ON THE NUMBER OF BOUND STATES FOR SCHRÖDINGER OPERATORS WITH OPERATOR-VALUED POTENTIALS

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Abstract. Cwikel's bound is extended to an operator-valued setting. One application of this result is a semi-classical bound for the number of negative bound states for Schrödinger operators with operator-valued potentials. We recover Cwikel's bound for the Lieb–Thirring constant $L_{0,3}$ which is far worse than the best available by Lieb (for scalar potentials). However, it leads to a uniform bound (in the dimension $d \geq 3$) for the quotient $L_{0,d}/L^{\mathrm{cl}}_{0,d}$, where $L^{\mathrm{cl}}_{0,d}$ is the so-called classical constant. This gives some improvement in large dimensions.

1. Introduction

The Lieb-Thirring inequalities bound certain moments of the negative eigenvalues of a one-particle Schrödinger operator by the corresponding classical phase space moment. More precisely, for "nice enough" potentials one has

$$\mathrm{tr}_{L^2(\mathbb{R}^d)}(-\Delta + V)_-^\gamma \leq \frac{C_{\gamma,d}}{(2\pi)^d} \iint\limits_{\mathbb{R}^d\mathbb{R}^d} d\xi dx\, (\xi^2 + V(x))_-^\gamma. \tag{1}$$

Here and in the following, $(x)_- = \frac{1}{2}(|x| - x)$ is the negative part of a real number or a self-adjoint operator. Doing the ξ integration explicitly with the help of scaling the above inequality is equivalent to its more often used form

$$\mathrm{tr}_{L^2(\mathbb{R}^d)}(-\Delta + V)_-^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} dx\, V(x)_-^{\gamma+d/2}, \tag{2}$$

where the Lieb-Thirring constant $L_{\gamma,d}$ is given by $L_{\gamma,d} = C_{\gamma,d} L^{\mathrm{cl}}_{\gamma,d}$ with the classical Lieb-Thirring constant

$$L^{\mathrm{cl}}_{\gamma,d} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} dp (1 - p^2)_+^\gamma. \tag{3}$$

This integral is, of course, explicitly given by a quotient of Gamma functions, but we will have no need for this. The Lieb-Thirring inequalities are valid as soon as the potential V is in $L^{\gamma+d/2}(\mathbb{R}^d)$.

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These inequalities are important tools in the spectral theory of Schrödinger operators and they are known to hold if and only if $\gamma \geq \frac{1}{2}$ if $d = 1$, $\gamma > 0$ if $d = 2$, and $\gamma \geq 0$ if $d \geq 3$. The bound for the critical case $\gamma = 0$, that is, the bound for the number of negative eigenvalues of a Schrödinger operator in three or more dimensions is the celebrated Cwikel-Lieb-Rozenblum bound [6, 17, 21]. Later, different proofs for this were given by Conlon and Li and Yau [5, 16]. The remaining case $\gamma = \frac{1}{2}$ in $d = 1$ was settled in [29]. The well-known Weyl asymptotic formula

$$\lim_{\lambda \to \infty} \operatorname{tr}(-\Delta + \lambda V)_-^\gamma = L^{\mathrm{cl}}_{\gamma,d} \int dx\, V(x)_-^{\gamma+d/2}$$

immediately gives the lower bound $C_{\gamma,d} \geq 1$. There are certain refined lower bounds [20, 9] for small values of γ. In particular, one always has $C_{\gamma,d} > 1$ for $\gamma < 1$; see [9]. In one dimension this even happens for $\gamma < 3/2$, and in two dimensions, one always has $C_{1,2} > 1$ [20].

Depending on the dimension there are certain conjectures for the optimal value of the constants in these inequalities [19, 20]. One part of the conjectures on the Lieb-Thirring constants is that, indeed, $C_{\gamma,d} = 1$ for $d \geq 3$ and moments $\gamma \geq 1$. For the physically most important case $\gamma = 1$, $d = 3$ this would imply, via a duality argument, that the kinetic energy of fermions is bounded below by the Thomas-Fermi ansatz for the kinetic energy, which in turn has certain consequences for the energy of large quantum Coulomb systems [17, 19].

Laptev and Weidl [14] realized that a, at first glance, purely technical extension of the Lieb-Thirring inequality from scalar to operator-valued potentials already suggested in [12] is a key in proving at least a part of the Lieb-Thirring conjecture. It allowed them to show that $C_{\gamma,d} = 1$ for all $d \in \mathbb{N}$ as long as $\gamma \geq 3/2$. To prove this they considered Schrödinger operators of the form $-\Delta \otimes \mathbf{1}_{\mathcal{G}} + V$ on the Hilbert space $L^2(\mathbb{R}^d, \mathcal{G})$ where V now is an operator-valued potential with values $V(x)$ in the bounded self-adjoint operators on the auxiliary Hilbert space $\mathcal{G}$. In this case the Lieb-Thirring inequalities (1) and (2) are modified to

$$\operatorname{tr}_{L^2(\mathbb{R}^d,\mathcal{G})}(-\Delta \otimes \mathbf{1}_{\mathcal{G}} + V)_-^\gamma \leq \frac{C_{\gamma,d}}{(2\pi)^d} \iint_{\mathbb{R}^d \mathbb{R}^d} dp dx\, \operatorname{tr}_{\mathcal{G}}(p^2 + V(x))_-^\gamma, \tag{4}$$

or, again doing the ξ integral explicitly with the help of the spectral theorem and scaling

$$\operatorname{tr}_{L^2(\mathbb{R}^d,\mathcal{G})}(-\Delta \otimes \mathbf{1}_{\mathcal{G}} + V)_-^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} dx\, \operatorname{tr}_{\mathcal{G}}(V(x)_-^{\gamma+d/2}). \tag{5}$$

Here we abused the notation slightly in using the same symbol for the constants as in the scalar case. But in the following, we will only consider the operator-valued case anyway. Laptev and Weidl realized that this extension of the Lieb-Thirring inequality gives rise to the possibility of an inductive proof for $C_{3/2,d} = 1$ as long as one has the a priori information $C_{3/2,1} = 1$ for operator-valued potentials. This idea together with ideas in [11] was then later used in [10] to prove improved bounds on $C_{\gamma,d}$ in the range $1/2 \leq \gamma \leq 3/2$; in particular, it was shown that $C_{1,d} \leq 2$ uniformly in $d \in \mathbb{N}$.

Unlike the scalar case, however, the range of parameters γ and d for which (4) or equivalently (5) holds is not known. The results in [10] only show that these inequalities are true for $\gamma \geq 1/2$ and all $d \in \mathbb{N}$. This shortcoming has to do with the way the Lieb-Thirring estimates are proven for operator-valued potentials: First, the estimate is shown to hold in one dimension. Then a suitable induction proof, using the one-dimensional result, is set up to prove the full result in all dimensions. This turns out to give good estimates for the coefficients $C_{\gamma,d}$ in the Lieb-Thirring inequality, for example, they are independend of the dimension. However, moments below $1/2$ cannot be addressed with this method, since the a priori estimate fails already for scalar potentials.

This led Ari Laptev [13], see also [15], to ask the question whether, in particular, the Cwikel-Lieb-Rozenblum estimate holds for Schrödinger operators with operator-valued potentials. In this note we answer his question affirmatively, that is, the Lieb-Thirring inequalities for operator-valued potentials are shown to hold also for $\gamma = 0$ as long as $d \geq 3$ and then, by a monotonicity argument also for all $\gamma \geq 0$. More precisely, we want to show that Cwikel's proof of the Cwikel-Lieb-Rozenblum bound can be adapted to the operator-valued setting. However, the bound for $C_{0,d}$ is far from being optimal since we use Cwikel's approach. But, nevertheless, reasoning similar to Laptev and Weidl, any a priori bound on $C_{0,3}$ implies the bound $C_{0,d} \leq C_{0,3}$ for $d \geq 3$, thus giving a *uniform* bound in the dimension, whereas the best available bound in the scalar case due to Lieb [17] grows like $\sqrt{\pi d}$, see [20].

2. Statement of the results

Let $\mathcal{G}$ be a (separable) Hilbert space with norm $\|.\|_{\mathcal{G}}$, scalar product $\langle ., . \rangle_{\mathcal{G}}$, and let $\mathbf{1}_{\mathcal{G}}$ be the identity operator on $\mathcal{G}$. We follow the convention that scalar products are linear in the second component. Furthermore, $\mathcal{B}(\mathcal{G})$ is the Banach space of bounded operators equipped with the operator norm $\|.\|_{\mathcal{B}(\mathcal{G})}$ and $\mathcal{K}(\mathcal{G})$ the (separable) ideal of the compact operators on $\mathcal{G}$. For a compact operator $A \in \mathcal{K}(\mathcal{G})$, the singular values $\mu_n(A)$, $n \in \mathbb{N}$ are the eigenvalues of $|A| := (A^*A)^{1/2}$ arranged in decreasing order counting multiplicity. A^* is the adjoint of A. $\mathcal{S}^q(\mathcal{G})$ denotes the ideal of compact operators $A \in \mathcal{K}(\mathcal{G})$ whose singular values are q-summable, that is, $\sum_n \mu_n(A)^q < \infty$. In particular, $\mathcal{S}^1(\mathcal{G})$ and $\mathcal{S}^2(\mathcal{G})$ are the trace class and Hilbert-Schmidt operators on $\mathcal{G}$. We will often write $\mathcal{B}$, $\mathcal{K}$, and $\mathcal{S}^q$ if there is no ambiguity. Of course, $A \in \mathcal{S}^q$ if and only if $\mathrm{tr}_{\mathcal{G}}(|A|^q) = \mathrm{tr}_{\mathcal{G}}((A^*A)^{q/2}) < \infty$, where $\mathrm{tr}_{\mathcal{G}}$ is the trace on $\mathcal{G}$.

The Hilbert space $L^2(\mathbb{R}^d, \mathcal{G})$ is the space of all measurable functions $\phi : \mathbb{R}^d \to \mathcal{G}$ such that

$$\|\psi\|^2_{L^2(\mathbb{R}^d,\mathcal{G})} := \int_{\mathbb{R}^d} dx \; \|\psi(x)\|^2_{\mathcal{G}} < \infty$$

and the Sobolev space $H^1(\mathbb{R}^d,\mathcal{G})$ consists of all functions $\psi\in L^2(\mathbb{R}^d,\mathcal{G})$ with finite norm

$$\|\psi\|^2_{H^1(\mathbb{R}^d,\mathcal{G})} := \sum_{l=1}^d \|\partial_l\psi\|^2_{L^2(\mathbb{R}^d,\mathcal{G})} + \|\psi\|^2_{L^2(\mathbb{R}^d,\mathcal{G})}\,.$$

As in the scalar case, the quadratic form

$$h_0(\psi,\psi) := \sum_{l=1}^d \|\partial_l\psi\|^2_{L^2(\mathbb{R}^d,\mathcal{G})}$$

is closed in $L^2(\mathbb{R}^d,\mathcal{G})$ on the domain $H^1(\mathbb{R}^d,\mathcal{G})$. Naturally, this form corresponds to the Laplacian $-\Delta\otimes\mathbf{1}_\mathcal{G}$ on $L^2(\mathbb{R}^d,\mathcal{G})$.

$L^q(\mathbb{R}^d,\mathcal{B}(\mathcal{G}))$ is the space of operator-valued functions $f\colon\mathbb{R}^d\to\mathcal{B}(\mathcal{G})$ with finite norm

$$\|f\|^q_q = \|f\|^q_{L^q(\mathbb{R}^d,\mathcal{B}(\mathcal{G}))} := \int_{\mathbb{R}^d} dx\ \|f(x)\|^q_{\mathcal{B}(\mathcal{G})}$$

and $L^q(\mathbb{R}^d,\mathcal{S}^r(\mathcal{G}))$ the space of operator-valued functions f whose norm

$$\|f\|^q_{q,r} = \|f\|^q_{L^q(\mathbb{R}^d,\mathcal{S}^r(\mathcal{G}))} := \int_{\mathbb{R}^d} dx\,\mathrm{tr}_\mathcal{G}(|f(x)|^r)^{q/r}$$

is finite. A potential is a function $V\in L^q(\mathbb{R}^d,\mathcal{B}(\mathcal{G}))$ such that $V(x)$ is a symmetric operator for almost every $x\in\mathbb{R}^d$. If

$$q\ge 1 \text{ for } d=1,\quad q>1 \text{ for } d=2,\quad \text{and } q\ge d/2 \text{ for } d\ge 3 \tag{6}$$

one sees, using Sobolev embedding theorems as in the scalar case, that the real-valued quadratic form

$$v[\psi,\psi] := \int_{\mathbb{R}^d} dx\,\langle\psi(x),V(x)\psi(x)\rangle_\mathcal{G}$$

is infinitesimally form-bounded with respect to h_0. Hence the form sum

$$h[\psi,\psi] := h_0[\psi,\psi] + v[\psi,\psi]$$

is closed and semi-bounded from below on $H^1(\mathbb{R}^d,\mathcal{G})$ and thus generates the self-adjoint operator

$$H = -\Delta\otimes\mathbf{1}_\mathcal{G} + V$$

on $L^2(\mathbb{R}^d,\mathcal{G})$ by the KLMN theorem [23]. It is easy to see that any potential $V\in L^q(\mathbb{R}^d,\mathcal{B}(\mathcal{G}))$ satisfying (6) for which $V(x)\in\mathcal{K}(\mathcal{G})$ for almost every $x\in\mathbb{R}^d$ is relatively form compact with respect to h_0. Hence by Weyl's theorem for such potentials, the negative eigenvalues $E_0\le E_1\le E_3\le\cdots\le 0$ are at most a countable set with accumulation point zero and their eigenspaces are finite-dimensional. In particular, this is the case for potentials $V\in L^q(\mathbb{R}^d,\mathcal{S}^r(\mathcal{G}))$.

Our first result is a generalized version of a basic observation of Laptev and Weidl: The two versions (4) and (5) of the Lieb-Thirring inequality give rise to two different monotonicity properties of $C_{\gamma,d}$ in d.

Theorem 2.1 (Sub-multiplicativity of $\mathbf{C}_{\gamma,\mathbf{d}}$)**.** *If, for dimensions n and $d-n$, the Lieb-Thirring inequality holds for operator-valued potentials then it also holds in dimension d. Moreover,*

$$C_{\gamma,d} \leq C_{\gamma,n} C_{\gamma,d-n} \qquad \textit{and} \tag{7}$$

$$C_{\gamma,d} \leq C_{\gamma,n} C_{\gamma+n/2,d-n}. \tag{8}$$

Remarks 2.2. i) In the scalar case Aizenman and Lieb [1] showed that the map $\gamma \to C_{\gamma,d} = L_{\gamma,d}/L^{\mathrm{cl}}_{\gamma,d}$ is decreasing. This monotonicity holds also in the general case, so, in fact, (8) implies (7). The monotonicity in γ is most easily seen in the phase space picture: By scaling one has, for $\gamma > \gamma_0 \geq 0$,

$$\int_0^\infty (s+t)_-^{\gamma_0}\, t^{\gamma-\gamma_0-1} dt = (s)_-^{\gamma}\, B(\gamma-\gamma_0, \gamma_0+1),$$

where $B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt$ is the Beta function. In other words, for each choice of $\gamma > \gamma_0 \geq 0$ there exists a positive measure μ on $\mathbb{R}_+$ with $(s)_-^\gamma = \int_{\mathbb{R}_+} (s+t)_-^{\gamma_0}\, d\mu(t)$. Using this, the functional calculus, and the Fubini-Tonelli theorem, we immediately get

$$\begin{aligned}
\mathrm{tr}_{L^2(\mathbb{R}^d,\mathcal{G})}(\Delta+V)_-^\gamma &= \int_0^\infty \mathrm{tr}_{L^2(\mathbb{R}^d,\mathcal{G})}(\Delta+V+t)_-^{\gamma_0}\, d\mu(t) \\
&\leq \frac{C_{\gamma_0,d}}{(2\pi)^d} \int_0^\infty d\mu(t) \iint d\xi dx\, \mathrm{tr}_{\mathcal{G}}(\xi^2+V(x)+t)_-^{\gamma_0} \\
&= \frac{C_{\gamma_0,d}}{(2\pi)^d} \iint d\xi dx \int_0^\infty d\mu(t)\, \mathrm{tr}_{\mathcal{G}}(\xi^2+V(x)+t)_-^{\gamma_0} \\
&= \frac{C_{\gamma_0,d}}{(2\pi)^d} \iint d\xi dx\, \mathrm{tr}_{\mathcal{G}}(\xi^2+V(x))_-^{\gamma}.
\end{aligned}$$

ii) Theorem 2.1 is a slight extension of a very nice observation of Laptev and Weidl [12, 14]. They used it to show $C_{\gamma,d} = 1$ as long as $\gamma \geq 3/2$. Basically this follows immediately by induction and the above monotonicity from (7) for $n = 1$ once one knows that $C_{3/2,1} = 1$. The beauty of this observation is that this bound is well-known in the scalar case [20] and Laptev and Weidl gave a proof for it in the general case. See also [2] for an elegant alternative proof which avoids the proof of Buslaev-Fadeev-Zhakarov type sum rules for matrix-valued potentials.

iii) Using $C_{\gamma,d} = 1$ for $\gamma \geq 3/2$ and (8), we get the bound

$$C_{\gamma,d} \leq C_{\gamma,3}$$

in $d \geq 3$ for all $\gamma \geq 0$. In particular, this implies a uniform bound (in d) for the constant in the Cwikel-Lieb-Rozenblum bound as soon as such an estimate is established in dimension three for operator-valued potentials. Below we will recover Cwikel's bound $C_{0,3} \leq 3^4 = 81$, see Corollary 2.4. It is, already for scalar potentials, known, that $C_{0,3} \geq 8/\sqrt{3} > 4.6188$, [7] [20, eq. (4.24)] (see also the discussion in [28, page 96–97]); in fact, it is conjectured to be the correct value [7, 20, 27]. In the *scalar case* Lieb's proof [17] of the CLR-bound gives by far the best estimate, $C_{0,3}^{\mathrm{scalar}} \le 6.87$. However, Lieb's estimate grows like $\sqrt{\pi d}$ for large dimensions [20, eq. (5.5)]. While we get a quite large bound on $C_{0,3}$ this at least furnishes the uniform bound $C_{0,d} \le 81$ for all $d \ge 3$. It would be nice to extend Lieb's or even Conlon's proof [5] of the CLR-bound to operator-valued potentials.

To state our second result, Cwikel's bound in the operator-valued case, we need some more notation: $L^q_w(\mathbb{R}^d, \mathcal{B}(\mathcal{G}))$, the analog of the weak L^q-space $L^q_w(\mathbb{R}^d)$, is given by all operator-valued functions $g\colon \mathbb{R}^d \to \mathcal{B}(\mathcal{G})$ for which

$$\|g\|^*_{q,w} = \|g\|^*_{L^q_w(\mathbb{R}^d,\mathcal{B}(\mathcal{G}))} := \sup_{t>0} \big(t\,|\{\|g(\cdot)\|_{\mathcal{B}(\mathcal{G})} > t\}|^{1/q}\big) < \infty.$$

Here $|B|$ is the d-dimensional Lebesgue measure of a Borel set $B \subset \mathbb{R}^d$. Note that $\|\cdot\|^*_{q,w}$ is not a norm since it fails to obey the triangle inequality already for scalar g. But, as in the scalar case, one can give a norm on $L^q_w(\mathbb{R}^d; \mathcal{B}(\mathcal{G}))$ which is equivalent to $\|\cdot\|^*_{L^q(\mathbb{R}^d;\mathcal{B}(\mathcal{G}))}$. However, we will not need this.

With p we abbreviate the operator $-i\nabla$ and similarly to the scalar case we define the operator $f(x)g(p)$ to be

$$\psi \to f(x)g(p)\psi(x) = f(x)\frac{1}{(2\pi)^{d/2}} \int \mathrm{e}^{ix\zeta} g(\zeta)\hat{\psi}(\zeta)\, d\zeta,$$

that is, $f(x)g(p) = M_f \mathcal{F}^{-1} M_g \mathcal{F}$ with M_f, M_g the "multiplication" operators by $f(x)$ and $g(\xi)$ and $\mathcal{F}$ the Fourier transform. A priori, $f(x)g(p)$ is well-defined only for simple functions, but it will turn out to be a compact operator for rather general "functions" f and g. The extension of Cwikel's bound to the operator-valued case is

Theorem 2.3 (Cwikel's bound, operator-valued case)**.** *Let f and g be operator-valued functions on an auxiliary Hilbert space $\mathcal{G}$. Assume that $f \in L^q(\mathbb{R}^d, \mathcal{S}^q(\mathcal{G}))$ and $g \in L^q_w(\mathbb{R}^d, \mathcal{B}(\mathcal{G}))$ for some $q > 2$. Then $f(x)g(p)$ is a compact operator on $L^2(\mathbb{R}^d, \mathcal{G})$. In fact, it is in the weak operator ideal $\mathcal{S}^q_w(L^2(\mathbb{R}^d, \mathcal{G}))$ and, moreover,*

$$\|f(x)g(p)\|^*_{q,w} := \sup_{n\ge 1} n^{1/q}\mu_n\big(f(x)g(p)\big) \le K_q\, \|f\|_{q,q}\, \|g\|^*_{q,w} \tag{9}$$

where the constant K_q is given by

$$K_q = (2\pi)^{-d/q}\frac{q}{2}\left(\frac{8}{q-2}\right)^{1-2/q}\left(1 + \frac{2}{q-2}\right)^{1/q}.$$

As in the scalar case Theorem 2.3 gives a bound for the number of negative eigenvalues of Schrödinger operators with operator-valued potentials.

Corollary 2.4. *Let $\mathcal{G}$ be some auxiliary Hilbert space and V a potential in $L^{d/2}(\mathbb{R}^d, \mathcal{S}^{d/2}(\mathcal{G}))$. Then the operator $-\Delta \otimes \mathbf{1}_{\mathcal{G}} + V$ has a finite number N of negative eigenvalues. Furthermore, we have the bound*

$$N \le L_{0,d} \int_{\mathbb{R}}^{d} \mathrm{tr}_{\mathcal{G}}(V(x)_-^{d/2})\, dx$$

with

$$L_{0,d} \leq (2\pi\, K_d)^d L_{0,d}^{\mathrm{cl}},$$

that is, $C_{0,d} \leq (2\pi\, K_d)^d$.

Proof. For completeness we explicitly derive the estimate for the number of negative eigenvalues of $-\Delta \otimes \mathbf{1}_{\mathcal{G}} + V$ from Theorem 2.3. Replacing V with $-(V)_-$ if necessary and using the min–max principle, we can assume V to be non-positive. Let N be the number of negative eigenvalues of $-\Delta \otimes +V$ and put $Y := |V|^{1/2}\,(|p|^{-1}\otimes \mathbf{1}_{\mathcal{G}})$. By the Birman-Schwinger principle [3, 25, 4, 26, 24] one has

$$1 \leq \mu_N(Y).$$

But $\xi \to |\xi|^{-1}\otimes \mathbf{1}_{\mathcal{G}}$ has weak $L^d(\mathbb{R}^d, \mathcal{B}(\mathcal{G}))$-norm $\tau_d^{1/d}$, τ_d being the volume of the unit ball in $\mathbb{R}^d$. With Theorem 2.3 we arrive at

$$1 \leq K_d \tau_d^{1/d} \| |V|^{1/2} \|_{d,d} N^{-1/d},$$

that is,

$$N \leq K_d^d \tau_d \| |V|^{1/2} \|_{d,d}^d = (2\pi K_d)^d L_{0,d}^{\mathrm{cl}} \int \mathrm{tr}_{\mathcal{G}}(|V(x)|^{d/2})\, dx,$$

since $L_{0,d}^{\mathrm{cl}} = \tau_d/(2\pi)^d$. ∎

Remark 2.5. Corollary 2.4 gives the a priori bound $C_{0,d} \leq (2\pi\, K_d)^d$ for $d \geq 3$. Using Theorem 2.1 and the fact that $C_{\gamma,d} = 1$ if $\gamma \geq 3/2$, [14], we know that $C_{0,d} \leq \min_{n=3,\dots,d} C_{0,n}$. Since the a priori bound given in Corollary 2.4 increases rather fast in the dimension, the best we can conclude is $C_{0,d} \leq (2\pi\, K_3)^3 = 3^4 = 81$.

3. Proof of the sub-multiplicativity of the Lieb-Thirring constants

We proceed very similarly to [14], but freeze the first $n < d$ variables. Let $x_< = (x_1, \dots, x_n)$, $x_> = (x_{n+1}, \dots, x_d)$ and $\xi_<$, $\xi_>$ similarly defined. Put

$$W(x_<) := (-\Delta_> + V(x_<, .))_-,$$

where $\Delta_>$ is the Laplacian in the $x_>$ variables. Clearly, by assumption on V, W is a non-negative compact operator on $L^2(\mathbb{R}^{d-n}, \mathcal{G})$ for almost all $x_< \in \mathbb{R}^n$ and, moreover,

$$\begin{aligned} \mathrm{tr}_{L^2(\mathbb{R}^d,\mathcal{G})}(-\Delta + V)_-^\gamma &\leq \mathrm{tr}_{L^2(\mathbb{R}^n, L^2(\mathbb{R}^{d-n},\mathcal{G}))}(-\Delta_< - W)_-^\gamma \\ &\leq \frac{C_{\gamma,n}}{(2\pi)^n} \iint\limits_{\mathbb{R}^n\mathbb{R}^n} d\xi_< dx_< \, \mathrm{tr}_{L^2(\mathbb{R}^{d-n},\mathcal{G}))}(\xi_<^2 - W(x_<))_-^\gamma. \end{aligned} \tag{10}$$

Since $(t - (s)_-)_- = (t+s)_-$ for $t \geq 0$, $s \in \mathbb{R}$, the spectral theorem gives

$$\begin{aligned} \mathrm{tr}_{L^2(\mathbb{R}^{d-n},\mathcal{G}))}(\xi_<^2 - W(x_<))_-^\gamma &= \mathrm{tr}_{L^2(\mathbb{R}^{d-n},\mathcal{G}))}(\xi_<^2 - \Delta_> + V(x_<, .))_-^\gamma \\ &\leq \frac{C_{\gamma,d-n}}{(2\pi)^{d-n}} \iint\limits_{\mathbb{R}^{d-n}\mathbb{R}^{d-n}} d\xi_> dx_> \, \mathrm{tr}_{\mathcal{G}}(\xi_<^2 + \xi_>^2 + V(x_<, x_>))_-^\gamma. \end{aligned}$$

This together with (10) and the Fubini-Tonelli theorem shows (7).
For the other inequality we use the more usual form (5) of the Lieb-Thirring inequality. Again, freezing the first n coordinates and proceeding as before, we immediately get

$$L_{\gamma,d} \leq L_{\gamma,n} L_{\gamma+n/2,d-n}, \tag{11}$$

where $L_{\gamma+n/2,d-n}$ enters now because in the first application of the Lieb-Thirring inequality (5) the exponent is raised from γ to $\gamma + n/2$. Using the definition (3) for the classical Lieb-Thirring constant together with the Fubini-Tonelli theorem and scaling, one easily sees

$$\begin{aligned} L^{\mathrm{cl}}_{\gamma,d} &= \int_{\mathbb{R}^d} dp\,(|p|^2 - 1)^{\gamma}_{-} \\ &= \int_{\mathbb{R}^n} dp_<\,(|p_<|^2 - 1)^{\gamma}_{-} \int_{\mathbb{R}^{(d-n)}} dp_>\,(|p_>|^2 - 1)^{\gamma+n/2}_{-} \\ &= L^{\mathrm{cl}}_{\gamma,n} L^{\mathrm{cl}}_{\gamma+n/2,d-n}. \end{aligned}$$

This together with (11) proves (8) and thus Theorem 2.1.

4. Proof of Cwikel's bound

The proof of Theorem 2.3 follows closely Cwikel's original proof. We first need a criterion for $f(x)g(p)$ to be a Hilbert-Schmidt operator.

Lemma 4.1. *Let $f \in L^2(\mathbb{R}^d, \mathcal{S}^2(\mathcal{G}))$ and assume g obeys $\|g(.)\|_{\mathcal{B}(\mathcal{G})} \in L^2(\mathbb{R}^d)$. Then the operator $f(x)g(p)$ is Hilbert-Schmidt and we have the estimate*

$$\begin{aligned} \|f(x)g(p)\|^2_{HS} &= (2\pi)^{-d} \iint \mathrm{tr}_{\mathcal{G}}[g^*(\xi) f(x)^* f(x) g(\xi)]\, dx d\xi \\ &\leq (2\pi)^{-d} \int \mathrm{tr}_{\mathcal{G}}[|f(x)|^2]\, dx \int \|g(\xi)\|^2_{\mathcal{G}}\, d\xi. \end{aligned}$$

Proof. In the scalar case this is well-known and is usually shown by noting that in this case $f(x)g(p)$ is a convolution operator. Another proof is by changing the basis: Let $\mathcal{F}$ be the Fourier transform on $L^2(\mathbb{R}^d, \mathcal{G})$, that is,

$$\mathcal{F}u(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathrm{e}^{-i\xi\cdot x} u(x)\, dx.$$

Then the Hilbert-Schmidt norms of $f(x)g(p)$ and $M_f\mathcal{F}^{-1}M_g$ are equal. The operator $M_f\mathcal{F}^{-1}M_g$ has "kernel" $(2\pi)^{-d/2}\mathrm{e}^{ix\cdot\xi}f(x)g(\xi)$ and thus by [22, Theorem VI.23] or [8, Section III.9]

$$\begin{aligned}\|f(x)g(p)\|_{HS}^2 &= \left\|f(x)\mathcal{F}^{-1}g(\xi)\right\|_{HS}^2\\ &= (2\pi)^{-d}\iint \mathrm{tr}_{\mathcal{G}}(g(\xi)^*f(x)^*f(x)g(\xi))\,dxd\xi\\ &= (2\pi)^{-d}\iint \mathrm{tr}_{\mathcal{G}}(|f(x)|^2|g(\xi)^*|^2)\,dxd\xi\\ &\le (2\pi)^{-d}\iint \mathrm{tr}_{\mathcal{G}}(|f(x)|^2)\,\|g(\xi)\|_{\mathcal{G}}^2)\,dxd\xi\\ &= (2\pi)^{-d}\int \mathrm{tr}_{\mathcal{G}}(|f(x)|^2)\,dx\int \|g(\xi)\|_{\mathcal{G}}^2)\,d\xi.\end{aligned}$$

■

The first step rests on splitting the operator $f(x)g(p)$ (which is a priori only defined on simple functions) into manageable pieces. Fix $t>0$, $r>1$ and assume that f and g are non-negative, in particular, self-adjoint, operator-valued functions. For a Borel subset B of $\mathbb{R}$ let $\chi_B(f(x))$ and $\chi_B(g(\xi))$ be the spectral projection operators of $f(x)$ and $g(\xi)$, respectively. By the functional calculus we have

$$\begin{aligned}f(x) &= \sum_{l\in\mathbb{Z}} f(x)\chi_{t(r^{l-1},r^l]}(f(x)) = \sum_{l\in\mathbb{Z}} f_l(x)\\ g(\xi) &= \sum_{l\in\mathbb{Z}} g(\xi)\chi_{(r^{l-1},r^l]}(g(\xi)) = \sum_{l\in\mathbb{Z}} g_l(\xi),\end{aligned} \tag{12}$$

where f_l (resp., g_m) are mutually orthogonal operators. We use this decomposition of f and g to split the operator $f(x)g(p)$ into

$$f(X)g(P) = B_t + H_t \tag{13}$$

with $B_t := \sum_{l+m\le 1} f_l(x)g_m(p)$, $H_t := \sum_{l+m>1} f_l(x)g_m(p)$. Note that this decomposition of $f(x)g(p)$ is slightly different from the one used by Cwikel. We have

Lemma 4.2. *Let f and g be non-negative operator-valued functions. If $q>2$ and $f\in L^q(\mathbb{R}^d,\mathcal{S}^q(\mathcal{G}))$, $g\in L^q_w(\mathbb{R}^d,\mathcal{B}(\mathcal{G}))$ with $\|f\|_{q,q}=1$ and $\|g\|_{q,w}^*=1$ then*
a) *B_t is a bounded operator with operator norm bounded by*

$$\|B_t\|_{L^2(\mathbb{R}^d,\mathcal{G})} \le t\frac{r}{1-r^{-1}}.$$

b) *H_t is a Hilbert-Schmidt operator with Hilbert-Schmidt norm bounded by*

$$\|H_t\|_{HS}^2 \le (2\pi)^{-d}t^{-(q-2)}\Big(1+\frac{2}{q-2}\Big).$$

Remarks 4.3. i) Due to our choice of B_t, H_t the bound in Lemma 4.2.b) is *independent* of r and in a) it easy to see that the choice $r=2$ is optimal.

ii) This lemma also shows that $f(x)g(p)$ is a compact operator since it is the norm limit for $t \to 0$ of the Hilbert-Schmidt operators H_t.

Proof. Part a) follows completely Cwikel's original proof: Since the f_l (resp., g_m) are orthogonal operators for different indices we get, for simple functions ψ and ϕ, say,

$$\begin{aligned}
|\langle\psi, B_t\phi\rangle| &\leq \sum_{l+m\leq 1} r^{l+m} \left\| r^{-l} f_l(x)\psi \right\|_2 \left\| r^{-m} g_m(p)\phi \right\|_2 \\
&\leq \sum_{s\leq 1} r^s \sum_{m\in\mathbb{N}} \left\| r^{-(s-m)} f_{s-m}(x)\psi \right\|_2 \left\| r^{-m} g_m(p)\phi \right\|_2 \\
&\leq \sum_{s\leq 1} r^s \left(\sum_{m\in\mathbb{N}} \left\| r^{-(s-m)} f_{s-m}(x)\psi \right\|_2^2 \right)^{1/2} \left(\sum_{m\in\mathbb{N}} \left\| r^{-m} g_m(p)\phi \right\|_2^2 \right)^{1/2} \\
&= \sum_{s\leq 1} r^s \left\| \sum_{m\in\mathbb{N}} r^{-(s-m)} f_{s-m}(x)\psi \right\|_2 \left\| \sum_{m\in\mathbb{N}} r^{-m} g_m(p)\phi \right\|_2 \\
&\leq r(1-r^{-1})^{-1} t \, \|\psi\| \, \|\phi\| ,
\end{aligned}$$

since $\sum_l r^{-l} f_l(x) \leq t\mathbf{1}_{\mathcal{G}}$ and $\sum_m r^{-m} g_m(\xi) \leq \mathbf{1}_{\mathcal{G}}$. Thus B_t extends to a bounded operator on $L^2(\mathbb{R}^d, \mathcal{G})$ with the given bound for its norm.

To prove part b) observe that by Lemma 4.1 and the cyclicity of the trace, we have

$$\|H_t\|_{\mathrm{HS}}^2 = \sum_{l+m>1} \iint \mathrm{tr}_{\mathcal{G}}[f_l(x) g_m(\xi)^2 f_l(x)] \, dx d\xi.$$

Assume for $x, \xi \in \mathbb{R}^d$ the operator inequality

$$\begin{aligned}
\sum_{l+m>1} f_l(x) g_m(\xi)^2 f_l(x) &\leq \Big(\|g(\xi)\| \, f(x) \chi_{(t,\infty)}(\|g(\xi)\| \, f(x)) \Big)^2 \\
&=: h(x,\xi)^2
\end{aligned} \tag{14}$$

on the Hilbert space $\mathcal{G}$. Note that the projection operator $\chi_{(t,\infty)}(\|g(\xi)\| \, f(x))$ (on $\mathcal{G}$) commutes with $f(x)$ for all $x, \xi \in \mathbb{R}^d$. Let $\lambda_j(x)$ be the j^{th} ordered eigenvalue of $f(x)$, and $E_j(\alpha) := \{\|g(.)\| \, \lambda_j(.) > \alpha\}$. Each E_j has $2d$ dimensional Lebesgue measure

$$|E_j(\alpha)|_{2d} = \int |\{\|g(.)\| > \alpha/\lambda_j(x)\}|_d \, dx \leq \alpha^{-q} \int \lambda_j(x)^q \, dx,$$

since $\|g\|_{q,w}^* = 1$ by assumption. Thus we see

$$\begin{aligned}
\|H_t\|_{HS}^2 &\le (2\pi)^{-d} \iint \operatorname{tr}_{\mathcal{G}}[h(x,\xi)^2]\, dx d\xi \\
&= (2\pi)^{-d} \sum_j 2 \int_0^\infty |E_j(\max(\alpha,t))|_{2d} \alpha\, d\alpha \\
&= (2\pi)^{-d} \left(\sum_j 2 \int_0^t |E_j(t)|_{2d} \alpha\, d\alpha + \sum_j 2 \int_t^\infty |E_j(\alpha)|_{2d} \alpha\, d\alpha \right) \\
&\le (2\pi)^{-d} t^{-(q-2)} \big(1 + 2/(q-2)\big) \sum_j \int \lambda_j(x)^q\, dx \\
&= (2\pi)^{-d} t^{-(q-2)} \big(1 + 2/(q-2)\big),
\end{aligned}$$

since $\sum_j \int \lambda_j(x)^q\, dx = \|f\|_{q,q}^q = 1$ by assumption. It remains to prove (14): Again, let $s = l + m$ and note that the $g_m(\xi) = g(\xi)\chi_{(r^{m-1}, r^m]}(g(\xi))$ are orthogonal operators for different indices. As operators on $\mathcal{G}$,

$$\begin{aligned}
\sum_{l+m>1} f_l(x) g_m(\xi)^2 f_l(x) &= \sum_{l\in\mathbb{Z}} \sum_{s\ge 2} f_l(x) g_{s-l}(\xi)^2 f_l(x) \\
&= \sum_{l\in\mathbb{Z}} f_l(x) \big(\sum_{s\ge 2} g_{s-l}(\xi)^2 \big) f_l(x) = \sum_{l\in\mathbb{Z}} f_l(x) g(\xi)^2 \chi_{(r^{1-l},\infty)}(g(\xi)) f_l(x) \\
&\le \sum_{l\in\mathbb{Z}} f_l(x)\, \|g(\xi)\|_{\mathcal{G}}^2\, \chi_{(r^{1-l},\infty)}(\|g(\xi)\|) f_l(x) \\
&= f(x)^2\, \|g(\xi)\|_{\mathcal{G}}^2 \sum_{l\in\mathbb{Z}} \underbrace{\chi_{(r^{1-l},\infty)}(\|g(\xi)\|) \chi_{t(r^{l-1}, r^l]}(f(x))}_{\le \chi_{(t,\infty)}(\|g(\xi)\| f(x)) \chi_{t(r^{l-1}, r^l]}(f(x))} \\
&\le f(x)^2\, \|g(\xi)\|_{\mathcal{G}}^2\, \chi_{(t,\infty)}(\|g(\xi)\|\, f(x)) \sum_{l\in\mathbb{Z}} \chi_{t(r^{l-1}, r^l]}(f(x)) \\
&= f(x)^2\, \|g(\xi)\|_{\mathcal{G}}^2\, \chi_{(t,\infty)}(\|g(\xi)\|\, f(x)),
\end{aligned}$$

which proves (14) and hence the lemma. ∎

Given the above bounds the proof of Theorem 2.3 is by now a standard interpolation argument. We give this argument for the sake of completeness:

Proof of Theorem 2.3. First, without loss of generality assume that f and g are non-negative operator-valued functions. Indeed, let $\mathcal{F}$ be the Fourier transform and M_f and M_g the operators of "multiplication" by f and g and note that $f(x)g(p)$ and $M_f \mathcal{F}^{-1} M_g$ have the same singular values. With the polar decompositions $f(x) = U_1(x)|f(x)|$ and $g(\xi) = |g^*(\xi)| U_2^*(\xi)$ in the Hilbert space $\mathcal{G}$ we have

$$M_f \mathcal{F}^{-1} M_g = U_1 M_{|f|} \mathcal{F}^{-1} M_{|g^*|} U_2^*,$$

where U_j, $j \in \{1,2\}$ are fibered partial isometries in the space $L^2(\mathbb{R}^d, \mathcal{G})$, for example, $(U_1\psi)(x) = U_1(x)\psi(x)$. Hence the singular values of $f(x)g(p)$ are bounded by the singular values of $M_{|f|}\mathcal{F}^{-1}M_{|g^*|}$ and $\||g^*|\|_{q,w}^* = \|g\|_{q,w}^*$.

By one of the consequences of Ky Fan's inequality [8] we have

$$\mu_n(f(x)g(p)) = \mu_n(B_t + H_t) \le \mu_1(B_t) + \mu_n(H_t) \le \|B_t\| + \frac{1}{\sqrt{n}}\|H_t\|_{HS}$$
$$\le t\frac{r}{1-r^{-1}} + (2\pi)^{-d/2}t^{-(q-2)/2}\big(1 + \frac{2}{q-2}\big)^{1/2}\frac{1}{\sqrt{n}}$$

using Lemma 4.2. Choosing t and r $(= 2)$ optimal gives

$$\mu_n(f(x)g(p)) \le (2\pi)^{-d/q}\frac{q}{2}\left(\frac{8}{q-2}\right)^{1-2/q}\left(1 + \frac{2}{q-2}\right)^{1/q} n^{-1/q}$$

which proves Theorem 2.3. ∎

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