction management geometries. We show that, for vanishing average diffraction, all solutions of the non-linear and non-local diffraction management equation decay super-exponentially. As a byproduct of our method, we also have a simple proof of existence of diffraction managed solitons in the case of vanishing average diffraction.

1. INTRODUCTION

Solitons, localized coherent structures resulting from a balance of non-linear and dispersive effects, have been the focus of an intense research activity over the last decades, see [35, 37]. Besides solitons in the continuum, discrete solitons have emerged in such diverse areas as solid states physics, some biological systems, Bose-Einstein condensation, and in discrete non-linear optics, e.g., optical waveguide arrays, [5, 9, 31, 32, 38]. The model describing this range of phenomena is given by the discrete non-linear Schrödinger equation

$$i\frac{\partial}{\partial\xi}u(x) + d(\xi)(\Delta u)(x) + |u(x)|^2u(x) = 0 \quad (1.1)$$

where for waveguide arrays ξ is the distance along the waveguide, $x \in \mathbb{Z}$ the location of the array element, Δ the discrete Laplacian given by $(\Delta f)(x) = f(x+1) + f(x-1) - 2f(x)$ for all $x \in \mathbb{Z}$, and $d(\xi)$ the total diffraction along the waveguide.

Nearly a decade after their theoretical prediction in [7], discrete solitons in an optical waveguide array were studied experimentally and, as in the continuous case localized stable non-linear waves were found [10]. Similar to the continuous case, i.e, non-linear fiber-optics, where the dispersion management technique introduced by [23] in 1980 turned out to be enormously successful in creating stable low power pulses by periodically varying the dispersion along the glass-fiber cable, see [1, 14, 15, 18, 21, 22, 28, 26, 29, 40] and the survey article [39], the diffraction management technique was proposed much more recently in [11] in order to create low power stable

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discrete pulses by periodically varying the diffraction in discrete optical waveguide arrays. In this case, the total diffraction $d(\xi)$ along the waveguide is given by

$$d(\xi) = \varepsilon^{-1} \tilde{d}(\xi/\varepsilon) + d_{av}. \quad (1.2)$$

Here $d_{av} \geq 0$ is the average component of the diffraction and \tilde{d} its mean zero part. Note that unlike in the continuum case, the diffraction management technique uses the *geometry* of the waveguide to achieve a periodically varying diffraction, see [11].

In the region of *strong diffraction management* ε is a small positive parameter. Rescaling $t = \xi/\varepsilon$, (1.1) is equivalent to

$$i \frac{\partial}{\partial t} u + \tilde{d}(t) \Delta u + \varepsilon (d_{av} \Delta u + |u|^2 u) = 0. \quad (1.3)$$

For small ε an average equation which describes the slow evolution of solutions of (1.3) was derived and numerical studies showed that this average equation possesses stable solutions which evolve nearly periodically when used as initial data in the diffraction managed non-linear discrete Schrödinger equation, [2, 3, 4]. Normalizing the period in the fast variable t to one, the average equation for the slow part v of solutions of (1.3) is given by

$$i \frac{\partial}{\partial t} v + \varepsilon d_{av} \Delta v + \varepsilon Q(v, v, v) = 0 \quad (1.4)$$

where

$$Q(v_1, v_2, v_3) := \int_0^1 T_s^{-1} [T_s v_1 \overline{T_s v_2} T_s v_3] ds \quad (1.5)$$

with $T_s := e^{iD(s)\Delta}$ and $D(s) = \int_0^s \tilde{d}(\xi) d\xi$ the solution operator for the free discrete Schrödinger equation with periodically varying diffraction

$$i \frac{\partial}{\partial t} v = -\tilde{d}(t) \Delta v. \quad (1.6)$$

One should keep in mind that the variable t denotes the distance along the waveguide. Physically it makes sense to assume that the diffraction profile \tilde{d} is bounded, or even piecewise constant along the waveguide. This assumption was made in [27, 30, 34]. For our results we need only to assume that its integral D is bounded over one period in the fast variable t ,

$$\tau := \sup_{t \in [0,1]} |D(t)| = \sup_{t \in [0,1]} \left| \int_0^t \tilde{d}(\xi) d\xi \right| < \infty, \quad (1.7)$$

where we normalized, without loss of generality, the period in t to one.

Using the same general method as in the continuum case, see, e.g., [43], the averaged equation (1.4) was derived in [2, 3, 4], where it was expressed in the Fourier space. The above formulation is from [27, 30]. Note that the non-linear and non-local equation (1.4) has an associated (averaged) Hamiltonian given by

$$H(v) := \varepsilon \left(\frac{d_{av}}{2} \langle v, -\Delta v \rangle - \frac{1}{4} Q(v, v, v) \right) \quad (1.8)$$

with $\langle g, f \rangle := \sum_{x \in \mathbb{Z}} \overline{g(x)} f(x)$ the usual scalar product on $l^2(\mathbb{Z})$, which, in our convention, is linear in the second component and anti-linear in the first and

$$\mathcal{Q}(v_1, v_2, v_3, v_4) := \int_0^1 \sum_{x \in \mathbb{Z}} \overline{(T_s v_1)(x)} (T_s v_2)(x) \overline{(T_s v_3)(x)} (T_s v_4)(x) ds. \quad (1.9)$$

Following the procedure for the continuous case in [43], it was shown in [27] that over long scales $0 \leq t \leq C\varepsilon^{-1}$ solutions of the non-autonomous equation (1.3) stay ε -close to solutions of the autonomous average equation (1.4) with the same initial condition. Thus it is interesting to find stationary solutions of (1.4), which are precisely the right initial conditions leading to breather-like nearly periodic solutions of (1.3) on long scales $0 \leq t \leq C\varepsilon^{-1}$. Making the ansatz $v(t, x) = e^{i\varepsilon\omega t} f(x)$ in (1.4) one arrives at the non-linear and non-local eigenvalue problem

$$-\omega f = -d_{\text{av}} \Delta f - Q(f, f, f). \quad (1.10)$$

Solutions of this equation can be found by minimizing the Hamiltonian H in (1.8) over functions $f \in l^2(\mathbb{Z})$ with a fixed l^2 -norm. The problem of constructing such minimizers for positive average diffraction $d_{\text{av}} > 0$ has been studied in [27, 30] using a discrete version of Lions' concentration compactness method [24]. Moreover, using by now classical arguments, see, [41, 42] or [6], it was noticed in [27, 30] that these minimizers are so-called *orbitally stable*, explaining at least in part the strong stability properties of diffraction management.

Similar as in the continuous case, see [19], proving existence of minimizers for vanishing average dispersion, $d_{\text{av}} = 0$, i.e., existence of weak solutions $f \in l^2(\mathbb{Z})$ of

$$\omega f = Q(f, f, f) \quad (1.11)$$

is much harder and has only recently been established in [34] using Ekeland's variational principle, [12, 13]. Moreover, it was shown in [34], that the corresponding minimizer is decaying faster than polynomial, which again yields the orbital stability of solutions of (1.4) for $d_{\text{av}} = 0$ and initial conditions close to a minimizer.

In this paper we continue our study of regularity properties of the dispersion management technique initiated in [16] and study the decay properties of diffraction management solitons for vanishing average dispersion, i.e., weak solutions of (1.11). Our main result is a significant strengthening of the super-polynomial decay result for diffraction managed solitons in [34].

Theorem 1.1 (Super-exponential decay). *Assume that the diffraction profile obeys (1.7). Then any weak solution $f \in l^2(\mathbb{Z})$ of (1.11) decays faster than any exponential. More precisely, with $c = 1 + \ln(8\tau)$,*

$$|f(x)| \lesssim e^{-\frac{1}{4}(|x|+1)(\ln((|x|+1)/2)-c)} \quad \text{for all } x \in \mathbb{Z}.$$

Remarks 1.2. (i) In particular, for any $0 < \mu < 1/4$ Theorem 1.1 yields the bound

$$|f(x)| \lesssim (|x| + 1)^{-\mu(|x|+1)} \quad \text{for all } x \in \mathbb{Z}.$$

(ii) The bound given in Theorem 1.1 rigorously justifies the theoretical and experimental conclusion of [11], that the diffraction management technique leads to optical soliton like pulses along a waveguide array which are extremely well-localized along

the array elements.

(iii) The super-exponential decay given in Theorem 1.1 is in stark contrast to the continuous case where one has, so far, only super-polynomial bounds on the decay of dispersion management solitons, see [16]. It is believed that the decay in the continuous case is exponential, see [25] for convincing but non-rigorous arguments.

(iv) Weak solutions of (1.11) are defined as

$$\omega \langle g, f \rangle = \langle g, Q(f, f, f) \rangle \quad (1.12)$$

for any $g \in l^2(\mathbb{Z})$. Recalling the definition (1.9) for the four-linear functional \mathcal{Q} , a short calculation gives

$$\langle f_1, Q(f_2, f_3, f_4) \rangle = \mathcal{Q}(f_1, f_2, f_3, f_4) \quad (1.13)$$

for any $f_j \in l^2(\mathbb{Z})$, $j = 1, 2, 3, 4$. Thus f is a weak solution of (1.11) if and only if

$$\omega \langle g, f \rangle = \mathcal{Q}(g, f, f, f) \quad (1.14)$$

for all $g \in l^2(\mathbb{Z})$.

(v) One easily sees that $\mathcal{Q}(f, f, f, f) > 0$ as soon as f is not the zero function. Thus $\omega = \mathcal{Q}(f, f, f, f) / \langle f, f \rangle > 0$ for any non-zero solution of (1.11).

Our second result is a simple proof of existence of weak solutions of (1.11) under the weak condition (1.7) on the diffraction profile.

Theorem 1.3. *Assume that the diffraction profile obeys (1.7). Let $\lambda > 0$. There is an $f \in l^2(\mathbb{Z})$ with $\|f\|_2^2 = \lambda$ such that*

$$\mathcal{Q}(f, f, f, f) = P_\lambda := \sup \{ \mathcal{Q}(g, g, g, g) \mid g \in l^2(\mathbb{Z}), \|g\|_2^2 = \lambda \}.$$

This maximizer f is also a weak solution of the diffraction management equation (1.11) where $\omega > 0$ is a suitable Lagrange-multiplier.

Remark 1.4. The existence of a maximizer is non-trivial, even for the corresponding problem with $d_{\text{av}} > 0$, since the equation (1.11), respectively (1.10), is invariant under translations and so is the corresponding energy functional H given by (1.8). Thus maximizing sequences for \mathcal{Q} , respectively minimizing sequences for H , can very easily converge to zero weakly. This was overcome using Lions' concentration compactness principle in [27, 30] for positive average diffraction and, for vanishing average dispersion, in [34] using Ekeland's variational principle [12, 13, 17], assuming that the diffraction profile is piecewise constant. Besides holding under much more general conditions, our proof is rather direct and, we believe, simple. We show that modulo translations any maximizing sequence has a strongly convergent subsequence, i.e., there is a sequence of shifts such that the shifted sequence, which by the translation invariance of the problem is also a maximizing sequence, has a strongly convergent subsequence. Our proof avoids the use of the concentration compactness principle but relies instead on a discrete version of multi-linear Strichartz estimates, see Corollary 2.9 and Lemma 4.4.

Our paper is organized as follows: In the next section we fix our notation and develop our basic technical estimates, the discrete versions of bilinear and multi-linear Strichartz estimates from Lemmata 2.7 and 2.8 and Corollary 2.9. All results

in Section 2 are valid in arbitrary dimension $d \geq 1$. The proof of Theorem 1.1 is given in Section 3, see Theorem 3.2 and Corollary 3.3. Similar to our study of decay properties of dispersion managed solitons in [16], the main tool in the proof of our super-exponential decay Theorem 1.1 is the self-consistency bound from Proposition 3.1 on the tail distribution of weak solutions of the diffraction management equation (1.11). Our existence proof for diffraction management solitons is given in Section 4. It relies heavily on Lemma 4.4, which follows from the enhanced multi-linear estimates of Corollary 2.9, and a simple characterization of strong convergence in $l^2(\mathbb{Z})$, or more generally, strong convergence in $l^p(\mathbb{Z}^d)$ for $1 \leq p < \infty$, given in Lemma 4.1.

2. BASIC ESTIMATES

In this section we consider \mathbb{Z}^d for arbitrary dimension $d \geq 1$. First we introduce some notation. By \mathbb{N} we denote the natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Given $n \in \mathbb{N}_0$, we denote $n!$ the factorial, $0! = 1$ and $(n+1)! = (n+1)n!$. The integers are denoted by \mathbb{Z} and \mathbb{Z}^d is the d -fold Euclidian product of \mathbb{Z} . $l^p(\mathbb{Z}^d)$ is the usual sequence space with norm

$$\|f\|_p = \left(\sum_{x \in \mathbb{Z}^d} |f(x)|^p \right)^{1/p} \quad \text{for } 1 \leq p < \infty \quad (2.1)$$

and

$$\|f\|_\infty = \sup_{x \in \mathbb{Z}^d} |f(x)|. \quad (2.2)$$

Of course, for $p = 2$ we get the Hilbert space of square summable sequences indexed by \mathbb{Z}^d . In this case we use

$$\langle f, g \rangle := \sum_{x \in \mathbb{Z}^d} \overline{f(x)} g(x) \quad (2.3)$$

for the scalar product on $l^2(\mathbb{Z}^d)$. Here \bar{z} is the complex conjugate of a complex number z . The real and imaginary parts of a complex number are given by $\text{Re}(z) = \frac{1}{2}(z + \bar{z})$ and $\text{Im}(z) = \frac{1}{2i}(z - \bar{z})$. Note that in our convention the scalar product given by (2.3) is linear in the second argument and anti-linear in the first. The discrete Laplacian on \mathbb{Z}^d is given by

$$\Delta f(x) = \sum_{|\nu|=1} f(x + \nu) - 2df(x) \quad (2.4)$$

where we take $|x| = \sum_{j=1}^d |x_j|$ for the norm on \mathbb{Z}^d . Since Δ is a bounded symmetric operator, $e^{it\Delta}$ is the unitary solution operator of the free discrete Schrödinger equation

$$i\partial_t u = -\Delta u \quad (2.5)$$

on $l^2(\mathbb{Z}^d)$; for any $f \in l^2(\mathbb{Z}^d)$ the function $u(t, x) = (e^{it\Delta} f)(x)$ solves (2.5) and $u(0, \cdot) = f$. Note that $e^{it\Delta}$ is unitary, in particular, $\|e^{it\Delta} f\|_2 = \|f\|_2$ for all $f \in l^2(\mathbb{Z}^d)$. For a diffraction profile \tilde{d} we set $D(t) = \int_0^t \tilde{d}(\xi) d\xi$ and $T_t := e^{iD(t)\Delta}$. Thus for any initial condition $f \in l^2(\mathbb{Z}^d)$ the function $u(t, \cdot) = T_t f$ solves

$$i\partial_t u = -\tilde{d}(t)\Delta u \quad (2.6)$$

with initial condition $u(0, \cdot) = f$. Again, T_t is a unitary operator on $l^2(\mathbb{Z}^d)$.

For a function $f : \mathbb{Z}^d \rightarrow \mathbb{C}$, its support is given by the set

$$\text{supp}(f) := \{x \in \mathbb{Z}^d : f(x) \neq 0\}.$$

For arbitrary $A \subset \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$ the distance from x to A is given by

$$\text{dist}(x, A) := \inf(|x - y| : y \in A)$$

and for subsets $A, B \subset \mathbb{Z}^d$ their distance is given by

$$\text{dist}(A, B) := \inf(\text{dist}(x, B) : x \in A) = \inf(|x - y| : x \in A, y \in B).$$

For any operator $T : l^2(\mathbb{Z}^d) \rightarrow l^2(\mathbb{Z}^d)$ we denote its kernel by

$$\langle x|T|y \rangle := \langle \delta_x, T\delta_y \rangle \quad (2.7)$$

where δ_y is the Kronecker δ -function

$$\delta_y(x) := \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}.$$

In particular,

$$Tf(x) = \sum_{y \in \mathbb{Z}^d} \langle x|T|y \rangle f(y).$$

We use the \lesssim notation in inequalities, if it is convenient not to specify any constants in the bounds: for two real-valued functions g, h defined on the same domain, $g \lesssim h$ means that there exists a non-negative constant C such that $g(x) \leq Ch(x)$ for all x .

The extension of the non-linear and non-local functional \mathcal{Q} to $l^2(\mathbb{Z}^d)$ is again denoted by \mathcal{Q} ,

$$\mathcal{Q}(f_1, f_2, f_3, f_4) = \int_0^1 \sum_{x \in \mathbb{Z}^d} \overline{(T_t f_1)(x)} (T_t f_2)(x) \overline{(T_t f_3)(x)} (T_t f_4)(x) dt. \quad (2.8)$$

The first problem is to show that \mathcal{Q} is well-defined on $l^2(\mathbb{Z}^d)$. Due to the next lemma this turns out to be easier than in the continuous case.

Lemma 2.1. *Let $1 \leq p \leq q \leq \infty$. Then, $l^p(\mathbb{Z}^d) \subset l^q(\mathbb{Z}^d)$ and*

$$\|f\|_q \leq \|f\|_p. \quad (2.9)$$

Proof. Clearly $\|f\|_\infty \leq \|f\|_p$ for all $f \in l^p(\mathbb{Z}^d)$ and all $1 \leq p \leq \infty$.

Let $0 \leq s \leq 1$. Then for all non-negative sequences $(a_n)_{n \in \mathbb{Z}^d}$,

$$\left(\sum_{n \in \mathbb{Z}^d} a_n \right)^s \leq \sum_{n \in \mathbb{Z}^d} a_n^s. \quad (2.10)$$

This follows from

$$(a_1 + a_2)^s = \frac{a_1}{(a_1 + a_2)^{1-s}} + \frac{a_2}{(a_1 + a_2)^{1-s}} \leq \frac{a_1}{a_1^{1-s}} + \frac{a_2}{a_2^{1-s}} = a_1^s + a_2^s$$

and induction. Now let $1 \leq p \leq q < \infty$ and $f \in l^p(\mathbb{Z}^d)$. Then with $s = p/q$,

$$\|f\|_p^p = \sum_{n \in \mathbb{Z}^d} |f(n)|^{qs} \geq \left(\sum_{n \in \mathbb{Z}^d} |f(n)|^q \right)^s = \|f\|_q^{qs} = \|f\|_q^p,$$

where we used (2.10). This finishes the proof of (2.9). \blacksquare

Corollary 2.2. *For any $f_j \in l^2(\mathbb{Z}^d)$, $j = 1, \dots, 4$, we have*

$$|\mathcal{Q}(f_1, f_2, f_3, f_4)| \leq \int_0^1 \sum_{x \in \mathbb{Z}^d} \prod_{j=1}^4 |T_t f_j(x)| dt \leq \prod_{j=1}^4 \|f_j\|_2 \quad (2.11)$$

Proof. Of course, the first inequality is just the triangle inequality. By Hölder's inequality followed by Lemma 2.1

$$\sum_{x \in \mathbb{Z}^d} \prod_{j=1}^4 |T_t f_j(x)| \leq \prod_{j=1}^4 \|T_t f_j\|_4 \leq \prod_{j=1}^4 \|T_t f_j\|_2 = \prod_{j=1}^4 \|f_j\|_2$$

where we also used that T_t is unitary on $l^2(\mathbb{Z}^d)$. Thus (2.11) follows by integrating this over t on $[0, 1]$. \blacksquare

Remark 2.3. The bound from Corollary 2.2 justifies the ad-hoc formal calculation

$$\langle f, \mathcal{Q}(f_1, f_2, f_3) \rangle = \mathcal{Q}(f, f_1, f_2, f_3) \quad (2.12)$$

for all $f, f_j \in l^2(\mathbb{Z}^d)$ with

$$\mathcal{Q}(f_1, f_2, f_3) := \int_0^1 T_t^{-1} [T_t f_1 \overline{T_t f_2} T_t f_3] dt. \quad (2.13)$$

In particular, this shows that $\mathcal{Q}(f_1, f_2, f_3) \in l^2(\mathbb{Z}^d)$ whenever $f_j \in l^2(\mathbb{Z}^d)$, and \mathcal{Q} defined in (2.13) is a bounded three linear map from $(l^2(\mathbb{Z}^d))^3$ to $l^2(\mathbb{Z}^d)$. This is in contrast to the continuous case, where it is bounded only for $d = 1, 2$, [16, 33, 43].

Lemma 2.4. *For the kernel of the free time evolution $e^{it\Delta}$ the bound*

$$\sup_{t \in [-\tau, \tau]} |\langle x | e^{it\Delta} | y \rangle| \leq \min(1, e^{4d\tau} \frac{(4d\tau)^{|x-y|}}{|x-y|!}) \quad (2.14)$$

holds for all $x, y \in \mathbb{Z}^d$ and $0 \leq \tau < \infty$.

Proof. The operator Δ is given by $\Delta f(x) = \sum_{|y-x|=1} f(y) - 2df(x)$ and, using, for example, the discrete Fourier transform, one sees that $0 \leq -\Delta \leq 4d$ and $\|\Delta\| = 4d$. Now, by symmetry of Δ , $e^{it\Delta}$ is unitary, hence one always has $|\langle x | e^{it\Delta} | y \rangle| \leq 1$ for any $x, y \in \mathbb{Z}^d$ and all t . Since Δ is bounded, the Taylor series for the exponential yields a strongly converging series for $e^{it\Delta}$. Thus

$$\langle x | e^{it\Delta} | y \rangle = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \langle x | \Delta^n | y \rangle = \sum_{n=|x-y|}^{\infty} \frac{(it)^n}{n!} \langle x | \Delta^n | y \rangle$$

since $\langle x | \Delta^n | y \rangle \neq 0$ if and only if x and y are connected by a path of length at most n , i.e., $|x-y| \leq n$. In particular, using $\|\Delta\| = 4d$,

$$|\langle x | e^{it\Delta} | y \rangle| \leq \sum_{n=|x-y|}^{\infty} \frac{|t|^n}{n!} \|\Delta\|^n = \sum_{l=0}^{\infty} \frac{(4d|t|)^{|x-y|+l}}{(|x-y|+l)!}$$

$$\leq \frac{(4d|t|)^{|x-y|}}{|x-y|!} \sum_{l=0}^{\infty} \frac{(4d|t|)^l}{l!} = \frac{(4d|t|)^{|x-y|}}{|x-y|!} e^{4d|t|}. \quad \blacksquare$$

Lemma 2.5. *Let $s \in \mathbb{N}$. For the free time-evolution $e^{it\Delta}$ generated by Δ the bound*

$$\sum_{y: |x-y| \geq s} \sup_{t \in [-\tau, \tau]} |\langle x | e^{it\Delta} | y \rangle|^2 \lesssim \frac{(4d\tau)^{2s} \max(s, 1)^{d-1}}{(s!)^2} \quad (2.15)$$

holds, where the implicit constant depends only on the dimension d and τ .

Proof. The number of points $y \in \mathbb{Z}^d$ with $|y| = \sum_{j=1}^d |y_j| = n$ can be estimated by $2^d n^{d-1}$ and with Lemma 2.4 we have

$$\begin{aligned} \sum_{y: |x-y| \geq s} \sup_{t \in [-\tau, \tau]} |\langle x | e^{it\Delta} | y \rangle|^2 &\leq e^{8d\tau} \sum_{|y| \geq s} \frac{(4d\tau)^{2|y|}}{(|y|!)^2} \leq e^{8d\tau} 2^d \sum_{n=0}^{\infty} (n+s)^{d-1} \frac{(4d\tau)^{2(n+s)}}{((n+s)!)^2} \\ &\leq \frac{(4d\tau)^{2s} \max(s, 1)^{d-1}}{(s!)^2} e^{8d\tau} 2^d \sum_{n=0}^{\infty} \left(1 + \frac{n}{\max(s, 1)}\right)^{d-1} \frac{(4d\tau)^{2n}}{(n!)^2} \\ &\leq \frac{(4d\tau)^{2s} s^{d-1}}{(s!)^2} e^{8d\tau} 2^d \sum_{n=0}^{\infty} (1+n)^{d-1} \frac{(4d\tau)^{2n}}{(n!)^2} \end{aligned}$$

where we also used $(n+s)! \geq n!s!$. Thus the inequality (2.15) holds with constant $C = e^{8d\tau} 2^d \sum_{n=0}^{\infty} (1+n)^{d-1} \frac{(4d\tau)^{2n}}{(n!)^2} < \infty$. \blacksquare

Remark 2.6. Estimating the number of points $y \in \mathbb{Z}^d$ with $|y| = n$ by $2^d n^{d-1}$ is, of course, a gross over-counting. A tighter estimate can be given as follows: Counting the number of different integers $x_j \geq 0$ with $\sum_{j=1}^d x_j = n$ is equal to distributing $d-1$ separators on $n+d-1$ places, i.e.,

$$\#\{x \in \mathbb{N}_0^d : \sum_{j=1}^d x_j = n\} = \binom{n+d-1}{d-1}$$

Since we have 2 choices for the sign, except when some coordinates are zero, we get the better bound

$$\begin{aligned} \#\{y \in \mathbb{Z}^d : \sum_{j=1}^d |y_j| = n\} &\leq 2^d \binom{n+d-1}{d-1} = \frac{2^d}{(d-1)!} \prod_{j=1}^{d-1} (n+j) \\ &= 2^d \frac{\prod_{j=1}^{d-1} (1+j/n)}{(d-1)!} n^{d-1} \leq \frac{2^d (1+\varepsilon)}{(d-1)!} n^{d-1} \end{aligned}$$

for some small $\varepsilon > 0$ when n is large. For our purpose, the rough estimate $2^d n^{d-1}$ is good enough.

In the formulation of the next lemma we need some more notation. For any $r \in \mathbb{R}$ let

$$[r] := \min\{z \in \mathbb{Z} : r \leq z\} \quad (2.16)$$

the smallest integer greater than or equal to r .

Lemma 2.7 (Strong Bilinear bound). *There exists a constant C depending only on the dimension d and τ such that for $f_1, f_2 \in l^2(\mathbb{Z}^d)$ and $s = \text{dist}(\text{supp}(f_1), \text{supp}(f_2))$*

$$\sup_{t \in [-\tau, \tau]} \|(e^{it\Delta} f_1)(e^{it\Delta} f_2)\|_2 \leq \min(1, C \frac{\max(\lceil s/2 \rceil, 1)^{d-1} (4d\tau)^{\lceil s/2 \rceil}}{\lceil s/2 \rceil!}) \|f_1\|_2 \|f_2\|_2. \quad (2.17)$$

Proof. First of all, note that

$$\|(e^{it\Delta} f_1)(e^{it\Delta} f_2)\|_2^2 = \sum_{x \in \mathbb{Z}^d} |e^{it\Delta} f_1(x)|^2 |e^{it\Delta} f_2(x)|^2 \leq \|e^{it\Delta} f_1\|_\infty^2 \|e^{it\Delta} f_2\|_2^2.$$

Hence, using Lemma 2.1 and the unicity of $e^{it\Delta}$ on $l^2(\mathbb{Z}^d)$, we see

$$\|(e^{it\Delta} f_1)(e^{it\Delta} f_2)\|_2^2 \leq \|e^{it\Delta} f_1\|_2^2 \|e^{it\Delta} f_2\|_2^2 = \|f_1\|_2^2 \|f_2\|_2^2 \quad (2.18)$$

uniformly in t . Now assume that $|t| \leq \tau$. Let $I_j = \text{supp}(f_j)$, $j = 1, 2$ and assume, without loss of generality that $s = \text{dist}(I_1, I_2) \geq 1$. Moreover we need the slightly enlarged sets

$$\begin{aligned} I'_1 &:= \{x : \text{dist}(x, I_1) \leq \text{dist}(x, I_2) - 1\}, \\ I'_2 &:= \{x : \text{dist}(x, I_1) \geq \text{dist}(x, I_2)\}. \end{aligned}$$

Note that $I_j \subset I'_j$, $j = 1, 2$, and $I'_2 = \mathbb{Z}^d \setminus I'_1$. The triangle inequality gives

$$s = \text{dist}(I_1, I_2) \leq \text{dist}(x, I_1) + \text{dist}(x, I_2) \leq \begin{cases} 2\text{dist}(x, I_2) - 1 & \text{if } x \in I'_1 \\ 2\text{dist}(x, I_1) & \text{if } x \in I'_2 \end{cases}$$

so, since the distance is always an integer, we have

$$\min(\text{dist}(I'_1, I_2), \text{dist}(I'_2, I_1)) \geq \lceil s/2 \rceil. \quad (2.19)$$

Certainly, since $I'_1 \cup I'_2 = \mathbb{Z}^d$,

$$\|(e^{it\Delta} f_1)(e^{it\Delta} f_2)\|_2^2 = \sum_{x \in I'_1} |e^{it\Delta} f_1(x)|^2 |e^{it\Delta} f_2(x)|^2 + \sum_{x \in I'_2} |e^{it\Delta} f_1(x)|^2 |e^{it\Delta} f_2(x)|^2. \quad (2.20)$$

Because of

$$e^{it\Delta} f_j(x) = \sum_{y \in I_j} \langle x | e^{it\Delta} | y \rangle f_j(y),$$

the Cauchy–Schwarz inequality implies

$$|e^{it\Delta} f_j(x)|^2 \leq \|f_j\|_2^2 \sum_{y \in I_j} |\langle x | e^{it\Delta} | y \rangle|^2.$$

Together with (2.19) and the bound from Lemma 2.4, this yields

$$\begin{aligned}
& \sup_{t \in [-\tau, \tau]} \sum_{x \in I'_1} |e^{it\Delta} f_1(x)|^2 |e^{it\Delta} f_2(x)|^2 \\
& \leq \sup_{t \in [-\tau, \tau]} \sum_{x \in I'_1} |e^{it\Delta} f_1(x)|^2 \|f_2\|_2^2 \sup_{x \in I'_1} \sum_{y \in I_2} |\langle x | e^{it\Delta} |y\rangle|^2 \\
& \leq \sup_{t \in [-\tau, \tau]} \sum_{x \in \mathbb{Z}^d} |e^{it\Delta} f_1(x)|^2 \|f_2\|_2^2 \sup_{x \in \mathbb{Z}^d} \sum_{y: |x-y| \geq \lceil s/2 \rceil} |\langle x | e^{it\Delta} |y\rangle|^2 \\
& \lesssim \|f_1\|_2^2 \|f_2\|_2^2 \frac{\max(\lceil s/2 \rceil, 1)^{d-1} (4d\tau)^{2\lceil s/2 \rceil}}{(\lceil s/2 \rceil!)^2}.
\end{aligned} \tag{2.21}$$

An identical argument gives

$$\sup_{t \in [-\tau, \tau]} \sum_{x \in I'_2} |e^{it\Delta} f_1(x)|^2 |e^{it\Delta} f_2(x)|^2 \lesssim \|f_1\|_2^2 \|f_2\|_2^2 \frac{\max(\lceil s/2 \rceil, 1)^{d-1} (4d\tau)^{2\lceil s/2 \rceil}}{(\lceil s/2 \rceil!)^2}. \tag{2.22}$$

The bounds (2.21) and (2.22) together with (2.20) finish the proof of the Lemma. \blacksquare

Lemma 2.8. *For $j \in \{1, 2, 3, 4\}$ let $f_j \in l^2(\mathbb{Z}^d)$. For any choice $j, k \in \{1, 2, 3, 4\}$ let $s = \text{dist}(\text{supp}(f_k), \text{supp}(f_j))$. Then*

$$\sup_{t \in [-\tau, \tau]} \sum_{x \in \mathbb{Z}^d} \prod_{j=1}^4 |(e^{it\Delta} f_j)(x)| \lesssim \frac{\max(\lceil s/2 \rceil, 1)^{d-1} (4d\tau)^{\lceil s/2 \rceil}}{\lceil s/2 \rceil!} \prod_{j=1}^4 \|f_j\|_2 \tag{2.23}$$

where the implicit constant depends only on the dimension d and τ .

Proof. Follows using Cauchy-Schwarz together with Corollary 2.2 and Lemma 2.7. \blacksquare

Corollary 2.9. *Let the diffraction profile obey the bound (1.7). Then with $c = 1 + \ln(8d\tau)$ we have*

$$|\mathcal{Q}(f_1, f_2, f_3, f_4)| \lesssim e^{-s(\ln s - c)/2 + (d-1)\ln(\max(s/2, 1))} \prod_{j=1}^4 \|f_j\|_2 \tag{2.24}$$

where $s = \text{dist}(\text{supp}(f_k), \text{supp}(f_j))$ for any choice $j, k \in \{1, 2, 3, 4\}$ and the implicit constant depends only on the dimension d and τ from (1.7).

Remark 2.10. Since for any $0 < \delta < 1/2$,

$$e^{-s(\ln s - c)/2 + (d-1)\ln(\max(s/2, 1))} \lesssim e^{-\delta s \ln s} = s^{-\delta s},$$

the bound (2.24) implies

$$|\mathcal{Q}(f_1, f_2, f_3, f_4)| \lesssim s^{-\delta s} \prod_{j=1}^4 \|f_j\|_2 \tag{2.25}$$

for all $0 < \delta < 1/2$, where the implicit constant depends only on d , δ , and τ .

Proof. Since (1.7) holds, there exists $0 < \tau < \infty$ such that $|D(t)| \leq \tau$ for all $0 \leq t \leq 1$. Thus, since $T_t = e^{iD(t)\Delta}$,

$$\begin{aligned} |\mathcal{Q}(f_1, f_2, f_3, f_4)| &\leq \int_0^1 \sum_{x \in \mathbb{Z}^d} \prod_{j=1}^4 |(T_t f_j)(x)| dt \leq \sum_{x \in \mathbb{Z}^d} \sup_{0 \leq t \leq 1} \prod_{j=1}^4 |(T_t f_j)(x)| \\ &\leq \sum_{x \in \mathbb{Z}^d} \sup_{t \in [-\tau, \tau]} \prod_{j=1}^4 |(e^{it\Delta} f_j)(x)| \end{aligned}$$

and the bound from Lemma 2.8 implies

$$|\mathcal{Q}(f_1, f_2, f_3, f_4)| \lesssim \frac{\max(\lceil s/2 \rceil, 1)^{d-1} (4d\tau)^{\lceil s/2 \rceil}}{\lceil s/2 \rceil!} \prod_{j=1}^4 \|f_j\|_2 \quad (2.26)$$

An easy proof by induction shows $n! \geq e^{n \ln n - n}$. Hence, using $\lceil s/2 \rceil \geq s/2$,

$$\begin{aligned} \frac{\max(\lceil s/2 \rceil, 1)^{d-1} (4d\tau)^{\lceil s/2 \rceil}}{\lceil s/2 \rceil!} &\lesssim e^{-s/2 \ln(s/2) + s/2 \ln(4d\tau) + s/2 + (d-1) \ln(\max(s/2, 1))} \\ &= e^{-s \ln(s)/2 + cs/2 + (d-1) \ln(\max(s/2, 1))} \end{aligned}$$

where $c = 1 + \ln(8d\tau)$. This proves the bound (2.24). \blacksquare

3. SELF-CONSISTENCY BOUND AND SUPER-EXPONENTIAL DECAY

As in the continuous case, see [16], the key idea is not to focus on the solution f directly, but to study its *tail distribution* defined, for $n \in \mathbb{N}_0$, by

$$\alpha(n) := \left(\sum_{|x| \geq n} |f(x)|^2 \right)^{1/2}. \quad (3.1)$$

The fundamental a-priori estimate for the tail distribution of weak solutions f of (1.11) is given by the following

Proposition 3.1 (Self-consistency bound). *Let f be a weak solution of $\omega f = Q(f, f, f)$. Then with $c = 1 + \ln(8\tau)$, where τ is from the bound (1.7) on the diffraction profile,*

$$\alpha(2n) \lesssim \alpha(n)^3 + e^{-(n+1)(\ln(n+1)-c)/2}. \quad (3.2)$$

In particular, for any $0 < \delta < 1/2$ the bound

$$\alpha(2n) \lesssim \alpha(n)^3 + (n+1)^{-\delta(n+1)} \quad (3.3)$$

holds. In (3.2) and (3.3) the implicit constants depend only on ω , δ , $\|f\|_2$, and τ .

Proof. Since f is a weak solution of $\omega f = Q(f, f, f)$, we have, by definition,

$$\omega \langle g, f \rangle = \mathcal{Q}(g, f, f, f) \quad \text{for all } g \in l^2(\mathbb{Z}).$$

Since

$$\alpha(2n) = \sup_{\substack{g \in l^2(\mathbb{Z}), \|g\|_2=1, \\ \text{supp}(g) \subset (-\infty, -2n] \cup [2n, \infty)}} |\langle g, f \rangle|$$

we need to estimate $\mathcal{Q}(g, f, f, f)$ for $g \in l^2(\mathbb{Z})$ with $\|g\|_2 = 1$ and $\text{supp}(g) \subset (-\infty, -2n] \cup [2n, \infty)$. Let $I_n = \{-n+1, \dots, n-1\}$, I_n^c its complement and split f into its low and high parts, $f = f_{<} + f_{>}$ with $f_{<} = f\chi_{I_n}$ and $f_{>} = f\chi_{I_n^c}$. Using the multi-linearity of \mathcal{Q} ,

$$\begin{aligned} \mathcal{Q}(g, f, f, f) &= \mathcal{Q}(g, f_{<}, f, f) + \mathcal{Q}(g, f_{>}, f, f) \\ &= \mathcal{Q}(g, f_{<}, f, f) + \mathcal{Q}(g, f_{>}, f_{<}, f) + \mathcal{Q}(g, f_{>}, f_{>}, f_{<}) + \mathcal{Q}(g, f_{>}, f_{>}, f_{>}). \end{aligned} \quad (3.4)$$

The last term is estimated by

$$|\mathcal{Q}(g, f_{>}, f_{>}, f_{>})| \lesssim \|g\|_2 \|f_{>}\|_2^3 = \alpha(n)^3.$$

For the first three terms in (3.4) we note that each of them contains one $f_{<}$. Since $s := \text{dist}(\text{supp}(g), \text{supp}(f_{<}))$ is at least $n+1$, the enhanced multi-linear estimate (2.24) from Corollary 2.9 applies and gives, since $d=1$, for the first term

$$|\mathcal{Q}(g, f_{<}, f, f)| \lesssim e^{-s(\ln s - c)/2} \|g\|_2 \|f_{<}\|_2 \|f\|_2^2.$$

Similar bounds hold for the second and third terms. Collecting terms and using $s \geq n+1$, we see

$$\begin{aligned} \alpha(2n) &\lesssim \omega^{-1} \left(\alpha(n)^3 + e^{-(n+1)(\ln(n+1)-c)/2} (\alpha(0)^3 + \alpha(0)^2 \alpha(n) + \alpha(0) \alpha(n)^2) \right) \\ &\lesssim \alpha(n)^3 + e^{-(n+1)(\ln(n+1)-c)/2} \end{aligned}$$

since α is a bounded decreasing function. This proves (3.2). Note that the implicit constant depends only on ω , τ , and $\alpha(0) = \|f\|_2$. To prove (3.3) one either argues as above, but uses (2.25) instead of (2.24), or simply notes that $e^{-(n+1)(\ln(n+1)-c)/2} \lesssim (n+1)^{-\delta(n+1)}$ for any $0 < \delta < 1/2$. \blacksquare

Theorem 3.2 (Super-exponential decay). *Let α be a decreasing non-negative function which obeys the self-consistency bound (3.2) of Proposition 3.1 and decays to zero at infinity. Then the bound*

$$\alpha(n) \lesssim e^{-\frac{n+1}{4}(\ln(\frac{n+1}{2})-c)}$$

holds for all $n \in \mathbb{N}_0$. Here one can choose $c = 1 + \ln(8\tau)$, with τ from the bound (1.7) on the diffraction profile.

Corollary 3.3 (= Theorem 1.1). *For any weak solution of $\omega f = Q(f, f, f)$ and any $0 < \mu < \frac{1}{4}$ the bound*

$$|f(x)| \lesssim e^{-\frac{|x|+1}{4}(\ln(\frac{|x|+1}{2})-c)} \quad (3.5)$$

holds for all $x \in \mathbb{Z}$.

Proof. Given Theorem 3.2, this follows immediately from $|f(x)| \leq \alpha(|x|)$. \blacksquare

It remains to prove Theorem 3.2. This is done in two steps. The first is a reduction of the full super-exponential decay to a slower but still super-exponential decay.

Lemma 3.4. *Let α be a non-negative decreasing function which obeys the self-consistency bound (3.2). Then the bounds*

$$\alpha(n) \lesssim e^{-\frac{n+1}{4}(\ln(\frac{n+1}{2})-c)} \quad (3.6)$$

and

$$\alpha(n) \lesssim (n+1)^{-\mu_0(n+1)} \quad (3.7)$$

for some $\mu_0 > 0$ are equivalent.

Proof. Of course, the bound (3.6) implies (3.7) for all $0 < \mu_0 < 1/4$. To prove the converse we will show that if $\alpha(n) \lesssim (n+1)^{-\mu(n+1)}$ for some $\mu > 0$ and if $3\mu < 1/2$ one can boost the decay to

$$\alpha(n) \lesssim (n+1)^{-\frac{5}{4}\mu(n+1)}. \quad (3.8)$$

Assume this for the moment and assume that (3.7) holds for some $\mu_0 > 0$. Let $l_0 \in \mathbb{N}_0$ such that $3(5/4)^{l_0-1}\mu_0 < 1/2 \leq 3(5/4)^{l_0}\mu_0$. We can iterate (3.8) exactly l_0 times to see

$$\alpha(n) \lesssim (n+1)^{-(5/4)^{l_0}\mu_0(n+1)}. \quad (3.9)$$

Plugging the estimate (3.9) into the self-consistency bound (3.2) yields

$$\alpha(2n) \lesssim (n+1)^{-3(5/4)^{l_0}\mu_0(n+1)} + e^{-(n+1)(\ln(n+1)-c)/2} \lesssim e^{-(n+1)(\ln(n+1)-c)/2}$$

since $3(5/4)^{l_0}\mu_0 \geq 1/2$, by assumption. Thus for even n we have the bound

$$\alpha(n) \lesssim e^{-(\frac{n}{2}+1)(\ln(\frac{n}{2}+1)-c)/2} = e^{-\frac{n+2}{4}(\ln(\frac{n+2}{2})-c)} \quad (3.10)$$

and, by monotonicity of α , for odd n the bound (3.10) yields

$$\alpha(n) \leq \alpha(n-1) \lesssim e^{-\frac{n+1}{4}(\ln(\frac{n+1}{2})-c)}. \quad (3.11)$$

The bounds (3.10) and (3.11) together show that (3.6) holds.

It remains to prove the boost in decay given in (3.9). If $\alpha(n) \lesssim (n+1)^{-\mu(n+1)}$ and $3\mu < 1/2$, the self-consistency bound (3.2) gives

$$\alpha(2n) \lesssim (n+1)^{-3\mu(n+1)} + e^{-(n+1)(\ln(n+1)-c)/2} \lesssim (n+1)^{-3\mu(n+1)}$$

as long as $3\mu < 1/2$. Thus, as before, for even n one gets

$$\alpha(n) \lesssim \left(\frac{n+2}{2}\right)^{-\frac{3}{2}\mu(n+2)} \lesssim (n+2)^{-(\frac{3}{2}-\varepsilon)(n+2)} \leq (n+1)^{-(\frac{3}{2}-\varepsilon)(n+1)} \quad (3.12)$$

for any $\varepsilon > 0$. For odd n the monotonicity of α and (3.12) give

$$\alpha(n) \leq \alpha(n-1) \lesssim (n+1)^{-(\frac{3}{2}-\varepsilon)(n+1)} \quad (3.13)$$

for any $\varepsilon > 0$. The bounds (3.12) and (3.13) together show

$$\alpha(n) \lesssim (n+1)^{-(\frac{3}{2}-\varepsilon)(n+1)} \quad (3.14)$$

for all $n \in \mathbb{N}_0$ and all $\varepsilon > 0$. Choosing $\varepsilon = 1/4$ yields (3.8). \blacksquare

Given Lemma 3.4, in order to prove the super-exponential decay of α given in Theorem 3.2, it is enough to show that $\alpha(n) \lesssim (n+1)^{-\mu_0(n+1)}$ for some arbitrarily small $\mu_0 > 0$. This is the content of the next proposition

Proposition 3.5. *Assume that α is a non-negative decreasing function which obeys the self-consistency bound (3.2) and goes to zero at infinity. Then there exists $\mu_0 > 0$ such that*

$$\alpha(n) \lesssim (n+1)^{-\mu_0(n+1)}$$

For the proof of Proposition 3.5 we need some more notation. Given $n \in \mathbb{N}_0$ let

$$F(n) := \begin{cases} (n+2)^{n+2} & \text{if } n \text{ is even} \\ (n+1)^{n+1} & \text{if } n \text{ is odd} \end{cases} \quad (3.15)$$

and, for $\varepsilon \geq 0$, its regularized version

$$F_\varepsilon(n) := \frac{F(n)}{1 + \varepsilon F(n)} = \frac{1}{F(n)^{-1} + \varepsilon}. \quad (3.16)$$

Finally, for $\mu \geq 0$ let

$$F_{\mu,\varepsilon}(n) := F_\varepsilon(n)^\mu = (F(n)^{-1} + \varepsilon)^{-\mu}. \quad (3.17)$$

Furthermore let

$$\|\alpha\|_{\mu,\varepsilon,b} := \sup_{n \geq b} F_{\mu,\varepsilon}(n) |\alpha(n)|. \quad (3.18)$$

Of course, the super-exponential decay given in Proposition 3.5 is equivalent to showing

$$\|\alpha\|_{\mu_0,0,b} < \infty \quad \text{for some } \mu_0 > 0 \text{ and some } b \in \mathbb{N}_0.$$

Since $\|\alpha\|_{\mu,0,b} = \sup_{0 < \varepsilon \leq 1} \|\alpha\|_{\mu,\varepsilon,b}$, see Lemma 3.6.vi below, it is enough to find an ε -independent bound on $\|\alpha\|_{\mu,\varepsilon,b}$, which is where the second self-consistency bound of Proposition 3.1 enters. First we gather some basic properties of $F_{\mu,\varepsilon}$ needed in the proof of Proposition 3.5.

Lemma 3.6. (i) *For any $\varepsilon \geq 0$ the function $n \mapsto F_\varepsilon(n)$ is increasing in n and, for fixed n , decreasing in $\varepsilon \geq 0$. Moreover, $F_\varepsilon(n) \leq \varepsilon^{-1}$ for all n .*

(ii) *For any $\mu, \varepsilon \geq 0$ the function $n \mapsto F_{\mu,\varepsilon}(n)$ is increasing and bounded by $\varepsilon^{-\mu}$. Moreover, $F_{\mu,\varepsilon}(n)$ is decreasing in $\varepsilon \geq 0$ and depends continuously on the parameters μ and ε for fixed $n \in \mathbb{N}_0$.*

(iii) *For any $0 \leq \mu \leq \delta/3$, the function $\mathbb{N}_0 \ni n \rightarrow F_{\mu,0}(2n)(n+1)^{-\delta(n+1)}$ is decreasing.*

(iv) *The bound*

$$F_{\mu,\varepsilon}(2n) \leq 4F_{\mu,\varepsilon}(n)^3 \quad (3.19)$$

holds for all $0 \leq \mu, \varepsilon \leq 1$ and $n \in \mathbb{N}_0$.

(v) *For fixed $b \in \mathbb{N}_0$ and an arbitrary bounded function α the map $(\mu, \varepsilon) \mapsto \|\alpha\|_{\mu,\varepsilon,b}$ is continuous on $[0, 1] \times (0, 1]$.*

(vi) *For fixed $0 < \mu, b \in \mathbb{N}_0$, and an arbitrary bounded function α ,*

$$\|\alpha\|_{\mu,0,b} = \lim_{\varepsilon \rightarrow 0} \|\alpha\|_{\mu,\varepsilon,b} = \sup_{0 < \varepsilon \leq 1} \|\alpha\|_{\mu,\varepsilon,b}.$$

Remark 3.7. The last part of the lemma shows that for fixed $0 < \mu \leq 1$ and $b \in \mathbb{N}_0$ the map $\varepsilon \mapsto \|\alpha\|_{\mu,\varepsilon,b}$ is continuous on $[0, 1]$. Here we interpret continuity in a generalized sense: One certainly has $\|\alpha\|_{\mu,\varepsilon,b} \leq \|\alpha\|_\infty / \varepsilon < \infty$ for all $\varepsilon > 0$. If $\|\alpha\|_{\mu,0,b} < \infty$, then $\lim_{\varepsilon \rightarrow 0} \|\alpha\|_{\mu,\varepsilon,b} = \|\alpha\|_{\mu,0,b}$, and if $\|\alpha\|_{\mu,0,b} = \infty$, then $\lim_{\varepsilon \rightarrow 0} \|\alpha\|_{\mu,\varepsilon,b} = \infty$.

We postpone the proof of Lemma 3.6 after the proof of Proposition 3.5.

Proof of Proposition 3.5. We assume that α decays monotonically to zero and obeys the second self-consistency bound given in Proposition 3.1. That is, for fixed $0 < \delta < 1/2$ there exists a constant C_0 such that

$$\alpha(2n) \leq C_0 \alpha(n)^3 + C_0 (n+1)^{-\delta(n+1)}. \quad (3.20)$$

Multiply this by $\chi_{[b,\infty)}$, the characteristic function of the set $[b, \infty)$, using

$$\chi_{[b,\infty)}(n) = \chi_{[b,\infty)}(2n) - \chi_{[b,2b)}(2n), \quad (3.21)$$

putting $\alpha_b = \chi_{[b,\infty)}\alpha$, and rearranging terms yields

$$\alpha_b(2n) \leq C_0 \alpha_b(n)^3 + C_0 (n+1)^{-\delta(n+1)} \chi_{[b,\infty)}(n) + \chi_{[b,2b)}(2n) \alpha(2n). \quad (3.22)$$

Now let $\mu \leq \delta/3$. Multiplying (3.22) by $F_{\mu,\varepsilon}(2n)$, using the bound (3.19) from Lemma 3.6 on the first term on the right hand side of (3.22) and $F_{\mu,\varepsilon}(2n) \leq F_{\mu,0}(2n)$ from Lemma 3.6.ii on the other two, assuming $b \geq 0$, we get

$$\begin{aligned} F_{\mu,\varepsilon}(2n) \alpha_b(2n) &\leq C_1 (F_{\mu,\varepsilon}(n) \alpha_b(n))^3 + C_0 \frac{F_{\mu,0}(2n)}{(n+1)^{\delta(n+1)}} \chi_{[b,\infty)}(n) \\ &\quad + F_{\mu,0}(2n) \chi_{[b,2b)}(2n) \alpha(2n) \\ &\leq C_1 (F_{\mu,\varepsilon}(n) \alpha_b(n))^3 + C_0 \frac{F_{\mu,0}(2b)}{(b+1)^{\delta(b+1)}} + F_{\mu,0}(2b) \alpha(b) \\ &\leq C_1 \|\alpha\|_{\mu,\varepsilon,b}^3 + F_{\mu,0}(2b) \left(\frac{C_0}{(b+1)^{\delta(b+1)}} + \alpha(b) \right). \end{aligned} \quad (3.23)$$

For the second inequality we used that α and, by Lemma 3.6.iii, $F_{\mu,0}(2n)(n+1)^{-\delta(n+1)}$ are decreasing and that $F_{\mu,0}(2n)$ is increasing in n . In the third inequality, we used the obvious bound

$$F_{\mu,\varepsilon}(n) \alpha_b(n) \leq \|\alpha\|_{\mu,\varepsilon,b} \quad \text{for all } n \in \mathbb{N}_0.$$

by the definition (3.18) for $\|\alpha\|_{\mu,\varepsilon,b}$.

The punchline is that, since $F_{\mu,\varepsilon}(2n) = F_{\mu,\varepsilon}(2n+1)$ by construction of $F_{\mu,\varepsilon}$, the monotonicity of α gives

$$\sup_{n \geq b} F_{\mu,\varepsilon}(n) \alpha(n) = \sup_{n \in \mathbb{N}_0} F_{\mu,\varepsilon}(2n) \alpha_b(2n) \quad (3.24)$$

whenever b is an even natural number. So, since (3.23) holds for any $n \in \mathbb{N}_0$, taking the supremum over n in (3.23) yields

$$\|\alpha\|_{\mu,\varepsilon,b} \leq C_1 \|\alpha\|_{\mu,\varepsilon,b}^3 + F_{\mu,0}(2b) \left(\frac{C_0}{(b+1)^{\delta(b+1)}} + \alpha(b) \right) \quad (3.25)$$

which is a *closed* inequality for $\|\alpha\|_{\mu,\varepsilon,b}$ and holds as long as b is even and $0 < \mu \leq \delta/3$ and $0 < \varepsilon \leq 1$ or $\mu = 0$ and $0 \leq \varepsilon \leq 1$.

Equivalently, with $G(\nu) := \nu - C_1 \nu^3$ defined for $\nu \geq 0$, we arrived at the a-priori bound

$$G(\|\alpha\|_{\mu,\varepsilon,b}) \leq F_{\mu,0}(2b) (C_0 (b+1)^{-\delta(b+1)} + \alpha(b)). \quad (3.26)$$

A simple exercise shows that the maximum of G is attained at $\nu_{\max} = (3C_1)^{-1/2}$ and is given by $G_{\max} = G(\nu_{\max}) = 2/(3\sqrt{3C_1})$. Let $0 < G_0 < G_{\max}$ and $0 < \nu_0 < \nu_{\max} < \nu_1$ with $G(\nu_0) = G(\nu_1) = G_0$. Since $G(\nu) < \nu$ for $\nu > 0$, we have $G(\nu_0) < \nu_0$. Moreover,

$$G^{-1}((-\infty, G_0]) = [0, \nu_0] \cup [\nu_1, \infty). \quad (3.27)$$

This situation is visualized in Figure 1. Now we finish the proof of the decay estimate:

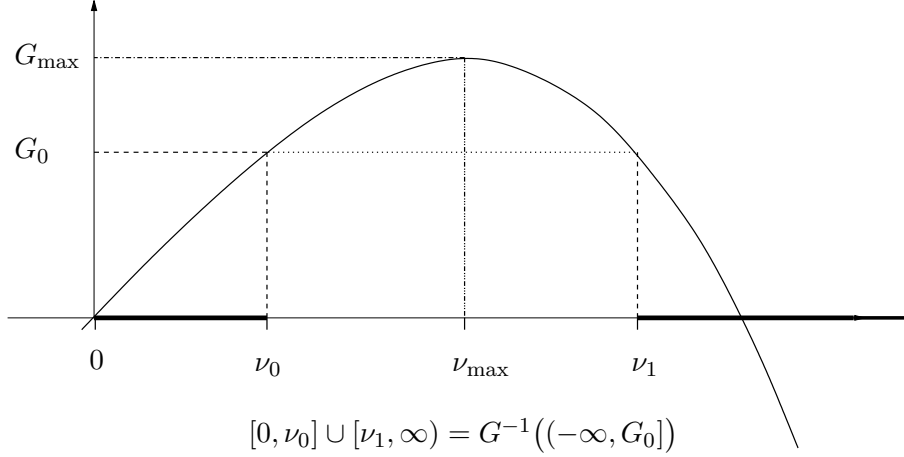


FIGURE 1. Graph of $G(\nu) = \nu - C_1\nu^3$ and the trapping region $G^{-1}((-\infty, G_0])$.

we need to show that $\|\alpha\|_{\mu,0,b}$ is finite for some $\mu > 0$ and $b \in \mathbb{N}_0$. Note that by Lemma 3.6.v the map $(\mu, \varepsilon) \mapsto \|\alpha\|_{\mu,\varepsilon,b}$ is continuous in $(\mu, \varepsilon) \in [0, 1] \times (0, 1]$, and, by Lemma 3.6.vi, for fixed $0 < \mu \leq 1$ $\|\alpha\|_{\mu,0,b} = \lim_{\varepsilon \rightarrow 0} \|\alpha\|_{\mu,\varepsilon,b}$. So it will be enough to find, for some $\mu > 0$ and $b \in \mathbb{N}_0$, a uniform in $0 < \varepsilon \leq 1$ estimate for $\|\alpha\|_{\mu,\varepsilon,b}$. This is where the bound (3.26) will enter.

Step 1: Choose an even b such that $C_0(b+1)^{-\delta(b+1)} + \alpha(b) < G_0 < G_{\max}$. This is possible since α goes to zero at infinity. Since α is monotone decreasing, it also guarantees $\|\alpha\|_{0,1,b} = \sup_{n \in \mathbb{N}_0} \alpha_b(n) = \alpha(b) < G_0 \leq \nu_0$.

Step 2: For the b fixed in Step 1, let $0 < \mu_0 \leq \delta/3$ such that $F_{\mu_0,0}(2b)(C_0(b+1)^{-\delta(b+1)} + \alpha(b)) \leq G_0 < G_{\max}$ and $\|\alpha\|_{\mu_0,1,b} < \nu_0$. This is possible since $F_{0,0}(2b) = 1$ and $F_{\mu,0}(2b)$ and $\|\alpha\|_{\mu,1,b}$ are continuous in $0 \leq \mu \leq \delta/3$.

Putting things together, (3.26) gives

$$G(\|\alpha\|_{\mu_0,\varepsilon,b}) \leq G_0 \quad \text{for all } 0 < \varepsilon \leq 1. \quad (3.28)$$

Since G is continuous and $\|\alpha\|_{\mu_0,\varepsilon,b}$ depends continuously on $\varepsilon > 0$, the bound (3.28) shows that $\|\alpha\|_{\mu_0,\varepsilon,b}$ is trapped in the same connected component of $G^{-1}((-\infty, G_0])$ as $\|\alpha\|_{\mu_0,1,b}$ for all $0 < \varepsilon \leq 1$. Thus, using $0 \leq \|\alpha\|_{\mu_0,1,b} < \nu_0$ and (3.27), we must have

$$\|\alpha\|_{\mu_0,\varepsilon,b} \leq \nu_0 \quad \text{for all } 0 < \varepsilon \leq 1. \quad (3.29)$$

Together with $\|\alpha\|_{\mu_0,0,b} = \lim_{\varepsilon \rightarrow 0} \|\alpha\|_{\mu_0,\varepsilon,b}$, the bound (3.29) shows $\|\alpha\|_{\mu_0,0,b} \leq \nu_0 < \infty$ which proves the estimate

$$\alpha(n) \leq \nu_0 F(n)^{-\mu_0}$$

for all large n . This finished our proof of Proposition 3.5. \blacksquare

It remains to prove the properties of $F_{\mu,\varepsilon}$ given in Lemma 3.6.

Proof of Lemma 3.6. The function F is clearly increasing in n and since $s \mapsto s/(1+\varepsilon s)$ is increasing in $s \geq 0$ for fixed $\varepsilon \geq 0$, we see that F_ε and hence also $F_{\mu,\varepsilon}$ is increasing in n . The other claims in part (i) and (ii) of Lemma 3.6 are obvious.

(iii): With $\lambda = \frac{\delta}{2\mu} - 1 \geq 1/2$, we have

$$F_{\mu,0}(2n)(n+1)^{-\delta(n+1)} = (2(n+1))^{2\mu(n+1)}(n+1)^{-\delta(n+1)} = \left[2^{n+1}(n+1)^{-\lambda(n+1)}\right]^{2\mu}.$$

Hence

$$\frac{F_{\mu,0}(2(n+1))(n+2)^{-\delta(n+2)}}{F_{\mu,0}(2n)(n+1)^{-\delta(n+1)}} = \left[\frac{2}{(n+2)^\lambda} \left(\left(1 + \frac{1}{n+1}\right)^{(n+1)} \right)^{-\lambda} \right]^{2\mu}.$$

Since the sequence $(1 + \frac{1}{n+1})^{n+1}$ is increasing, one has $(1 + \frac{1}{n+1})^{n+1} \geq 2$ and

$$\frac{F_{\mu,0}(2(n+1))(n+2)^{-\delta(n+2)}}{F_{\mu,0}(2n)(n+1)^{-\delta(n+1)}} \leq \left[\frac{2}{(n+2)^\lambda} 2^{-\lambda} \right]^{2\mu} \leq 2^{(1-2\lambda)2\mu} \leq 1$$

so the function $F_{\mu,0}(2n)(n+1)^{-\delta(n+1)}$ is decreasing on \mathbb{N}_0 .

(iv): Put

$$f(n, \varepsilon) := \frac{F_\varepsilon(2n)}{F_\varepsilon(n)^3} = \frac{(F(n)^{-1} + \varepsilon)^3}{F(2n)^{-1} + \varepsilon}. \quad (3.30)$$

We claim that

$$\sup_{n \in \mathbb{N}_0, 0 \leq \varepsilon \leq 1} f(n, \varepsilon) = 4, \quad (3.31)$$

which obviously yields $F_\varepsilon(2n) \leq 4F_\varepsilon(n)^3$ and

$$F_{\mu,\varepsilon}(2n) \leq 4^\mu F_{\mu,\varepsilon}(n)^3 \quad \text{for all } 0 \leq \mu \leq 1.$$

So in order to prove (3.19) it is enough to show that (3.31) holds. The partial derivative of f with respect to ε is given by

$$\frac{\partial f}{\partial \varepsilon} = \frac{(F(n)^{-1} + \varepsilon)^2}{(F(2n)^{-1} + \varepsilon)^2} [3F(2n)^{-1} - F(n)^{-1} + 2\varepsilon].$$

In the case $3F(2n)^{-1} - F(n)^{-1} \geq 0$ one has $\frac{\partial f}{\partial \varepsilon} \geq 0$ for all $\varepsilon \geq 0$ and the case $3F(2n)^{-1} - F(n)^{-1} < 0$ one has $\frac{\partial f}{\partial \varepsilon} < 0$ as long as $0 < \varepsilon < (F(n)^{-1} - 3F(2n)^{-1})/2$ and $\frac{\partial f}{\partial \varepsilon} > 0$ for $\varepsilon > (F(n)^{-1} - 3F(2n)^{-1})/2$. Altogether, as a function of ε , $f(n, \varepsilon)$ is either increasing on $[0, \infty)$ or it has a single minimum and no maximum in $(0, \infty)$. Hence, for fixed $n \in \mathbb{N}_0$, its maximum in $\varepsilon \in [0, 1]$ is attained at the boundary,

$$\sup_{n \in \mathbb{N}_0, 0 \leq \varepsilon \leq 1} f(n, \varepsilon) = \max(\sup_{n \in \mathbb{N}_0} f(n, 0), \sup_{n \in \mathbb{N}_0} f(n, 1)). \quad (3.32)$$

Since $F_1(n) = F(n)/(1 + F(n)) = (1 + F(n)^{-1})^{-1}$ and $F(n) \geq 4$ for all $n \in \mathbb{N}_0$,

$$f(n, 1) = \frac{F_1(2n)}{F_1(n)^3} = \frac{F(2n)}{1 + F(2n)} (1 + F(n)^{-1})^3 \leq \left(\frac{5}{4}\right)^3 < 2 \quad (3.33)$$

and using $(n+1)^{n+1} \leq F(n) \leq (n+2)^{n+2}$ one sees

$$f(n, 0) = \frac{F(2n)}{F(n)^3} \leq \frac{(2(n+1))^{2(n+1)}}{(n+1)^{3(n+1)}} = \left(\frac{4}{n+1}\right)^{n+1} \leq 4 \quad (3.34)$$

for all $n \in \mathbb{N}_0$. Putting (3.32), (3.33), and (3.34) together and noticing $f(1, 0) = 4$ yields (3.31).

(v): Continuity of $\|\alpha\|_{\mu, \varepsilon, b}$ in $(\mu, \varepsilon) \in [0, 1] \times (0, 1]$: First note that the triangle inequality implies

$$\begin{aligned} \left| \|\alpha\|_{\mu_1, \varepsilon_1, b} - \|\alpha\|_{\mu_2, \varepsilon_2, b} \right| &\leq \sup_{n \geq b} |(F_{\mu_1, \varepsilon_1}(n) - F_{\mu_2, \varepsilon_2}(n))\alpha(n)| \\ &\leq \|\alpha\|_\infty \sup_{n \in \mathbb{N}_0} \left| (F(n)^{-1} + \varepsilon_1)^{-\mu_1} - (F(n)^{-1} + \varepsilon_2)^{-\mu_2} \right| \\ &\leq \|\alpha\|_\infty \sup_{0 \leq x \leq 1/4} \left| (x + \varepsilon_1)^{-\mu_1} - (x + \varepsilon_2)^{-\mu_2} \right| \end{aligned} \quad (3.35)$$

since $4 \leq F(n)$ for all $n \in \mathbb{N}_0$. Let $h(x, \mu, \varepsilon) := (x + \varepsilon)^{-\mu}$. For any $0 < \varepsilon' < 1$, h is continuous on the compact set $[0, 1/4] \times [0, 1] \times [\varepsilon', 1]$ and hence uniformly continuous. Thus for any $\eta > 0$ there exists $\delta > 0$ such that for $(x_j, \mu_j, \varepsilon_j) \in [0, 1] \times [0, 1] \times [\varepsilon', 1]$, $j = 1, 2$ with $|x_1 - x_2|, |\mu_1 - \mu_2|, |\varepsilon_1 - \varepsilon_2| < \delta$ we have $|h(x_1, \mu_1, \varepsilon_1) - h(x_2, \mu_2, \varepsilon_2)| < \eta$. In particular,

$$\sup_{0 \leq x \leq 1/4} |h(x, \mu_1, \varepsilon_1) - h(x, \mu_2, \varepsilon_2)| < \eta$$

which, together with (3.35), shows that $(\mu, \varepsilon) \mapsto \|\alpha\|_{\mu, \varepsilon, b}$ is uniformly continuous on any compact subset of $[0, 1] \times (0, 1]$, hence continuous on $[0, 1] \times (0, 1]$.

(vi): Fix $\mu > 0$. Recall that $\|\alpha\|_{\mu, \varepsilon, b}$ is decreasing in ε . Thus

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \|\alpha\|_{\mu, \varepsilon, b} &= \sup_{0 < \varepsilon \leq 1} \|\alpha\|_{\mu, \varepsilon, b} = \sup_{0 < \varepsilon \leq 1} \sup_{n \geq b} F_{\mu, \varepsilon}(n) |\alpha(n)| \\ &= \sup_{n \geq b} \sup_{0 < \varepsilon \leq 1} F_{\mu, \varepsilon}(n) |\alpha(n)| = \sup_{n \geq b} F_{\mu, 0}(n) |\alpha(n)| = \|\alpha\|_{\mu, 0, b} \end{aligned}$$

which finishes the proof of Lemma 3.6. ■

4. EXISTENCE OF DIFFRACTION MANAGED SOLITONS FOR ZERO AVERAGE DIFFRACTION

Here we want to give a simple proof of existence of diffraction managed solitons, i.e., weak solutions of (1.11), via the direct method from the calculus of variations. This hinges on the fact that the diffraction management equation is the Euler–Lagrange equation for the functional

$$\varphi(f) := \mathcal{Q}(f, f, f, f) \quad (4.1)$$

on $l^2(\mathbb{Z})$. The corresponding constraint maximization problem is given by

$$P_\lambda = \sup \{ \varphi(f) : f \in l^2(\mathbb{Z}), \|f\|_2^2 = \lambda \} \quad (4.2)$$

where $\lambda > 0$. Up to some minor technical details it is clear that any maximizer f of the variational problem (4.2), that is, any $f \in l^2(\mathbb{Z})$ with $\|f\|_2^2 = \lambda$ and

$$\mathcal{Q}(f, f, f, f) = P_\lambda \quad (4.3)$$

is by the Lagrange multiplier theorem a weak solution of the diffraction management equation (1.11).

The usual way to show existence of a maximizer is to identify it as the limit of a suitable maximizing sequence, i.e., a sequence $(f_n)_n$ with $\|f_n\|_2^2 = \lambda$ and $\lim_{n \rightarrow \infty} \varphi(f_n) = P_\lambda$. Such a sequence always exists, the problem, due to the translation invariance of the time evolution $T_t = e^{iD(t)\Delta}$, is that the functional φ is invariant under translation; if f_n is a maximizing sequence for (4.2) and ξ_n any sequence in \mathbb{Z} , then the shifted sequence

$$f_{n,\xi_n}(x) := f_n(x - \xi_n)$$

is also a maximizing sequence. That is, the problem (4.2) is invariant under shifts, yielding a loss of compactness since maximizing sequences can very easily converge weakly to zero. The usual way to overcome this is the use of Lions' concentration compactness method [24] which, for positive average dispersion has been used in [27, 30]. For vanishing average diffraction, and under much more restrictive conditions on the diffraction profile than (1.7), the existence of a maximizer of (4.2) has been shown in [34] with the help of Ekeland's variational principle, [12, 13], see also [17]. We will give a different approach, which avoids the use of Lions' concentration compactness method or Ekeland's variational principle by using the translation invariance of the problem to show that for any maximizing sequence f_n there exists a sequence of shifts ξ_n such that the shifted sequence f_{n,ξ_n} is tight in the sense of (4.4) below. Then, since f_{n,ξ_n} is bounded in $l^2(\mathbb{Z})$, it has a weakly convergent and hence, by the compactness Lemma 4.1 below, also a strongly convergent subsequence. The limit of this subsequence is then a natural candidate for the maximizer of (4.2), see Theorem 4.7.

In the following, we will always assume that the diffraction profile obeys the bound (1.7). Our main tools are the multi-linear bound from Corollary 2.9 and the following simple compactness result.

Lemma 4.1. *Let $1 \leq p < \infty$. A sequence $(f_n)_{n \in \mathbb{N}} \subset l^p(\mathbb{Z}^d)$ is strongly converging to f in $l^p(\mathbb{Z}^d)$ if and only if it is weakly convergent to f and the sequence is tight, i.e.,*

$$\lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{|x| > L} |f_n(x)|^p = 0. \quad (4.4)$$

Proof. For $L \geq 1$, let K_L be the operator of multiplication with the characteristic function of $[-L, L]^d \cap \mathbb{Z}^d$, that is,

$$K_L f(x) = \begin{cases} f(x) & \text{for } |x| \leq L \\ 0 & \text{for } |x| > L \end{cases}$$

and $\overline{K}_L := \mathbf{1} - K_L$. Note that both K_L is an operator with finite dimensional range. In particular, for any $1 \leq L < \infty$, $K_L : l^p(\mathbb{Z}^d) \rightarrow l^p(\mathbb{Z}^d)$ is a compact operator. Furthermore, both K_L and \overline{K}_L are bounded operators with operator norm one since

$$\|f\|_p^p = \|K_L f\|_p^p + \|\overline{K}_L f\|_p^p \geq [\max(\|K_L f\|_p, \|\overline{K}_L f\|_p)]^p$$

for all $f \in l^p(\mathbb{Z}^d)$ and K_L , respectively \overline{K}_L , acts as the identity on its range. Moreover, the tightness-condition (4.4) is equivalent to

$$\lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} \|\overline{K}_L f_n\|_p = 0. \quad (4.5)$$

Now assume that f_n converges strongly to f in $l^p(\mathbb{Z}^d)$. Then it clearly converges also weakly to f and

$$\|\overline{K}_L f_n\|_p \leq \|\overline{K}_L f\|_p + \|\overline{K}_L(f_n - f)\|_p \leq \|\overline{K}_L f\|_p + \|f_n - f\|_p.$$

Thus, by strong convergence of f_n to f ,

$$\limsup_{n \rightarrow \infty} \|\overline{K}_L f_n\|_p \leq \|\overline{K}_L f\|_p.$$

Since $f \in l^p(\mathbb{Z}^d)$ and $p < \infty$, we have $\lim_{L \rightarrow \infty} \|\overline{K}_L f\|_p = 0$, so the sequence f_n is tight.

For the converse assume that f_n converges to f weakly in $l^p(\mathbb{Z}^d)$ and is tight, that is, (4.5) holds. Then

$$\|f - f_n\|_p \leq \|K_L(f - f_n)\|_p + \|\overline{K}_L(f - f_n)\|_p \leq \|K_L(f - f_n)\|_p + \|\overline{K}_L f\|_p + \|\overline{K}_L f_n\|_p.$$

Since K_L is a compact operator on L^p , it maps weakly convergent sequences into strongly converging sequences. Hence

$$\limsup_{n \rightarrow \infty} \|f - f_n\|_p \leq \|\overline{K}_L f\|_p + \limsup_{n \rightarrow \infty} \|\overline{K}_L f_n\|_p$$

Now using $f \in l^p(\mathbb{Z}^d)$ and the condition (4.5), take the limit $L \rightarrow \infty$ in this inequality to get

$$\limsup_{n \rightarrow \infty} \|f - f_n\|_p \leq 0.$$

Thus f_n converges to f strongly in $l^p(\mathbb{Z}^d)$. ■

Remark 4.2. The proof above breaks down for $p = \infty$, since for an arbitrary $f \in l^\infty(\mathbb{Z}^d)$, one does not, in general, have

$$\lim_{L \rightarrow \infty} \|\overline{K}_L f\|_\infty = 0.$$

However, for the closed subspace $l_0^\infty(\mathbb{Z}^d) \subset l^\infty(\mathbb{Z}^d)$ consisting of bounded sequences indexed by \mathbb{Z}^d vanishing at infinity, $\lim_{x \rightarrow \infty} f(x) = 0$ for any $f \in l_0^\infty(\mathbb{Z}^d)$, the above proof immediately generalizes and yields the analogous compactness statement: $(f_n)_{n \in \mathbb{N}}$ converges strongly in $l_0^\infty(\mathbb{Z}^d)$ if and only if it converges weakly and

$$\lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} \sup_{|x| > L} |f_n(x)| = 0.$$

In the next Lemma we gather some simple bounds for the non-linear functional φ and the associated constraint maximization problem (4.2).

Lemma 4.3. *For any $\lambda > 0$ one has $0 < P_\lambda \leq 1$ and the scaling property $P_\lambda = P_1 \lambda^2$ holds. In particular, for any $f \in l^2(\mathbb{Z})$ the bound*

$$\varphi(f) \leq P_1 \|f\|_2^4$$

holds.

Proof. Since $\varphi(f) = \mathcal{Q}(f, f, f, f) = \int_0^1 \|T_t f\|_4^4 dt$ we obviously have $P_\lambda \geq \varphi(f) \geq 0$. With Corollary 2.2, we see $P_\lambda \leq 1$. By scaling, replacing f with $\tilde{f} = f/\sqrt{\lambda}$, one gets

$$\mathcal{Q}(f, f, f, f) = \lambda^2 \mathcal{Q}(\tilde{f}, \tilde{f}, \tilde{f}, \tilde{f})$$

and taking the supremum over f with $\|f\|_2^2 = \lambda$ we see $P_\lambda = P_1 \lambda^2$. If $P_1 = 0$ then $0 = \mathcal{Q}(f, f, f, f) = \int_0^1 \|T_t f\|_4^4 dt$ for all $f \in l^2(\mathbb{Z})$ with $\|f\|_2 = 1$. Thus, for almost every $t \in [0, 1]$ one would have $T_t f(x) = 0$ for all $x \in \mathbb{Z}$. Hence $f = 0$ in contradiction to $\|f\|_2 = 1$. Thus $P_1 > 0$. \blacksquare

The next lemma is key in our proof that maximizing sequences have strongly convergent subsequences modulo translations. Its proof is inspired by the proof of Lemma 2.3 in [20].

Lemma 4.4. *Assume that the diffraction profile obeys the bound (1.7). Then there is a constant C such that if $f \in l^2(\mathbb{Z})$, $\varepsilon^2 < \|f\|_2^2$ and $a, b \in \mathbb{Z}$, $a \leq b$ with*

$$\sum_{x < a} |f(x)|^2 \geq \frac{\varepsilon^2}{2} \text{ and } \sum_{x > b} |f(x)|^2 \geq \frac{\varepsilon^2}{2}, \quad (4.6)$$

then

$$\varphi(f) \leq P_1(\|f\|_2^4 - \varepsilon^4/2) + \frac{C\|f\|_2^4}{((b-a+1)^{1/2} - 1)^{1/2}} \quad (4.7)$$

Proof. The number of points in $[a, b] \cap \mathbb{Z}$ is given by $b - a + 1$. Choose $l \in \mathbb{N}$ such that

$$l \leq (b - a + 1)^{1/2} < l + 1 \quad (4.8)$$

and let I be a subinterval of $\{a, a + 1, \dots, b\}$ consisting of l^2 consecutive points. By pigeonholing, there must be a subset $I_0 = \{a', \dots, b'\} \subset I$ consisting of l consecutive points with

$$\sum_{x \in I_0} |f(x)|^2 \leq \frac{\|f\|_2^2}{l}. \quad (4.9)$$

We split f into three parts, $f_{-1} := f|_{(-\infty, a')}$, $f_1 := f|_{(b', \infty)}$, and $f_0 := f|_{I_0}$. Then $f = f_{-1} + f_0 + f_1$, $\|f\|_2^2 = \|f_{-1}\|_2^2 + \|f_0\|_2^2 + \|f_1\|_2^2$, and (4.6) and (4.9) give

$$\|f_{-1}\|_2^2 \geq \frac{\varepsilon^2}{2}, \quad \|f_1\|_2^2 \geq \frac{\varepsilon^2}{2}, \quad \text{and } \|f_0\|_2 \leq \frac{\|f\|_2}{\sqrt{l}}. \quad (4.10)$$

Using the multi-linearity of \mathcal{Q} ,

$$\begin{aligned}
\mathcal{Q}(f, f, f, f) &= \sum_{\substack{j_l \in \{-1, 0, 1\} \\ l=1, \dots, 4}} \mathcal{Q}(f_{j_1}, f_{j_2}, f_{j_3}, f_{j_4}) \\
&= \sum_{\substack{j_l \in \{-1, 1\} \\ l=1, \dots, 4}} \mathcal{Q}(f_{j_1}, f_{j_2}, f_{j_3}, f_{j_4}) + \sum_{\substack{j_l \in \{-1, 0, 1\} \\ l=1, \dots, 4 \\ \text{some } j_l=0}} \mathcal{Q}(f_{j_1}, f_{j_2}, f_{j_3}, f_{j_4}) \\
&= \mathcal{Q}(f_{-1}, f_{-1}, f_{-1}, f_{-1}) + \mathcal{Q}(f_1, f_1, f_1, f_1) \\
&+ \sum_{\substack{j_l \in \{-1, 1\} \\ \exists k, l: j_k = -1, j_l = 1}} \mathcal{Q}(f_{j_1}, f_{j_2}, f_{j_3}, f_{j_4}) + \sum_{\substack{j_l \in \{-1, 0, 1\} \\ l=1, \dots, 4 \\ \text{some } j_l=0}} \mathcal{Q}(f_{j_1}, f_{j_2}, f_{j_3}, f_{j_4}).
\end{aligned} \tag{4.11}$$

Lemma 4.3 shows

$$\mathcal{Q}(f_{-1}, f_{-1}, f_{-1}, f_{-1}) + \mathcal{Q}(f_1, f_1, f_1, f_1) = \varphi(f_{-1}) + \varphi(f_1) \leq P_1(\|f_{-1}\|_2^4 + \|f_1\|_2^4) \tag{4.12}$$

and, utilizing that the supports of f_{-1} and f_1 have distance at least $l+1 \geq 2$, the enhanced multi-linear bound (2.25) with the choice $\delta = 1/4$ gives

$$\sum_{\substack{j_l \in \{-1, 1\} \\ \exists k, l: j_k = -1, j_l = 1}} |\mathcal{Q}(f_{j_1}, f_{j_2}, f_{j_3}, f_{j_4})| \lesssim \frac{1}{(l+1)^{\delta(l+1)}} \|f\|_2^4 \leq \frac{\|f\|_2^4}{\sqrt{l}}. \tag{4.13}$$

Also,

$$\sum_{\substack{j_l \in \{-1, 0, 1\} \\ l=1, \dots, 4 \\ \text{some } j_l=0}} |\mathcal{Q}(f_{j_1}, f_{j_2}, f_{j_3}, f_{j_4})| \leq \sum_{\substack{j_l \in \{-1, 0, 1\} \\ l=1, \dots, 4 \\ \text{some } j_l=0}} \prod_{l=1}^4 \|f_{j_l}\|_2 \leq \frac{\|f\|_2^4}{\sqrt{l}} \tag{4.14}$$

by the a-priori-bound on \mathcal{Q} given in Corollary 2.2 and (4.10) since each term in the above sum contains at least one factor $\|f_0\|_2$. Thus (4.11) together with (4.12), (4.13), and (4.14) gives

$$\varphi(f) = \mathcal{Q}(f, f, f, f) \leq P_1(\|f_{-1}\|_2^4 + \|f_1\|_2^4) + \frac{C}{\sqrt{l}} \|f\|_2^4 \tag{4.15}$$

for some constant C . Since $\|f_{-1}\|_2^2, \|f_1\|_2^2 \geq \varepsilon^2/2$ and $\|f_{-1}\|_2^2 + \|f_1\|_2^2 \leq \|f\|_2^2$, we have

$$\|f_{-1}\|_2^4 + \|f_1\|_2^4 \leq \|f\|_2^4 - 2\|f_{-1}\|_2^2 \|f_1\|_2^2 \leq \|f\|_2^4 - \varepsilon^4/2.$$

Together with (4.8) the bound (4.15) yields (4.7). \blacksquare

Lemma 4.5. *Assume that the diffraction profile obeys the bound (1.7) and let $(f_n)_{n \in \mathbb{N}}$ be a maximizing sequence for the constraint maximization problem (4.2). Then there exists a sequence $(\xi_n)_{n \in \mathbb{N}}$ and, for each $0 < \varepsilon < \sqrt{\lambda}$, there exists R_ε , which is independent of n , such that*

$$\limsup_{n \rightarrow \infty} \sum_{|x - \xi_n| > R_\varepsilon} |f_n(x)|^2 \leq \varepsilon^2. \tag{4.16}$$

Proof. Given $0 < \varepsilon < \sqrt{\lambda}$ as in the Lemma, we will show that there are corresponding intervals $I_{n,\varepsilon} = \{c_{n,\varepsilon}, \dots, d_{n,\varepsilon}\}$ which are nested,

$$I_{n,\varepsilon} \subset I_{n,\varepsilon'} \quad \text{for all } 0 < \varepsilon' \leq \varepsilon, \quad (4.17)$$

have bounded length,

$$\sup_{n \in \mathbb{N}} (d_{n,\varepsilon} - c_{n,\varepsilon}) < \infty, \quad (4.18)$$

and contain most of the l^2 norm of f_n ,

$$\limsup_{n \rightarrow \infty} \sum_{x \notin I_{n,\varepsilon}} |f_n(x)|^2 \leq \varepsilon^2. \quad (4.19)$$

Given (4.17), (4.18), and (4.19), ξ_n and R_ε can be constructed as follows: Fix some $0 < \varepsilon_0 < \sqrt{\lambda}$. For each $n \in \mathbb{N}$ define ξ_n by simply choosing some point from the set I_{n,ε_0} and define R_ε by

$$R_\varepsilon := \sup_{n \in \mathbb{N}} (d_{n,\varepsilon} - c_{n,\varepsilon})$$

for $\varepsilon \leq \varepsilon_0$ and $R_\varepsilon = R_{\varepsilon_0}$ for $\varepsilon_0 < \varepsilon < \sqrt{\lambda}$. The bound (4.18) guarantees that R_ε is finite for all $0 < \varepsilon < \sqrt{\lambda}$. Since the intervals $I_{n,\varepsilon}$ are nested in ε , the point $\xi_n \in I_{n,\varepsilon}$ for any $0 < \varepsilon \leq \varepsilon_0$. In particular, $I_{n,\varepsilon} \subset \{x : |x - \xi_n| \leq R_\varepsilon\}$ for all $0 < \varepsilon \leq \varepsilon_0$ by definition of R_ε . Moreover, again since the sets $I_{n,\varepsilon}$ are nested, $I_{n,\varepsilon} \subset I_{n,\varepsilon_0} \subset \{x : |x - \xi_n| \leq R_\varepsilon\}$ for all $\varepsilon_0 < \varepsilon < \sqrt{\lambda}$, too.

Putting everything together, (4.19) shows that with this choice of ξ_n and R_ε the bound (4.16) holds. Thus it is enough to prove (4.17), (4.18), and (4.19): Let

$$\begin{aligned} a_{n,\varepsilon} &:= \min \left(z \in \mathbb{Z} : \sum_{x < z} |f_n(x)|^2 \geq \frac{\varepsilon^2}{2} \right) \\ b_{n,\varepsilon} &:= \max \left(z \in \mathbb{Z} : \sum_{x > z} |f_n(x)|^2 \geq \frac{\varepsilon^2}{2} \right). \end{aligned} \quad (4.20)$$

Both exist and are finite, since $f_n \in l^2(\mathbb{Z})$. Put $c_{n,\varepsilon} = \min(a_{n,\varepsilon}, b_{n,\varepsilon}) - 1$ and $d_{n,\varepsilon} = \max(a_{n,\varepsilon}, b_{n,\varepsilon}) + 1$. With this choice (4.17) and (4.19) certainly hold and we only have to check (4.18), which is where Lemma 4.4 enters.

Either $a_{n,\varepsilon} > b_{n,\varepsilon}$, in which case $d_{n,\varepsilon} - c_{n,\varepsilon} \leq 2$, or Lemma 4.4 applies and gives the bound

$$P_1 \frac{\varepsilon^2}{2} - (P_\lambda - \varphi(f_n)) \leq \frac{C \|f_n\|_2^4}{\left((d_{n,\varepsilon} - c_{n,\varepsilon} - 1)^{1/2} - 1 \right)^{1/2}} \quad (4.21)$$

where we rearranged (4.7) a bit and used the scaling $P_1 \|f_n\|_2^4 = P_1 \lambda^2 = P_\lambda$. Since f_n is a maximizing sequence for (4.2) we have $\lim_{n \rightarrow \infty} \varphi(f_n) = P_\lambda$. Thus taking the limit $n \rightarrow \infty$ in (4.21) gives

$$P_1 \frac{\varepsilon^2}{2} \leq \liminf_{n \rightarrow \infty} \frac{C \lambda^2}{\left((d_{n,\varepsilon} - c_{n,\varepsilon} - 1)^{1/2} - 1 \right)^{1/2}}$$

or

$$\limsup_{n \rightarrow \infty} (d_{n,\varepsilon} - c_{n,\varepsilon}) \leq \left[\left(\frac{2C\lambda^2}{P_1\varepsilon^2} \right)^2 + 1 \right]^2 + 1 < \infty.$$

which proves (4.18) and hence the Lemma. \blacksquare

A simple reformulation of Lemma 4.5 is

Corollary 4.6. *Assume that the diffraction profile obeys the bound (1.7) and let $\lambda > 0$. Then, given any maximizing sequence $(f_n)_n$ of the variational problem (4.2), there exists a sequence of translations ξ_n such that the translated sequence f_{n,ξ_n} with $f_{n,\xi_n}(x) := f_n(x - \xi_n)$ is tight, that is,*

$$\lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{|x| > L} |f_{n,\xi_n}(x)|^2 = 0. \quad (4.22)$$

Moreover, this shifted sequence is still a maximizing sequence for the variational problem (4.2).

Proof. By Lemma 4.5, the sequence f_{n,ξ_n} is certainly tight. On the other hand it is also a maximizing sequence for 4.2, since the time evolution $T_t = e^{iD(t)\Delta}$ commutes with translation, $T_t f_{n,\xi_n}(x) = T_t f_n(x - \xi_n)$ for all x , and hence

$$\varphi(f_{n,\xi_n}) = \mathcal{Q}(f_{n,\xi_n}, f_{n,\xi_n}, f_{n,\xi_n}, f_{n,\xi_n}) = \mathcal{Q}(f_n, f_n, f_n, f_n) = \varphi(f_n)$$

for all $n \in \mathbb{N}$. \blacksquare

Theorem 4.7 (= Theorem 1.3). *Let $\lambda > 0$ and assume that the diffraction profile obeys the bound (1.7). Then there is an $f \in l^2(\mathbb{Z})$ with $\|f\|_2^2 = \lambda$ such that*

$$\varphi(f) = P_\lambda := \sup_{g: \|g\|_2^2 = \lambda} \varphi(g).$$

This maximizer f is also a weak solution of the dispersion management equation

$$\omega f = Q(f, f, f).$$

where $\omega = P_\lambda/\lambda > 0$ is the Lagrange-multiplier.

Proof. Let $\lambda > 0$ and $(f_n)_n$ be a maximizing sequence for (4.2) with $\|f_n\|_2^2 = \lambda$. By Corollary 4.6, we can, without loss of generality, assume that this maximizing sequence is already tight in the sense of equation (4.22).

Since the unit ball in $l^2(\mathbb{Z})$ is weakly compact, there is a subsequence $(f_{n_j})_{j \in \mathbb{N}}$ of $(f_n)_{n \in \mathbb{N}}$ which converges weakly to some $f \in l^2(\mathbb{Z})$. From Lemma 4.1 we know that f_{n_j} converges even strongly to f in $l^2(\mathbb{Z})$. Thus $\|f\|_2^2 = \lambda$ and hence f is a good candidate for the maximizer. To conclude that f is a maximizer for the variational problem, we need to show that $\varphi(f) = P_\lambda$. Since $\|f\|_2^2 = \lambda$ one certainly has

$$\varphi(f) \leq P_\lambda = \lim_{n \rightarrow \infty} \varphi(f_n),$$

so one only needs upper semi-continuity of φ at f , i.e.,

$$\limsup_{n \rightarrow \infty} \varphi(f_{n_j}) \leq \varphi(f). \quad (4.23)$$

By Lemma 4.8 below, the map $f \mapsto \varphi(f)$ is even continuous on $l^2(\mathbb{Z})$, in particular, (4.23) is true which finished the proof that the variational problem (4.2) has a maximizer.

The proof that the above maximizer is a weak solution of the associated Euler–Lagrange equation (1.11) is standard in the calculus of variations, we sketch it for the convenience of the reader: Lemma 4.9 below shows that the derivative of the functional φ at any $f \in l^2(\mathbb{Z})$ is given by the linear map $D\varphi(f)[h] = 4\operatorname{Re}\mathcal{Q}(h, f, f, f)$. Similarly, one can check that the derivative of $\psi(f) = \|f\|_2^2 = \langle f, f \rangle$ is given by $D\psi(f)[h] = 2\operatorname{Re}\langle h, f \rangle$. Note that although, in our convention for the inner product, the map $h \mapsto \langle h, f \rangle$ is anti-linear, the map $h \mapsto \operatorname{Re}\langle h, f \rangle$ is linear. Similarly, one easily checks that for fixed f the map $h \mapsto \operatorname{Re}\mathcal{Q}(h, f, f, f)$ is linear.

Now let f be any maximizer of the constraint variational problem (4.2) and $h \in l^2(\mathbb{Z})$ arbitrary. Define, for any $(s, t) \in \mathbb{R}^2$,

$$\begin{aligned} F(s, t) &:= \varphi(f + sf + th), \\ G(s, t) &:= \psi(f + sf + th). \end{aligned}$$

Note that

$$\begin{aligned} \nabla F(s, t) &= \begin{pmatrix} D\varphi(f + sf + th)[f] \\ D\varphi(f + sf + th)[h] \end{pmatrix} \\ &= 4 \begin{pmatrix} \operatorname{Re}\mathcal{Q}(f, f + sf + th, f + sf + th, f + sf + th) \\ \operatorname{Re}\mathcal{Q}(h, f + sf + th, f + sf + th, f + sf + th) \end{pmatrix} \end{aligned}$$

and

$$\nabla G(s, t) = \begin{pmatrix} D\psi(f + sf + th)[f] \\ D\psi(f + sf + th)[h] \end{pmatrix} = 2 \begin{pmatrix} \operatorname{Re}\langle f, f + sf + th \rangle \\ \operatorname{Re}\langle h, f + sf + th \rangle \end{pmatrix}.$$

Since $\langle f, f \rangle = \lambda \neq 0$,

$$\nabla G(0, 0) = 2 \begin{pmatrix} \langle f, f \rangle \\ \operatorname{Re}\langle h, f \rangle \end{pmatrix}$$

is not the zero vector in \mathbb{R}^2 and since $\nabla G(s, t)$ depends multi-linearly, in particular continuously, on (s, t) , the implicit function theorem [36] shows that there exists an open interval $I \subset \mathbb{R}$ containing 0 and a differentiable function ϕ on I with $\phi(0) = 0$ such that

$$\lambda = \|f\|_2^2 = G(0, 0) = G(\phi(t), t)$$

for all $t \in I$. Consider the function $I \ni t \mapsto F(\phi(t), t)$. Since f is a maximizer for the constraint variational problem (4.2), $F(\phi(t), t)$ has a local maximum at $t = 0$. Hence, using the chain rule,

$$0 = \left. \frac{dF(\phi(t), t)}{dt} \right|_{t=0} = \nabla F(0, 0) \cdot \begin{pmatrix} \phi'(0) \\ 1 \end{pmatrix} = 4\mathcal{Q}(f, f, f, f)\phi'(0) + 4\operatorname{Re}\mathcal{Q}(h, f, f, f)$$

Since $\lambda = G(\phi(t), t)$, the chain rule also yields

$$0 = \left. \frac{dG(\phi(t), t)}{dt} \right|_{t=0} = \nabla G(0, 0) \cdot \begin{pmatrix} \phi'(0) \\ 1 \end{pmatrix} = 2\langle f, f \rangle\phi'(0) + 2\operatorname{Re}\langle h, f \rangle.$$

Solving this for $\phi'(0)$ and plugging it back into the expression for the derivative of F , we see that

$$\frac{\mathcal{Q}(f, f, f, f)}{\langle f, f \rangle} \operatorname{Re}\langle h, f \rangle = \operatorname{Re}\mathcal{Q}(h, f, f, f).$$

In other words, with $\omega := \mathcal{Q}(f, f, f, f)/\langle f, f \rangle = P_\lambda/\lambda > 0$ and f any maximizer of (4.2), we have

$$\operatorname{Re}(\omega\langle h, f \rangle) = \operatorname{Re}\mathcal{Q}(h, f, f, f) \quad (4.24)$$

for any $h \in l^2(\mathbb{Z})$. Replacing h by ih in (4.24), one gets

$$\operatorname{Im}(\omega\langle h, f \rangle) = \operatorname{Im}\mathcal{Q}(h, f, f, f) \quad (4.25)$$

for all $h \in l^2(\mathbb{Z})$. (4.24) and (4.25) together show

$$\omega\langle h, f \rangle = \mathcal{Q}(h, f, f, f)$$

for any $h \in l^2(\mathbb{Z})$, that is, f is a weak solution of the diffraction management equation (1.11). \blacksquare

In the proof of Theorem 4.7, we needed the following two Lemmata.

Lemma 4.8. *The map $f \mapsto \varphi(f) = \mathcal{Q}(f, f, f, f)$ is locally Lipschitz continuous on $l^2(\mathbb{Z})$.*

Proof. Given f, g , one has

$$\begin{aligned} \mathcal{Q}(f, f, f, f) - \mathcal{Q}(g, g, g, g) &= \int_0^1 \|T_t f\|_4^4 - \|T_t g\|_4^4 dt \\ &= \int_0^1 \sum_{j=0}^3 \|T_t f\|_4^{3-j} (\|T_t f\|_4 - \|T_t g\|_4) \|T_t g\|_4^j dt. \end{aligned} \quad (4.26)$$

The above together with the triangle inequality, the bound $\|h\|_4 \leq \|h\|_2$ for all $h \in l^2(\mathbb{Z})$, see Lemma 2.1, and the unicity of T_t on $l^2(\mathbb{Z})$ gives

$$\begin{aligned} |\mathcal{Q}(f, f, f, f) - \mathcal{Q}(g, g, g, g)| &\leq \int_0^1 \sum_{j=0}^3 \|T_t f\|_2^{3-j} \|\|T_t f\|_2 - \|T_t g\|_2\| \|T_t g\|_2^j dt \\ &\leq \int_0^1 \sum_{j=0}^3 \|T_t f\|_2^{3-j} \|T_t(f - g)\|_2 \|T_t g\|_2^j dt \\ &\leq \sum_{j=0}^3 \|f\|_2^{3-j} \|f - g\|_2 \|g\|_2^j \\ &\leq 4 \max(1, \|f\|_2^3, \|g\|_2^3) \|f - g\|_2. \end{aligned} \quad (4.27)$$

We need one more technical result, about the differentiability of the non-linear functional φ .

Lemma 4.9. *The functional φ is continuously differentiable with derivative $D\varphi(f)[h] = 4\operatorname{Re}\mathcal{Q}(h, f, f, f)$*

Proof. Using the multi-linearity of \mathcal{Q} , one can check that for any $f, h \in l^2(\mathbb{Z})$

$$\begin{aligned}\varphi(f+h) &= \varphi(f) + \mathcal{Q}(f, f, f, h) + \mathcal{Q}(f, f, h, f) + \mathcal{Q}(f, h, f, f) + \mathcal{Q}(h, f, f, f) + O(\|h\|_2^2) \\ &= \varphi(f) + 4\operatorname{Re}\mathcal{Q}(h, f, f, f) + O(\|h\|_2^2)\end{aligned}\quad (4.28)$$

where in the term $O(\|h\|_2^2)$ we gathered expressions of the form $\mathcal{Q}(h, f, f, h)$, and $\mathcal{Q}(h, f, f, f+h)$, and similar, which by Corollary 2.2 are bounded by $C\|h\|_2^2$. This shows that φ is differentiable with derivative $D\varphi(f)[h] = 4\operatorname{Re}\mathcal{Q}(h, f, f, f)$. Moreover,

$$\begin{aligned}D\varphi(f)[h] - D\varphi(g)[h] &= 4\operatorname{Re}(\mathcal{Q}(h, f, f, f) - \mathcal{Q}(h, g, g, g)) \\ &= 4\operatorname{Re}(\mathcal{Q}(h, f-g, f, f) + \mathcal{Q}(h, g, f-g, f) + \mathcal{Q}(h, g, g, f-g)).\end{aligned}$$

Hence using the bound from Corollary 2.2 again, we see

$$\sup_{\|h\|_2 \leq 1} |D\varphi(f)[h] - D\varphi(g)[h]| \leq 4(\|f\|_2^2 + \|f\|_2\|g\|_2 + \|g\|_2^2)\|f-g\|_2 \quad (4.29)$$

which shows that the derivative $D\varphi$ is even locally Lipschitz continuous. \blacksquare

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