ON THE NUMBER OF BOUND STATES FOR SCHRÖDINGER OPERATORS WITH OPERATOR-VALUED POTENTIALS

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Abstract. Cwikel’s bound is extended to an operator-valued setting. One application of this result is a semi-classical bound for the number of negative bound states for Schrödinger operators with operator-valued potentials. We recover Cwikel’s bound for the Lieb-Thirring constant \( L_{0,3} \) which is far worse than the best available by Lieb (for scalar potentials). However, it leads to a uniform bound (in the dimension \( d \geq 3 \)) for the quotient \( L_{d,d}/L_{0,d}^{cl} \), where \( L_{0,d}^{cl} \) is the so-called classical constant. This gives some improvement in large dimensions.

1. Introduction

The Lieb-Thirring inequalities bound certain moments of the negative eigenvalues of a one-particle Schrödinger operator by the corresponding classical phase space moment. More precisely, for “nice enough” potentials one has

\[
\text{tr}_{L^{2}(\mathbb{R}^{d})}(-\Delta + V)_{+}^{\gamma} \leq C_{\gamma,d} \frac{1}{(2\pi)^{d}} \int_{\mathbb{R}^{d}} d\xi d\xi (\xi^{2} + V(x))_{+}^{\gamma},
\]

(1)

Here and in the following, \( (x)_{+} = \frac{1}{\pi}(|x| - x) \) is the negative part of a real number or a self-adjoint operator. Doing the \( \xi \) integration explicitly with the help of scaling the above inequality is equivalent to its more often used form

\[
\text{tr}_{L^{2}(\mathbb{R}^{d})}(-\Delta + V)_{+}^{\gamma} \leq L_{\gamma,d} \int_{\mathbb{R}^{d}} dx V(x)_{+}^{\gamma + d/2},
\]

(2)

where the Lieb-Thirring constant \( L_{\gamma,d} \) is given by \( L_{\gamma,d} = C_{\gamma,d} L_{\gamma,d}^{cl} \) with the classical Lieb-Thirring constant

\[
L_{\gamma,d}^{cl} = \frac{1}{(2\pi)^{d}} \int_{\mathbb{R}^{d}} dp (1 - p^{2})_{+}^{\gamma}.
\]

(3)

This integral is, of course, explicitly given by a quotient of Gamma functions, but we will have no need for this. The Lieb-Thirring inequalities are valid as soon as the potential \( V \) is in \( L^{\gamma + d/2}(\mathbb{R}^{d}) \).
These inequalities are important tools in the spectral theory of Schrödinger operators and they are known to hold if and only if \( \gamma \geq \frac{1}{2} \) if \( d = 1 \), \( \gamma > 0 \) if \( d = 2 \), and \( \gamma > 0 \) if \( d \geq 3 \). The bound for the critical case \( \gamma = 0 \), that is, the bound for the number of negative eigenvalues of a Schrödinger operator in three or more dimensions is the celebrated Cwikel-Lieb-Rozenblum bound \([6, 17, 21]\). Later, different proofs for this were given by Cwikel and Li and Yau \([5, 16]\). The remaining case \( \gamma = \frac{1}{2} \) in \( d = 1 \) was settled in \([29]\). The well-known Weyl asymptotic formula

\[
\lim_{\lambda \to \infty} \text{tr}(-\Delta + \lambda V)^{-\gamma} = I^\text{cl}_{\gamma,d} \int dx \ V(x)^{\gamma + d/2}
\]

immediately gives the lower bound \( C_{\gamma,d} \geq 1 \). There are certain refined lower bounds \([20, 9]\) for small values of \( \gamma \). In particular, one always has \( C_{\gamma,d} > 1 \) for \( \gamma < 1 \); see \([9]\). In one dimension this even happens for \( \gamma < 3/2 \), and in two dimensions, one always has \( C_{1,2} > 1 \) \([20]\).

Depending on the dimension there are certain conjectures for the optimal value of the constants in these inequalities \([19, 20]\). One part of the conjectures on the Lieb-Thirring constants is that, indeed, \( C_{\gamma,d} = 1 \) for \( d \geq 3 \) and moments \( \gamma \geq 1 \). For the physically most important case \( \gamma = 1, \ d = 3 \) this would imply, via a duality argument, that the kinetic energy of fermions is bounded below by the Thomas-Fermi ansatz for the kinetic energy, which in turn has certain consequences for the energy of large quantum Coulomb systems \([17, 19]\).

Laptev and Weidl \([14]\) realized that a, at first glance, purely technical extension of the Lieb-Thirring inequality from scalar to operator-valued potentials already suggested in \([12]\) is a key in proving at least a part of the Lieb-Thirring conjecture. It allowed them to show that \( C_{\gamma,d} = 1 \) for all \( d \in \mathbb{N} \) as long as \( \gamma \geq 3/2 \). To prove this they considered Schrödinger operators of the form \(-\Delta \otimes 1_G + V\) on the Hilbert space \( L^2(\mathbb{R}^d, G) \) where \( V \) now is an operator-valued potential with values \( V(x) \) in the bounded self-adjoint operators on the auxiliary Hilbert space \( G \). In this case the Lieb-Thirring inequalities \( (1) \) and \( (2) \) are modified to

\[
\text{tr}_{L^2}(-\Delta \otimes 1_G + V)^{-\gamma} \leq \frac{C_{\gamma,d}}{(2\pi)^d} \int_{\mathbb{R}^d} dp dx \ \text{tr}_G(p^2 + V(x))^{-\gamma},
\]

or, again doing the \( \xi \) integral explicitly with the help of the spectral theorem and scaling

\[
\text{tr}_{L^2}(-\Delta \otimes 1_G + V)^{-\gamma} \leq L_{\gamma,d} \int_{\mathbb{R}^d} dx \ \text{tr}_G(V(x)^{\gamma + d/2}).
\]

Here we abused the notation slightly in using the same symbol for the constants as in the scalar case. But in the following, we will only consider the operator-valued case anyway. Laptev and Weidl realized that this extension of the Lieb-Thirring inequality gives rise to the possibility of an inductive proof for \( C_{3/2,d} = 1 \) as long as one has the a priori information \( C_{3/2,1} = 1 \) for operator-valued potentials. This idea together with ideas in \([11]\) was then later
used in [10] to prove improved bounds on $C_{\gamma,d}$ in the range $1/2 \leq \gamma \leq 3/2$; in particular, it was shown that $C_{1,d} \leq 2$ uniformly in $d \in \mathbb{N}$.

Unlike the scalar case, however, the range of parameters $\gamma$ and $d$ for which (4) or equivalently (5) holds is not known. The results in [10] only show that these inequalities are true for $\gamma \geq 1/2$ and all $d \in \mathbb{N}$. This shortcoming has to do with the way the Lieb-Thirring estimates are proven for operator-valued potentials. First, the estimate is shown to hold in one dimension. Then a suitable induction proof, using the one-dimensional result, is set up to prove the full result in all dimensions. This turns out to give good estimates for the coefficients $C_{\gamma,d}$ in the Lieb-Thirring inequality, for example, they are independent of the dimension. However, moments below $1/2$ cannot be addressed with this method, since the a priori estimate fails already for scalar potentials.

This led Ari Laptev [13], see also [15], to ask the question whether, in particular, the Cwikel-Lieb-Rozenblum estimate holds for Schrödinger operators with operator-valued potentials. In this note we answer his question affirmatively, that is, the Lieb-Thirring inequalities for operator-valued potentials are shown to hold also for $\gamma = 0$ as long as $d \geq 3$ and then, by a monotonicity argument also for all $\gamma \geq 0$. More precisely, we want to show that Cwikel’s proof of the Cwikel-Lieb-Rozenblum bound can be adapted to the operator-valued setting. However, the bound for $C_{0,d}$ is far from being optimal since we use Cwikel’s approach. But, nevertheless, reasoning similar to Laptev and Weidl, any a priori bound on $C_{0,3}$ implies the bound $C_{0,d} \leq C_{0,3}$ for $d \geq 3$, thus giving a uniform bound in the dimension, whereas the best available bound in the scalar case due to Lieb [17] grows like $\sqrt{\pi d}$, see [20].

2. Statement of the results

Let $\mathcal{G}$ be a (separable) Hilbert space with norm $||.,||_{\mathcal{G}}$, scalar product $\langle .,. \rangle_{\mathcal{G}}$, and let $1_{\mathcal{G}}$ be the identity operator on $\mathcal{G}$. We follow the convention that scalar products are linear in the second component. Furthermore, $\mathcal{B}(\mathcal{G})$ is the Banach space of bounded operators equipped with the operator norm $||.||_{\mathcal{B}(\mathcal{G})}$ and $\mathcal{K}(\mathcal{G})$ the (separable) ideal of the compact operators on $\mathcal{G}$. For a compact operator $A \in \mathcal{K}(\mathcal{G})$, the singular values $\mu_n(A)$, $n \in \mathbb{N}$ are the eigenvalues of $|A| := (A^*A)^{1/2}$ arranged in decreasing order counting multiplicity. $A^*$ is the adjoint of $A$. $\mathcal{S}^q(\mathcal{G})$ denotes the ideal of compact operators $A \in \mathcal{K}(\mathcal{G})$ whose singular values are $q$-summable, that is, $\sum_n \mu_n(A)^q < \infty$. In particular, $\mathcal{S}^1(\mathcal{G})$ and $\mathcal{S}^2(\mathcal{G})$ are the trace class and Hilbert-Schmidt operators on $\mathcal{G}$. We will often write $\mathcal{B}$, $\mathcal{K}$, and $\mathcal{S}^q$ if there is no ambiguity. Of course, $A \in \mathcal{S}^q$ if and only if $\text{tr}_\mathcal{G}(|A|^q) = \text{tr}_\mathcal{G}((A^*A)^{q/2}) < \infty$, where $\text{tr}_\mathcal{G}$ is the trace on $\mathcal{G}$.

The Hilbert space $L^2(\mathbb{R}^d, \mathcal{G})$ is the space of all measurable functions $\phi : \mathbb{R}^d \to \mathcal{G}$ such that

$$||\psi||_{L^2(\mathbb{R}^d, \mathcal{G})}^2 := \int_{\mathbb{R}^d} dx \, ||\psi(x)||_{\mathcal{G}}^2 < \infty$$
and the Sobolev space $H^1(\mathbb{R}^d, G)$ consists of all functions $\psi \in L^2(\mathbb{R}^d, G)$ with finite norm
$$
\|\psi\|_{H^1(\mathbb{R}^d, G)}^2 := \sum_{l=1}^d \|\partial_l \psi\|_{L^2(\mathbb{R}^d, G)}^2 + \|\nabla \psi\|_{L^2(\mathbb{R}^d, G)}^2.
$$
As in the scalar case, the quadratic form
$$
h_0(\psi, \psi) := \sum_{l=1}^d \|\partial_l \psi\|^2_{L^2(\mathbb{R}^d, G)}
$$
is closed in $L^2(\mathbb{R}^d, G)$ on the domain $H^1(\mathbb{R}^d, G)$. Naturally, this form corresponds to the Laplacian $-\Delta \otimes 1_G$ on $L^2(\mathbb{R}^d, G)$.

$L^q(\mathbb{R}^d, B(G))$ is the space of operator-valued functions $f: \mathbb{R}^d \to B(G)$ with finite norm
$$
\|f\|_q^q := \|f\|_{L^q(\mathbb{R}^d, B(G))}^q := \int_{\mathbb{R}^d} dx \|f(x)\|_{B(G)}^q
$$
and $L^q(\mathbb{R}^d, \mathcal{S}^r(G))$ the space of operator-valued functions $f$ whose norm
$$
\|f\|_{q,r}^q := \|f\|_{L^q(\mathbb{R}^d, \mathcal{S}^r(G))}^q := \int_{\mathbb{R}^d} dx \text{tr}_G([f(x)]^r)^{q/r}
$$
is finite. A potential is a function $V \in L^q(\mathbb{R}^d, B(G))$ such that $V(x)$ is a symmetric operator for almost every $x \in \mathbb{R}^d$. If

$$
q \geq 1 \text{ for } d = 1, \quad q > 1 \text{ for } d = 2, \quad \text{and } q \geq d/2 \text{ for } d \geq 3 \quad (6)
$$
one sees, using Sobolev embedding theorems as in the scalar case, that the real-valued quadratic form
$$
v[\psi, \psi] := \int_{\mathbb{R}^d} dx \langle \psi(x), V(x) \psi(x) \rangle_G
$$
is infinitesimally form-bounded with respect to $h_0$. Hence the form sum
$$
h[\psi, \psi] := h_0[\psi, \psi] + v[\psi, \psi]
$$
is closed and semi-bounded from below on $H^1(\mathbb{R}^d, G)$ and thus generates the self-adjoint operator
$$
H = -\Delta \otimes 1_G + V
$$
on $L^2(\mathbb{R}^d, G)$ by the KLMN theorem [23]. It is easy to see that any potential $V \in L^q(\mathbb{R}^d, B(G))$ satisfying (6) for which $V(x) \in \mathcal{K}(G)$ for almost every $x \in \mathbb{R}^d$ is relatively form compact with respect to $h_0$. Hence by Weyl’s theorem for such potentials, the negative eigenvalues $E_0 \leq E_1 \leq E_2 \leq \cdots \leq 0$ are at most a countable set with accumulation point zero and their eigenspaces are finite-dimensional. In particular, this is the case for potentials $V \in L^q(\mathbb{R}^d, \mathcal{S}^r(G))$.

Our first result is a generalized version of a basic observation of Laptev and Weidl: The two versions (4) and (5) of the Lieb-Thirring inequality give rise to two different monotonicity properties of $C_{\gamma,d}$ in $d$. 


Theorem 2.1 (Sub-multiplicativity of $C_{\gamma,d}$). If, for dimensions $n$ and $d-n$, the Lieb-Thirring inequality holds for operator-valued potentials then it also holds in dimension $d$. Moreover,

$$C_{\gamma,d} \leq C_{\gamma,n} C_{\gamma,d-n} \quad \text{and} \quad C_{\gamma,d} \leq C_{\gamma,n} C_{\gamma+n/2,d-n}.$$  \hfill (7)

Remarks 2.2. i) In the scalar case Aizenman and Lieb [1] showed that the map $\gamma \rightarrow C_{\gamma,d} = L_{\gamma,d}/L_{\gamma,d}^1$ is decreasing. This monotonicity holds also in the general case, so, in fact, (8) implies (7). The monotonicity in $\gamma$ is most easily seen in the phase space picture: By scaling one has, for $\gamma > \gamma_0 \geq 0$,

$$\int_0^\infty (s + t)^{\gamma - \gamma_0 - 1} dt = (s)^\gamma B(\gamma - \gamma_0, \gamma_0 + 1),$$

where $B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt$ is the Beta function. In other words, for each choice of $\gamma > \gamma_0 \geq 0$ there exists a positive measure $\mu$ on $\mathbb{R}_+$ with $(s)^\gamma = \int_{\mathbb{R}_+} (s + t)^{\gamma - \gamma_0} d\mu(t)$. Using this, the functional calculus, and the Fubini-Tonelli theorem, we immediately get

$$\text{tr}_{L^2(\mathbb{R}^d, \mathcal{G})}(\Delta + V)^\gamma \leq \int_0^\infty \text{tr}_{L^2(\mathbb{R}^d, \mathcal{G})}(\Delta + V + t)^{\gamma - \gamma_0} d\mu(t) \leq \frac{C_{\gamma,0,d}}{(2\pi)^d} \int_0^\infty d\mu(t) \int d\xi dx \text{tr}_\mathcal{G}(\xi^2 + V(x) + t)^{\gamma - \gamma_0} = \frac{C_{\gamma,0,d}}{(2\pi)^d} \int d\xi dx \text{tr}_\mathcal{G}(\xi^2 + V(x))^\gamma,$$

ii) Theorem 2.1 is a slight extension of a very nice observation of Laptev and Weidl [12, 14]. They used it to show $C_{\gamma,d} = 1$ as long as $\gamma \geq 3/2$. Basically this follows immediately by induction and the above monotonicity from (7) for $n = 1$ once one knows that $C_{3/2,1} = 1$. The beauty of this observation is that this bound is well-known in the scalar case [20] and Laptev and Weidl gave a proof for it in the general case. See also [2] for an elegant alternative proof which avoids the proof of Buslaev-Faddeev-Zhakarov type sum rules for matrix-valued potentials.

iii) Using $C_{\gamma,d} = 1$ for $\gamma \geq 3/2$ and (8), we get the bound

$$C_{\gamma,d} \leq C_{\gamma,3}$$

in $d \geq 3$ for all $\gamma \geq 0$. In particular, this implies a uniform bound (in $d$) for the constant in the Cwikel-Lieb-Rozenblum bound as soon as such an estimate is established in dimension three for operator-valued potentials. Below we will recover Cwikel’s bound $C_{0,3} \leq 3^4 = 81$, see Corollary 2.4. It is, already for scalar potentials, known, that $C_{0,3} \geq 8/\sqrt{3} > 4.6188$, [7] [20, eq. (4.24)] (see also the discussion in [28, page 96–97]); in fact, it is conjectured to be the correct value [7, 20, 27]. In the scalar case Lieb’s proof [17] of the CLR-bound
gives by far the best estimate, $C_{0.3}^{\text{scalar}} \leq 6.87$. However, Lieb’s estimate grows like $\sqrt{\pi d}$ for large dimensions [20, eq. (5.5)]. While we get a quite large bound on $C_{0.3}$ this at least furnishes the uniform bound $C_{0,d} \leq 81$ for all $d \geq 3$. It would be nice to extend Lieb’s or even Conlon’s proof [5] of the CLR-bound to operator-valued potentials.

To state our second result, Cwikel’s bound in the operator-valued case, we need some more notation: \( L_w^q(\mathbb{R}^d, \mathcal{B}(\mathcal{G})) \), the analog of the weak \( L^q \)-space \( L_w^q(\mathbb{R}^d) \), is given by all operator-valued functions \( g: \mathbb{R}^d \rightarrow \mathcal{B}(\mathcal{G}) \) for which

$$
\|g\|_{q, w}^* = \|g\|_{L_w^q(\mathbb{R}^d, \mathcal{B}(\mathcal{G}))}^* = \sup_{t > 0} \left\{ t \left( \|g(\cdot)\|_{\mathcal{B}(\mathcal{G})} > t \right)^{1/q} \right\} < \infty.
$$

Here \( |B| \) is the \( d \)-dimensional Lebesgue measure of a Borel set \( B \subset \mathbb{R}^d \). Note that \( \|\cdot\|_{q, w}^* \) is not a norm since it fails to obey the triangle inequality already for scalar \( g \). But, as in the scalar case, one can give a norm on \( L_w^q(\mathbb{R}^d; \mathcal{B}(\mathcal{G})) \) which is equivalent to \( \|\cdot\|_{L_w^q(\mathbb{R}^d, \mathcal{B}(\mathcal{G}))}^* \). However, we will not need this.

With \( p \) we abbreviate the operator \(-i\nabla\) and similarly to the scalar case we define the operator \( f(x)g(p) \) to be

$$
\psi \rightarrow f(x)g(p)\psi(x) = f(x) \frac{1}{(2\pi)^{d/2}} \int e^{ix\xi} g(\xi) \hat{\psi}(\xi) \, d\xi,
$$

that is, \( f(x)g(p) = M_f \mathcal{F}^{-1} M_g \mathcal{F} \) with \( M_f, M_g \) the “multiplication” operators by \( f(x) \) and \( g(\xi) \) and \( \mathcal{F} \) the Fourier transform. A priori, \( f(x)g(p) \) is well-defined only for simple functions, but it will turn out to be a compact operator for rather general “functions” \( f \) and \( g \). The extension of Cwikel’s bound to the operator-valued case is

**Theorem 2.3** (Cwikel’s bound, operator-valued case). Let \( f \) and \( g \) be operator-valued functions on an auxiliary Hilbert space \( \mathcal{G} \). Assume that \( f \in L^q(\mathbb{R}^d, \mathcal{S}^q(\mathcal{G})) \) and \( g \in L_w^q(\mathbb{R}^d, \mathcal{B}(\mathcal{G})) \) for some \( q > 2 \). Then \( f(x)g(p) \) is a compact operator on \( L^2(\mathbb{R}^d, \mathcal{G}) \). In fact, it is in the weak operator ideal \( \mathcal{S}_w^q(L^2(\mathbb{R}^d, \mathcal{G})) \) and, moreover,

$$
\|f(x)g(p)\|_{q, w}^* := \sup_{n \geq 1} n^{1/q} \mu_n(f(x)g(p)) \leq K_q \|f\|_{q, q} \|g\|_{q, w}^* \tag{9}
$$

where the constant \( K_q \) is given by

$$
K_q = (2\pi)^{-d/q} q \left( \frac{8}{q - 2} \right)^{1-2/q} \left( 1 + \frac{2}{q - 2} \right)^{1/q}.
$$

As in the scalar case Theorem 2.3 gives a bound for the number of negative eigenvalues of Schrödinger operators with operator-valued potentials.

**Corollary 2.4.** Let \( \mathcal{G} \) be some auxiliary Hilbert space and \( V \) a potential in \( L^{d/2}(\mathbb{R}^d, \mathcal{S}^{d/2}(\mathcal{G})) \). Then the operator \(-\Delta \otimes 1_G + V\) has a finite number \( N \) of negative eigenvalues. Furthermore, we have the bound

$$
N \leq L_{0,d} \int_{\mathbb{R}^d} \text{tr}_\mathcal{G}(V(x)^{d/2}) \, dx
$$
with

\[ L_0, d \leq (2\pi K_d)^d L_{0, d}^3 \]

that is, \( C_{0, d} \leq (2\pi K_d)^d \).

**Proof.** For completeness we explicitly derive the estimate for the number of negative eigenvalues of \(-\Delta \otimes 1_G + V\) from Theorem 2.3. Replacing \( V \) with \(- (V)_-\) if necessary and using the min–max principle, we can assume \( V \) to be non-positive. Let \( N \) be the number of negative eigenvalues of \(-\Delta \otimes + V\) and put \( Y := |V|^{1/2} (|p|^{-1} \otimes 1_G)\). By the Birman-Schwinger principle [3, 25, 4, 26, 24] one has

\[ 1 \leq \mu_N(Y). \]

But \( \xi \to |\xi|^{-1} \otimes 1_G \) has weak \( L^d(\mathbb{R}^d, \mathcal{B}(G))\)-norm \( \tau_d^{1/d} \), \( \tau_d \) being the volume of the unit ball in \( \mathbb{R}^d \). With Theorem 2.3 we arrive at

\[ 1 \leq K_d \tau_d^{1/d} |||V|||^{1/2} |||d|| d N^{-1/d}, \]

that is,

\[ N \leq K_d \tau_d |||V|||^{1/2} |||d|| d = (2\pi K_d)^d L_{0, d}^3 \int \text{tr}_G(|V(x)|^{d/2}) \, dx, \]

since \( L_{0, d}^3 = \tau_d / (2\pi)^d \).

**Remark 2.5.** Corollary 2.4 gives the a priori bound \( C_{0, d} \leq (2\pi K_d)^d \) for \( d \geq 3 \). Using Theorem 2.1 and the fact that \( C_{\gamma, d} = 1 \) if \( \gamma \geq 3/2 \), [14], we know that \( C_{0, d} \leq \min_{n=3, \ldots, d} C_{0, n} \). Since the a priori bound given in Corollary 2.4 increases rather fast in the dimension, the best we can conclude is \( C_{0, d} \leq (2\pi K_3)^3 = 3^4 = 81 \).

3. **Proof of the sub-multiplicativity of the Lieb-Thirring constants**

We proceed very similarly to [14], but freeze the first \( n < d \) variables. Let \( x_\prec = (x_1, \ldots, x_n), \ x_\succ = (x_{n+1}, \ldots, x_d) \) and \( \xi_\prec, \xi_\succ \) similarly defined. Put

\[ W(x_\prec) := (-\Delta_\succ + V(x_\prec, ))_-, \]

where \( \Delta_\succ \) is the Laplacian in the \( x_\succ \) variables. Clearly, by assumption on \( V \), \( W \) is a non-negative compact operator on \( L^2(\mathbb{R}^{d-n}, G) \) for almost all \( x_\prec \in \mathbb{R}^n \) and, moreover,

\[ \text{tr}_{L^2(\mathbb{R}^d, G)}(-\Delta + V)_\gamma \leq \text{tr}_{L^2(\mathbb{R}^n, L^2(\mathbb{R}^{d-n}, G))}(-\Delta_\prec - W)_\gamma \]

\[ \leq \frac{C_{\gamma, n}}{(2\pi)^n} \int_{\mathbb{R}^n} d\xi_\prec d\xi_\prec \text{tr}_{L^2(\mathbb{R}^{d-n}, G)}(\xi_\prec^2 - W(x_\prec))_\gamma. \quad (10) \]

Since \( (t - (s)_-) = (t + s)_- \) for \( t \geq 0, s \in \mathbb{R} \), the spectral theorem gives

\[ \text{tr}_{L^2(\mathbb{R}^{d-n}, G)}(\xi_\prec^2 - W(x_\prec))^\gamma = \text{tr}_{L^2(\mathbb{R}^{d-n}, G)}(\xi_\prec^2 - \Delta_\succ + V(x_\succ))^\gamma \]

\[ \leq \frac{C_{\gamma, d-n}}{(2\pi)^{d-n}} \int_{\mathbb{R}^{d-n}} d\xi_\succ d\xi_\succ \text{tr}_G(\xi_\prec^2 + \xi_\prec^2 + V(x_\prec, x_\succ))^\gamma. \]
This together with (10) and the Fubini-Tonelli theorem shows (7). For the other inequality we use the more usual form (5) of the Lieb-Thirring inequality. Again, freezing the first \( n \) coordinates and proceeding as before, we immediately get

\[
L_{\gamma,d} \leq L_{\gamma,n} L_{\gamma+n/2,d-n},
\]

where \( L_{\gamma+n/2,d-n} \) enters now because in the first application of the Lieb-Thirring inequality (5) the exponent is raised from \( \gamma \) to \( \gamma + n/2 \). Using the definition (3) for the classical Lieb-Thirring constant together with the Fubini-Tonelli theorem and scaling, one easily sees

\[
L_{\gamma,d}^{\text{cl}} = \int_{\mathbb{R}^d} dp \ (|p|^2 - 1)^\gamma \\
= \int_{\mathbb{R}^n} dp_\omega (|p_\omega|^2 - 1)^\gamma \int_{\mathbb{R}^{d-n}} dp_\zeta (|p_\zeta|^2 - 1)^{\gamma+n/2} \\
= L_{\gamma,n}^{\text{cl}} L_{\gamma+n/2,d-n}^{\text{cl}}.
\]

This together with (11) proves (8) and thus Theorem 2.1.

4. Proof of Cwikel’s bound

The proof of Theorem 2.3 follows closely Cwikel’s original proof. We first need a criterion for \( f(x)g(p) \) to be a Hilbert-Schmidt operator.

**Lemma 4.1.** Let \( f \in L^2(\mathbb{R}^d, \mathcal{S}(\mathcal{G})) \) and assume \( g \) obeys \( \|g(.)\|_{\mathcal{B}(\mathcal{G})} = L^2(\mathbb{R}^d) \). Then the operator \( f(x)g(p) \) is Hilbert-Schmidt and we have the estimate

\[
\|f(x)g(p)\|_{HS}^2 = (2\pi)^{-d} \int \text{tr}_{\mathcal{G}}[g^*(\xi)f(x)^*f(x)g(\xi)] \, dx d\xi \\
\leq (2\pi)^{-d} \int \text{tr}_{\mathcal{G}}[|f(x)|^2] \, dx \int \|g(\xi)|^2_\mathcal{G} \, d\xi.
\]

**Proof.** In the scalar case this is well-known and is usually shown by noting that in this case \( f(x)g(p) \) is a convolution operator. Another proof is by changing the basis: Let \( \mathcal{F} \) be the Fourier transform on \( L^2(\mathbb{R}^d, \mathcal{G}) \), that is,

\[
\mathcal{F}u(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} u(x) \, dx.
\]
Then the Hilbert-Schmidt norms of \( f(x)g(p) \) and \( M_f \mathcal{F}^{-1} M_g \) are equal. The
operator \( M_f \mathcal{F}^{-1} M_g \) has “kernel” \( (2\pi)^{-d/2} e^{iz\cdot \xi} f(x)g(\xi) \) and thus by [22, Theorem VI.23] or [8, Section III.9]
\[
\| f(x)g(p) \|_{HS}^2 = \| f(x) \mathcal{F}^{-1} g(\xi) \|_{HS}^2 \\
= (2\pi)^{-d} \int \mathcal{G} (g(\xi)^* f(x)^* f(x) g(\xi)) \, dx d\xi \\
= (2\pi)^{-d} \int \mathcal{G} (|f(x)|^2 |g(\xi)|^2) \, dx d\xi \\
\leq (2\pi)^{-d} \int \mathcal{G} (|f(x)|^2) \| g(\xi) \|_{L^2(\mathcal{G})}^2 \, dx d\xi \\
= (2\pi)^{-d} \int \mathcal{G} (|f(x)|^2) \, dx \int \| g(\xi) \|_{L^2(\mathcal{G})}^2 \, d\xi.
\]

The first step rests on splitting the operator \( f(x)g(p) \) (which is a priori
only defined on simple functions) into manageable pieces. Fix \( t > 0, r > 1 \) and
assume that \( f \) and \( g \) are non-negative, in particular, self-adjoint, operator-valued functions. For a Borel subset \( B \) of \( \mathbb{R} \) let \( \chi_B(f(x)) \) and \( \chi_B(g(\xi)) \) be the
spectral projection operators of \( f(x) \) and \( g(\xi) \), respectively. By the functional
calculus we have
\[
f(x) = \sum_{l \in \mathbb{Z}} f_l(x) \chi_{[l^{-1}, l]}(f(x)) = \sum_{l \in \mathbb{Z}} f_l(x) \\
g(\xi) = \sum_{l \in \mathbb{Z}} g_l(\xi) \chi_{[l^{-1}, l]}(g(\xi)) = \sum_{l \in \mathbb{Z}} g_l(\xi),
\]
where \( f_l \) (resp., \( g_m \)) are mutually orthogonal operators. We use this decomposition of \( f \) and \( g \) to split the operator \( f(x)g(p) \) into
\[
f(X)g(P) = B_t + H_t
\]
with \( B_t := \sum_{l+m \leq 1} f_l(x)g_m(p), H_t := \sum_{l+m > 1} f_l(x)g_m(p) \). Note that this
decomposition of \( f(x)g(p) \) is slightly different from the one used by Cwikel.
We have

**Lemma 4.2.** Let \( f \) and \( g \) be non-negative operator-valued functions. If \( q > 2 \) and \( f \in L^q(\mathbb{R}^d, \mathcal{S}(\mathcal{G})), g \in L^q_w(\mathbb{R}^d, \mathcal{B}(\mathcal{G})) \) with \( \| f \|_{q,q} = 1 \) and \( \| g \|_{q,w}^* = 1 \) then
a) \( B_t \) is a bounded operator with operator norm bounded by
\[
\| B_t \|_{L^2(\mathbb{R}^d, \mathcal{G})} \leq t \frac{r}{1 - r^{-1}}.
\]
b) \( H_t \) is a Hilbert-Schmidt operator with Hilbert-Schmidt norm bounded by
\[
\| H_t \|_{HS}^2 \leq (2\pi)^{-d(r-(q-2))} \left( 1 + \frac{2}{q-2} \right) 
\]

**Remarks 4.3.** i) Due to our choice of \( B_t, H_t \) the bound in Lemma 4.2.b) is
independent of \( r \) and in a) it easy to see that the choice \( r=2 \) is optimal.
ii) This lemma also shows that $f(x)g(p)$ is a compact operator since it is the norm limit for $t \to 0$ of the Hilbert-Schmidt operators $H_t$.

Proof. Part a) follows completely Cwikel’s original proof: Since the $f_t$ (resp., $g_m$) are orthogonal operators for different indices we get, for simple functions $\psi$ and $\phi$, say,

$$
|\langle \psi, B_t \phi \rangle| \leq \sum_{l+m \leq 1} r^{l+m} r^{-l} f_t(x) \psi \| r^{-m} g_m(p) \phi \|_2 \\
\leq \sum_{s \leq 1} r^s \left( \sum_{m \in \mathbb{N}} \| r^{-m} f_{s-m}(x) \psi \|_2 \right) \left( \sum_{m \in \mathbb{N}} \| r^{-m} g_m(p) \phi \|_2 \right)^{1/2} \\
\leq \sum_{s \leq 1} r^s \left( \sum_{m \in \mathbb{N}} \| r^{-m} f_{s-m}(x) \psi \|_2 \right) \sum_{m \in \mathbb{N}} \| r^{-m} g_m(p) \phi \|_2 \\
\leq r(1 - r^{-1})^{-1} t \| \psi \| \| \phi \|,
$$

since $\sum_l r^{-l} f_t(x) \leq t 1_G$ and $\sum_m r^{-m} g_m(\xi) \leq 1_G$. Thus $B_t$ extends to a bounded operator on $L^2(\mathbb{R}^d, \mathcal{G})$ with the given bound for its norm.

To prove part b) observe that by Lemma 4.1 and the cyclicity of the trace, we have

$$
\| H_t \|_{HS}^2 = \sum_{l+m \geq 1} \iiint tr[ f_t(x) g_m(\xi)^2 f_t(x) ] \, dx \, d\xi.
$$

Assume for $x, \xi \in \mathbb{R}^d$ the operator inequality

$$
\sum_{l+m \geq 1} f_t(x) g_m(\xi)^2 f_t(x) \leq \left( \| g(\xi) \| f(x) \chi_{(t,\infty)}(\| g(\xi) \| f(x)) \right)^2 \\
= : h(x, \xi)^2
$$

(14)

on the Hilbert space $\mathcal{G}$. Note that the projection operator $\chi_{(t,\infty)}(\| g(\xi) \| f(x))$ (on $\mathcal{G}$) commutes with $f(x)$ for all $x, \xi \in \mathbb{R}^d$. Let $\lambda_j(x)$ be the $j$th ordered eigenvalue of $f(x)$, and $E_j(\alpha) := \{ \| g(\xi) \| > \lambda_j(x) \}$. Each $E_j$ has 2$d$ dimensional Lebesgue measure

$$
|E_j(\alpha)|_{2d} = \int \{ \| g(\xi) \| > \lambda_j(x) \} \, dx \leq \alpha^{-q} \int \lambda_j(x)^q \, dx,
$$
since \( \|g\|_{q,w}^q = 1 \) by assumption. Thus we see

\[
\|H_t\|_{HS}^2 \leq (2\pi)^{-d} \int \operatorname{tr}_G [h(x, \xi)^2] \, dx \, d\xi \\
= (2\pi)^{-d} \sum_j 2 \int_0^\infty |E_j(\max(\alpha, t))|^2_2 d\alpha \\
= (2\pi)^{-d} \left( \sum_j 2 \int_0^t |E_j(t)|^2_2 d\alpha + \sum_j 2 \int_t^\infty |E_j(\alpha)|^2_2 d\alpha \right) \\
\leq (2\pi)^{-d} \left( 2^{-q-2} \left( 1 + 2/(q-2) \right) \sum_j \int \lambda_j(x)^q \, dx \right) \\
= (2\pi)^{-d} \left( 2^{-q-2} \left( 1 + 2/(q-2) \right) \right) 
\]

since \( \sum_j \int \lambda_j(x)^q \, dx = \|f\|_{q,q}^q = 1 \) by assumption. It remains to prove (14): Again, let \( s = l + m \) and note that the \( g_m(\xi) = g(\xi) \chi_{(r^m-1,r^m]}(g(\xi)) \) are orthogonal operators for different indices. As operators on \( G \),

\[
\sum_{l+m>1} f_l(x)g_m(\xi)^2f_l(x) = \sum_{l \in \mathbb{Z}} \sum_{s \geq 2} f_l(x)g_{s-l}(\xi)^2f_l(x) \\
= \sum_{l \in \mathbb{Z}} f_l(x) \left( \sum_{s \geq 2} g_{s-l}(\xi)^2 \right) f_l(x) = \sum_{l \in \mathbb{Z}} f_l(x)g(\xi)^2 \chi_{(r^l-1,r^l]}(g(\xi))f_l(x) \\
\leq \sum_{l \in \mathbb{Z}} f_l(x) \|g(\xi)\|_G^2 \chi_{(r^l-1,r^l]} \chi_{(r^l-1,r^l]}(f(x)) \\
= f(x)^2 \|g(\xi)\|_G^2 \sum_{l \in \mathbb{Z}} \chi_{(r^l-1,r^l]}(\|g(\xi)\|f(x)) \chi_{(r^l-1,r^l]}(f(x)) \\
\leq f(x)^2 \|g(\xi)\|_G^2 \chi_{(r^l-1,r^l]}(\|g(\xi)\|f(x)) \sum_{l \in \mathbb{Z}} \chi_{(r^l-1,r^l]}(f(x)) \\
= f(x)^2 \|g(\xi)\|_G^2 \chi_{(r^l-1,r^l]}(\|g(\xi)\|f(x)),
\]

which proves (14) and hence the lemma. 

Given the above bounds the proof of Theorem 2.3 is by now a standard interpolation argument. We give this argument for the sake of completeness:

**Proof of Theorem 2.3.** First, without loss of generality assume that \( f \) and \( g \) are non-negative operator-valued functions. Indeed, let \( \mathcal{F} \) be the Fourier transform and \( M_f \) and \( M_g \) the operators of “multiplication” by \( f \) and \( g \) and note that \( f(x)g(p) \) and \( M_f \mathcal{F}^{-1} M_g \) have the same singular values. With the polar decompositions \( f(x) = U_1(x)f(x) \) and \( g(\xi) = |g^*(\xi)|U_2^*(\xi) \) in the Hilbert space \( G \) we have

\[
M_f \mathcal{F}^{-1} M_g = U_1 M_f \mathcal{F}^{-1} M_{g^*} U_2^*,
\]


where $U_j$, $j \in \{1, 2\}$ are fibered partial isometries in the space $L^2(\mathbb{R}^d, G)$, for example, $(U_1 \psi)(x) = U_1(x)\psi(x)$. Hence the singular values of $f(x)g(p)$ are bounded by the singular values of $M_1^T F^{-1} M_{g^*}$ and $\|g^*\|_{q,w} = \|g\|_{q,w}$.

By one of the consequences of Ky Fan’s inequality [8] we have

$$\mu_n(f(x)g(p)) = \mu_n(B_t + H_t) \leq \mu_1(B_t) + \mu_n(H_t) \leq \|B_t\| + \frac{1}{\sqrt{n}} \|H_t\|_{HS}$$

Using Lemma 4.2. Choosing $t$ and $r (= 2)$ optimal gives

$$\mu_n(f(x)g(p)) \leq (2\pi)^{-d/2} q \left( \frac{8}{q - 2} \right)^{1/2} \left( 1 + \frac{2}{q - 2} \right)^{1/4} n^{-1/4}$$

which proves Theorem 2.3.

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**References**


