BOUNDS ON THE SPECTRAL SHIFT FUNCTION AND THE DENSITY OF STATES

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Abstract. We study spectra of Schrödinger operators on $\mathbb{R}^d$. First we consider a pair of operators which differ by a compactly supported potential, as well as the corresponding semigroups. We prove almost exponential decay of the singular values μ_n of the difference of the semigroups as $n \to \infty$ and deduce bounds on the spectral shift function of the pair of operators.

Thereafter we consider alloy type random Schrödinger operators. The single site potential u is assumed to be non-negative and of compact support. The distributions of the random coupling constants are assumed to be Hölder continuous. Based on the estimates for the spectral shift function, we prove a Wegner estimate which implies Hölder continuity of the integrated density of states.

1. Introduction and Results

In this paper we analyze the spectral properties of multi dimensional Schrödinger operators. First, we consider a pair of operators H_1, H_2 which differ by a compactly supported potential u. The singular values μ_n of the difference of the corresponding exponentials $V_{\text{eff}} := e^{-H_1} - e^{-H_2}$ are shown to decay almost exponentially in n.

This result allows us to deduce a bound on the Lifshitz-Krein spectral shift function (SSF) of the operator pair H_1, H_2. We give a bound on the SSF when integrated over the energy axis against a bounded, compactly supported function. In turn, the bound on the SSF is used to prove a Wegner estimate for random Schrödinger operators of alloy type and Hölder continuity of the integrated density of states (IDS). Our estimates have a better continuity in the energy parameter than previously known bounds. Moreover, we are able to treat random coupling constants, whose distribution does not have a density. In particular, for Hölder continuous distributions we prove that the IDS is Hölder continuous, too.

We will treat magnetic Schrödinger operators

$$H = H_A + V = (-i\nabla - A)^2 + V \tag{1}$$

acting on $\mathbb{R}^d$ whose potentials (magnetic and electric) obey the following hypotheses: Each component of A is L^2_{loc}. The positive part of the electric potential, $V_+ := \max(0, V)$, belongs to L^1_{loc} and the negative part, $V_- := \max(0, -V)$, is in the Kato class. Notice that under our convention, $V = V_+ - V_-$.

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For a general discussion of the Kato-class, see [15]; for its relevance to the Feynman-Kac formula see, e.g., [2, 6, 49]. In particular, V_- is in the Kato-class, if

$$\|V_-\|_{L^p_{\mathrm{loc,unif}}(\mathbb{R}^d)} = \sup_{x\in\mathbb{R}^d} \Big(\int_{|x-y|\le 1} |V_-(y)\, dy| \Big)^{1/p} < \infty$$

where $p = 1$ if $d = 1$ and $p > d/2$ if $d \ge 2$. Thus the allowed potentials cover all physically relevant cases.

Under these hypotheses, one may define H via the corresponding quadratic form (with core C_c^∞). By the same method, one can define the Dirichlet restriction of H to the cube $\Lambda_l = [-l/2, l/2]^d, l \ge 1$. This will be denoted H^l.

Let H_1 be a Schrödinger operator of the form just described and let $H_2 = H_1 + u$ where $u = u_+ - u_-$ obeys the hypotheses for electric potentials just described.

The starting point for our analysis is an estimate on the singular values of $V_{\mathrm{eff}} := e^{-H_1} - e^{-H_2}$ and on the corresponding object in the finite volume case, namely $V^l_{\mathrm{eff}} := e^{-H^l_1} - e^{-H^l_2}$. Recall that the singular values of a compact operator A are the square-roots of the eigenvalues of A^*A. We will enumerate them as $\mu_1(A) \ge \mu_2(A) \ge \cdots \ge 0$ according to multiplicity.

Theorem 1. *There are finite positive constants c and C such that the singular values of the operator V^l_{eff} obey*

$$\mu_n \le C\, e^{-cn^{1/d}}. \tag{2}$$

In fact, c may be chosen depending only on the dimension, while C depends on the Kato-class norms of u_-, V_- and on the diameter of the support of u_+.

The same estimate holds for the singular values of V_{eff}.

Remarks: i) In particular $\|V^l_{\mathrm{eff}}\|_{\mathcal{J}_p} := \sum_n \mu_n^p$ is finite and thus V^l_{eff} is an element of the operator ideal $\mathcal{J}_p := \{A \text{ compact} \mid \|A\|_{\mathcal{J}_p} < \infty\}$ for any $p > 0$. Thus our result can be understood as a sharpening and generalization of norm, Hilbert Schmidt, and trace bounds on the difference of semigroups, derived e.g. in [7, 8, 9, 16, 17, 18, 19, 48, 51, 52].

ii) Note that the estimate (2) depends on the positive part of u only through $\operatorname{supp} u$. Thus, for $u = \lambda\tilde{u}$ where $\tilde{u} \ge 0$ and λ is a non-negative coupling constant, the estimate is independent of the choice of λ. Moreover

(1) u may be taken to $+\infty$ on its support. In this case H_2 equals the restriction of H_1 to $\mathbb{R}^d \setminus \operatorname{supp} u$ with Dirichlet boundary conditions, provided the boundary of $\operatorname{supp} u$ obeys some mild regularity conditions; see, e.g., [53].

(2) Similarly, H_1 may be defined on a set strictly smaller than $\mathbb{R}^d$: Let $D \subset$ be open, H_A^D the Dirichlet restriction of H_A on D, and $H_1 = H_A^D + V$ where V satisfies the same conditions as before. In this case H^l_j is the Dirichlet restriction of $H_j, j = 1, 2$ to the set $\Lambda_l \cap D$.

iii) The proof of Theorem 1 is surprisingly simple. Morally, the result is an immediate consequence of Weyl's law for the eigenvalue asymptotic of Dirichlet Laplacians on compact domains. This suggest that the decay rate of the singular values of $e^{-H_1} - e^{-H_2}$ is, in fact, given by $\exp(-cn^{2/d})$. One might ask, however, whether the singular values could not typically decay at a much faster rate. It turns out that a decay rate like $\exp(-cn^\alpha)$ for $\alpha > 2/d$ is impossible, see Remark ii) after Theorem 2 below. This leaves open the cases $\alpha \in (1/d, 2/d]$. We conjecture that, in fact, the true bound is of the form $\mu_n \le C \exp(-cn^{2/d})$.

Our interest in Theorem 1 comes from the fact that it allows us to derive an integral bound on the the SSF $\xi(\lambda, H_2, H_1)$ of the pair of operators H_1, H_2, which shows that the SSF can have only very mild local singularities. (See Section 2.1 for a precise definition of the SSF.) The SSF plays a role in different areas of mathematical physics, for instance in scattering theory, cf. e.g. [59], and the study of surface potentials, see [10, 36]. Various of its properties are discussed in the literature: monotonicity and concavity in [20, 22, 35], the asymptotic behaviour in the large coupling constant [43, 46, 44] and semiclassical limit [42, 40]. See [5, 34] for surveys.

For $t > 0$ let $F_t : [0, \infty) \to [0, \infty)$ be defined by

$$F_t(x) = \int_0^x (\exp(ty^{1/d}) - 1)\, dy. \tag{3}$$

As the integrand is increasing, F_t is a convex function.

Theorem 2. *Let ξ be the spectral shift function for the pair H_1, H_2 or H_1^l, H_2^l.*
i) Let F_t be defined as above. There exisits a constant K_1, depending on t, such that for small enough $t > 0$,

$$\int_{-\infty}^{T} F_t(|\xi(\lambda)|)\, d\lambda \le K_1 e^T < \infty \tag{4}$$

for all $T < \infty$.
ii) There exists constants K_1, K_2 depending only on d, diam supp u_+ *and the Kato class norms of V_-, u_-, such that for any bounded compactly supported function f,*

$$\int f(\lambda)\, \xi(\lambda)\, d\lambda \le K_1 e^b + K_2 \{\log(1 + \|f\|_\infty)\}^d \|f\|_1 \tag{5}$$

with $b = \sup \operatorname{supp}(f)$.

Remarks: i) Note that the function F_t defined in (3) has the asymptotic behavior

$$F_t(x) \sim d\, x^{(d-1)/d} \exp(tx^{1/d}) \qquad \text{for large } x.$$

Thus, by part i) of Theorem 2, the spectral shift function can have at most mild logarithmic local singularities. It is tempting to think that, at least for non-negative compactly supported perturbations, the spectral shift function should always be bounded. However, this is not the case. For a perturbation of the free Schrödinger operator with a constant magnetic field by a compactly supported potential, Raikov and Warzel showed that spectral shift function diverges at each Landau level E_q like

$$|\xi(E_q + \lambda)| \sim \Big(\frac{|\ln(\lambda)|}{\ln|\ln\lambda|}\Big)^{d/2} \qquad \text{as } \lambda \downarrow 0, \tag{6}$$

see [45] for the case $d = 2$ and [39] for the generalization to even dimensions. Thus, setting $F_{t,\alpha}(x) := \int_0^x (\exp(ty^\alpha) - 1)\, dy$, the asymptotic (6) implies that $F_{t,\alpha}(|\xi|)$ is locally integrable if and only if $0 \le \alpha \le 2/d$, whereas Theorem 2 guarantees it only for $\alpha \le 1/d$ (and t small enough, if $\alpha = 1/d$).

ii) The proof of Theorem 2 shows that if Theorem 1 holds in the form $\mu_n \le c_1 \exp(-c_2 n^\alpha)$, then $F_{t,\alpha}(|\xi|) \in L^1(-\infty, T)$ for small enough t (and all finite T). Thus the Raikov-Warzel result shows that the estimate in Theorem 1 *cannot* be improved beyond $C \exp(-cn^{2/d})$.

iii) An example without magnetic fields, where the SSF shows unexpected divergencies, was studied by Kirsch in [28, 29]. It is related to the one with a constant magnetic field in that the high degeneracy of eigenvalues plays a crucial role. Let $E > 0$, u non-negative, bounded with compact support, and not identically equal to zero, $a\colon [0,\infty) \to (0,\infty)$, and $\xi_l(\cdot) := \xi(\cdot, -\Delta^l, (-\Delta + a(l)u)^l)$. Then $\limsup_{l\to\infty} \xi_l(E) = \infty$, for any E, a and u as above. This result relies on the degeneracy of eigenvalues of the pure Dirichlet Laplacian on a cube. There is, however, a set of full measure $\mathcal{E} \subset \mathbb{R}$ with dense complement such that $\lim_{\mathbb{N}\ni l\to\infty} \xi_l(E) = 0$, for all $E \in \mathcal{E}$, if $a(l) \leq l^{-k}, k > 3$.

iv) In contrast to the above unboundedness results, Sobolev, [50] showed that for the pair $H_1 = -\Delta$ and $H_2 = -\Delta + u$ with $|u(x)| \leq \mathit{const.}\,(1+|x|)^{-\alpha}$ and $\alpha > d$, the spectral shift function ξ is, indeed, locally bounded. However, this type of result seems to require very strong hypotheses on H_1, for example, a limiting absorption principle and in particular, that H_1 has absolutely continuous spectrum on the positive real axis.

Theorem 2.ii) has a nice consequence in the theory of random Schrödinger operators. In this case, we take f to be the the derivative of a smooth, monotone switch function $\rho := \rho_{E,\varepsilon}\colon \mathbb{R} \to [-1, 0]$. By a switch function we mean that for a positive $\varepsilon \leq 1/2$ it has the following properties: $\rho \equiv -1$ on $(-\infty, E - \varepsilon]$, $\rho \equiv 0$ on $[E + \varepsilon, \infty)$ and $\|\rho'\|_\infty \leq 1/\varepsilon$. Theorem 2.ii) and the Krein trace identity, see §2.1, then imply that there is a constant C_E such that

$$\mathrm{Tr}\,[\rho(H_2) - \rho(H_1)] \leq C_E\,|\log(\varepsilon)|^d. \tag{7}$$

The estimate (7) improves upon the bound derived by Combes, Hislop and Nakamura in [14]. They prove that for any exponent $\alpha < 1$, there is a constant $\tilde{C}_E(\alpha)$ depending only on d, C_0, $\mathrm{diam\,supp}\,u$, $E + \varepsilon$ and α such that

$$\mathrm{Tr}\,[\rho(H_2) - \rho(H_1)] \leq \tilde{C}_E(\alpha)\ \varepsilon^{-\alpha}. \tag{8}$$

An *alloy type model* is a random Schrödinger operator $H_\omega = H_0 + V_\omega$, where $H_0 = H_A + V_{\mathrm{per}}$ with a periodic potential V_{per}. The random part of the potential has the form $V_\omega(x) = \sum_{k\in\mathbb{Z}^d} \omega_k\, u(x - k)$. The *coupling constants* $\omega_k, k \in \mathbb{Z}^d$, are a sequence of bounded random variables, which are independent and identically distributed with distribution μ. The expectation of the product measure $\bigotimes_{k\in\mathbb{Z}^d} \mu$ is denoted by $\mathbb{E}$. The single *site potential* $u \not\equiv 0$ is of compact support. Denote for $\varepsilon > 0$

$$s(\mu, \varepsilon) = \sup\{\mu([E - \varepsilon, E + \varepsilon]) \mid E \in \mathbb{R}\}. \tag{9}$$

With this definition, we have

Theorem 3. *Let H_ω be an alloy type model and $u \geq \kappa\chi_{[-1/2,1/2]^d}$ for some positive κ. Then for each $E_0 \in \mathbb{R}$ there exists a constant C_W such that, for all $E \leq E_0$ and $\varepsilon \leq 1/2$*

$$\mathbb{E}\{\mathrm{Tr}[\chi_{[E-\varepsilon,E+\varepsilon]}(H_\omega^l)]\} \leq C_W\ s(\mu, \varepsilon)\,(\log \tfrac{1}{\varepsilon})^d\ |\Lambda_l| \tag{10}$$

In particular, if μ is Hölder continuous with exponent α, then the ε-dependence of the RHS of (10) is $\varepsilon^\alpha(\log \frac{1}{\varepsilon})^d$. In [54], Stollmann proved a weaker version of (10) with RHS equal to $C_W\ s(\mu, \varepsilon)\,|\Lambda_l|^2$.

Bounds like (10) are called *Wegner estimates.* They were first deduced by physical reasoning by Wegner in [58] for the *Anderson model*, the finite difference analogue of the alloy type model. Wegner estimates are important *a priori* estimates, used to derive regularity properties of the integrated density of states (IDS) and to prove localization for random Schrödinger operators. In this context, localization means the existence of an energy region where the random operator has almost surely dense pure point spectrum with exponentially decaying eigenfunctions. See, e.g., [55] for a monograph exposition and, e.g., [21] (and the references therein) for more recent developments.

The proof of Theorem 3 can be directly applied to Anderson type models, i.e. random Schrödinger operators on $\ell^2(\mathbb{Z}^d)$. In this case, a compactly supported potential is a finite rank operator. For such perturbations the supremum-norm of the induced SSF is bounded by the rank of the operator, see e.g. [5, 14]. The uniform bound on the SSF yields a bound like (10) with the RHS side equal to $C_W\ s(\mu,\varepsilon)\,|\Lambda|$ and the constant C_W independent of the energy E. This gives an easy proof of a Wegner estimate for Hölder continuous single site distributions μ.

The IDS $N(E)$ is defined as the limit of the distribution functions,

$$N^l_\omega(E) := |\Lambda_l|^{-1}\ \#\{\text{ eigenvalues of } H^l_\omega \text{ not greater than } E\},$$

as l tends to infinity. For almost all $\omega\in\Omega$ the limit exists and is independent of ω. As a consequence of Theorem 3, the IDS of the above alloy-type model satisfies

$$|N(E_1)-N(E_2)| \le C_I\ s(\mu,|E_1-E_2|)\left(\log\frac{1}{|E_1-E_2|}\right)^d, \quad |E_1-E_2|\le 1/2,$$

where the constant C_I may be chosen uniformly if E_1 and E_2 vary in a compact interval I. Note that this continuity result cannot be obtained from a Wegner estimate with quadratic dependence on the volume of the box Λ_l. It also shows that, up to a logarithmic correction, the IDS enjoys the same regularity properties as the distribution of the random potential.

Let us return to the discussion of the regularity of the SSF. Denote by $H_1 = H_\omega - \omega_0 u$ and $H_2 = H_1 + u$ the alloy type operators where the value of the coupling constant at $k=0$ is frozen and equal to 0 and 1 respectively. The other coupling constants are still random. Despite the examples given above it is still possible that the *average* of the SSF $\bar\xi(\lambda) := \mathbb{E}\{\xi(\lambda,H_1,H_2)\}$ over the *random background* environment is locally bounded. This then implies no logarithmic loss in the Wegner estimate, and thus the Lipschitz continuity of the integrated density of states.

Indeed, such a bound on $\bar\xi$ for $d\le 3$ has recently been announced by Combes and Hislop, [23]. They have to assume that the single site distribution has a bounded density with respect to Lebesgue measure. So far, the averaging techniques at our disposal do not seem to be sufficient enough to prove that $\bar\xi(\lambda)$ is locally bounded for *rough* single site distributions, even if they are Hölder continuous.

In this context we would like to mention bounds on averaged fractional powers of the SSF derived in [1].

As a final remark, we discuss how Theorem 2 can be used to improve Wegner estimates for alloy type models with somewhat different properties than in Theorem 3. First we present an improvement of a recent Wegner estimate by Combes, Hislop and Klopp [12] for single site potentials with small support.

Theorem 4. *Let H_ω be an alloy type model as defined in the paragraph follwing* (8)*. Assume that V_{per} has the unique continuation property and is bounded below, ω is distributed according to a bounded density and $0\le u\in L^\infty$ is strictly positive* *on an open set. Then for each $E_0 \in \mathbb{R}$ there exists a constant C_W such that*

$$\mathbb{E}\{\mathrm{Tr}\chi_{[E-\varepsilon,E+\varepsilon]}(H_\omega^l)\} \le C_W\ \varepsilon\ \left(\log\frac{1}{\varepsilon}\right)^d\ |\Lambda| \tag{11}$$

for all $E \le E_0$ and $\varepsilon \le 1/2$.

This follows directly if one uses Theorem 2 instead of the L^p-estimates on the SSF in the Appendix of [12].

We mention three more disorder regimes where Theorem 2 may be used to simplify proofs of earlier Wegner estimates and to improve the dependence of the estimate in the energy interval length.
(1) *Single site potentials with small support and singular coupling constants.* Using the perturbation technique of Kirsch, Stollmann and Stolz in [32], we can extend the result from Theorem 3 to single site potentials $u \ge \kappa\chi_{[-s,s]^d}$ for some $\kappa, s > 0$ in the case of zero magnetic field for energies near spectral edges. In this case no assumption on the unique continuation property is needed.
(2) *Coupling constants whose distribution is continuous merely at the extreme values.* In [33], Kirsch and Veselić prove a Wegner estimate for alloy type potentials with non-positive single site potentials and coupling constants which have merely in a neighborhood of their maximal value a continuous distribution with bounded density. The estimate applies to energies at the bottom of the spectrum. Using Theorem 2 in the present paper, one can improve the Wegner estimate in [33].
(3) *Single site potentials with changing sign.* In [24], Hislop and Klopp studied alloy type models with continuous, compactly supported single site potentials, which may take values with both signs, and bounded coupling constants which are distributed according to a bounded, piecewise absolutely continuous density. They prove a Wegner estimate which is Hölder continuous in the energy variable and applies to energies below the spectrum of the non-random, unperturbed operator H_0. The result extends to internal spectral boundaries in the weak disorder regime. An extension of Theorem 2 in the present paper to the case where the perturbation is equal to a potential sandwiched between the square roots of the resolvent would make it possible to improve the Wegner estimate in [24] with regards to the energy interval length.

Further results on Wegner estimates for alloy type Schrödinger operators can be inferred from [4, 13, 27, 30, 31, 36, 56, 57] and the references therein. Let us mention specifically, that if μ has bounded density *and* $u \ge \kappa\chi_{\Lambda_1}$ the IDS is actually Lipschitz-continuous, see [37, 11]. However, the proof of this result is based on quite different methods than ours. It does not use estimates on the SSF; instead the residue theorem is applied to obtain an uniform bound on averaged resolvents. Due to the use of complex analysis it is not clear whether this method can be extended to the case when the single site distribution μ does not have a density.

Let us sketch the outline of the paper: The next section contains the definition of the SSF and the proofs of Theorems 1 and 2. In Section 3 we prove a lemma which is needed to deal with singular coupling constants and complete the proof of the Wegner estimate, Theorem 3.

2. Bounds on the SSF

2.1. **Definition of the SSF.** We define the SSF in three steps. Each of them extends the definition to a larger class of operators. For proofs see, e.g., [5] or [59].

Assume first that H_1, H_2 are selfadjoint, lower-semibounded with purely discrete spectrum. Then the SSF is defined as the difference of the eigenvalue counting functions,

$$\xi(\lambda) := \#\{n \mid \lambda_n(H_2) \leq \lambda\} - \#\{n \mid \lambda_n(H_1) \leq \lambda\},$$

where $\lambda_n(H)$ enumerates the spectrum of H, including multiplicity, in increasing order. Consider now a pair of selfadjoint, lower-semibounded operators such that the difference $H_2 - H_1$ is trace class. Then there is a unique function ξ such that *Krein's trace identity*

$$\operatorname{Tr}\left[\rho(H_2) - \rho(H_1)\right] = \int \rho'(\lambda)\, \xi(\lambda, H_2, H_1)\, d\lambda \tag{12}$$

holds for all $\rho \in \mathcal{C}^\infty$ with compactly supported derivative (actually, ρ can be taken to lie in a certain Besov space, see [41]). If the operators have discrete spectrum, this definition of ξ coincides with the previous one. It can be recovered choosing a sequence of ρ_ε which converges to a step function as $\varepsilon \to 0$.

Finally, we weaken the trace class assumption on the operator difference. Let $g\colon \mathbb{R} \to [0,\infty)$ be a monotone, smooth function such that $g(H_2) - g(H_1)$ is trace class. Assume that g is bounded on the spectra of H_1 and H_2. Then the SSF for the operator pair $g(H_1), g(H_2)$ is well defined and we may set

$$\xi(\lambda, H_2, H_1) := \operatorname{sign}(g')\ \xi\big(g(\lambda), g(H_2), g(H_1)\big). \tag{13}$$

This definition is independent of the choice of g. Formula (13) is called the *invariance principle*. This last definition will be sufficiently general to cover the Schrödinger operators we are considering. In the sequel we will choose $g(x) = e^{-x}$.

Alternatively, the SSF can be defined via the perturbation determinant from scattering theory by

$$\xi(\lambda, H_1, H_2) := \frac{1}{\pi} \lim_{\varepsilon \searrow 0} \arg\det\left[1 + (H_1 - H_2)(H_2 - \lambda - i\varepsilon)^{-1})\right]$$

if $(H_1 - H_2)(H_2 + i)^{-1}$ is trace class.

2.2. **Decay of singular values.** Weyl's asymptotic law gives the asymptotic behaviour of the n^{th} eigenvalue of the Laplacian on an open ball B for large n. The following simple lemma provides a robust lower bound, very much in the spirit of Weyl's law, but valid for all n and for general magnetic Schrödinger operators. It is the starting point for our proof of Theorem 1.

As it costs us nothing in clarity, the lemma will be presented under weaker hypotheses than those described in the introduction.

Lemma 5. *Let $H = H_A + V = (-i\nabla - A)^2 + V$ as in the introduction (cf. (1)), except that we now require that V_- is merely $-\Delta$ bounded with relative bound $\delta < 1$. Furthermore, let $H^{\mathcal{U}}$ be the Dirichlet restriction of H to an arbitrary open set $\mathcal{U}$ with finite volume $|\mathcal{U}|$ (also defined via the corresponding quadratic forms). Then, for some constant C, the n^{th} eigenvalue of $H^{\mathcal{U}} = H_A + W$ satisfies*

$$E_n \geq \frac{2\pi(1-\delta)d}{e} \left(\frac{n}{|\mathcal{U}|}\right)^{2/d} - C \qquad \text{for all } n \in \mathbb{N}. \tag{14}$$

Proof. Since the Dirichlet Sobolev space $H_0^1(\mathcal{U})$ is a natural subset of $H^1(\mathbb{R}^d)$, V_- is also relatively form bounded w.r.t. $-\Delta^{\mathcal{U}}$, the Dirichlet Laplacian on $\mathcal{U}$ with relative bound δ. The diamagnetic inequality, [48], then implies that V_- is also relative form bounded w.r.t. to the Dirichlet restriction of H_A to $\mathcal{U}$. That is, there exists a $C \in \mathbb{R}$ such that, as quadratic forms,

$$V_- \le \delta H_A + C.$$

In particular, since V_+ is non-negative,

$$H^{\mathcal{U}} \ge H_A - V_- \ge (1-\delta)H_A - C,$$

which implies the bound

$$\begin{aligned}\operatorname{Tr}(e^{-2tH^{\mathcal{U}}}) &\le e^{2tC}\operatorname{Tr}(e^{-2t(1-\delta)H_A}) = e^{2tC}\|e^{-t(1-\delta)H_A}\|_{\mathrm{HS}}^2 \\ &= e^{2tC}\iint_{\mathcal{U}\times\mathcal{U}} |e^{-t(1-\delta)H_A}(x,y)|^2 dx\, dy,\end{aligned}$$

where $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm. Again, using the diamagnetic inequality for the Schrödinger semigroup, e.g., [48, 26], one has the pointwise bound $|e^{-t(1-\delta)H_A}(x,y)| \le e^{t(1-\delta)\Delta^{\mathcal{U}}}(x,y)$. In particular,

$$\|e^{-t(1-\delta)H_A}\|_{\mathrm{HS}}^2 \le \|e^{t(1-\delta)\Delta^{\mathcal{U}}}\|_{\mathrm{HS}}^2 = \operatorname{Tr}(e^{2t(1-\delta)\Delta^{\mathcal{U}}}) \le |\mathcal{U}|\left(8\pi t(1-\delta)\right)^{-d/2}.$$

In the last line we used the fact that the kernel of the Dirichlet semigroup $e^{\beta\Delta^{\mathcal{U}}}$ on the diagonal is bounded by the free kernel, i.e.,

$$e^{\beta\Delta^{\mathcal{U}}}(x,x) \le e^{\beta\Delta}(x,x) = (4\pi\beta)^{-d/2} \quad \text{for all } \beta > 0 \text{ and } x \in \mathcal{U},$$

which follows immediately from the probabilistic representation of the Dirichlet semigroup [3, 47]. Thus

$$\operatorname{Tr}(e^{-2tH^{\mathcal{U}}}) \le |\mathcal{U}|\left(8\pi t(1-\delta)\right)^{-d/2}.$$

Let $\mathcal{N}^{\mathcal{U}}(E)$ be the number of eigenvalues of $H^{\mathcal{U}}$ smaller or equal to E. By Čebyšev's inequality and the above bound,

$$\begin{aligned}\mathcal{N}^{\mathcal{U}}(E) &\le e^{2tE}\int_{-\infty}^{E} e^{2ts}d\mathcal{N}^{\mathcal{U}}(s) \le e^{2tE}\operatorname{Tr}(e^{-2tH^{\mathcal{U}}}) \\ &\le |\mathcal{U}|\,(8\pi(1-\delta))^{-d/2}\, t^{-d/2}e^{2t(E+C)} = |\mathcal{U}|\left(\frac{e(E+C)}{2\pi(1-\delta)d}\right)^{d/2}\end{aligned}$$

where, in the last equality, we choose $t := \frac{d}{4(E+C)}$. Since $n \le \mathcal{N}^{\mathcal{U}}(E_n)$, this, in turn, implies the lower bound

$$E_n \ge \frac{2\pi(1-\delta)d}{e}\left(\frac{n}{|\mathcal{U}|}\right)^{2/d} - C$$

on the eigenvalues. □

Proof of Theorem 1. We give the proof for V_{eff}, the adaption to V_{eff}^l requires only minor changes. We will use the symbols c and C for constants that vary from line to line; however, their dependence on H_1 and H_2 will always be as stated in the Theorem.

Without loss of generality, we can assume that the origin is contained in the support of u. We will estimate the n^{th} singular value by Dirichlet decoupling at an n-dependent radius R. To this end, let R be sufficiently large that $\operatorname{supp}(u)$ is contained strictly inside the ball of radius R centered at the origin, which we will denote by B_R.

Let H_j^R ($j = 1$ or 2) be the Dirichlet restriction of H_j to the B_R, and let

$$A_R := e^{-H_2^R} - e^{-H_1^R} \quad \text{and} \quad D_R := V_{\text{eff}} - A_R. \tag{15}$$

As any Kato-class potential is relatively form bounded with respect to the Laplacian with relative bound zero, we may apply Lemma 5 to deduce that $\mu_n(e^{-H_j^n}) \leq C\exp(-cn^{2/d}R^{-2})$ for both $j = 1$ and $j = 2$. Since A_R is the difference of two *non-negative* operators by the min-max theorem its singular values obey the same type of bound:

$$\mu_n(A_R) \leq C\exp(-cn^{2/d}R^{-2}). \tag{16}$$

If D_n is bounded, then $\mu_n(V_{\text{eff}}) \leq \mu_n(A_R) + \|D_n\|$. We now proceed to estimate the norm of D_n by using the Feynman-Kac-Itô formula for magnetic Schrödinger semigroups with Dirichlet boundary conditions, see [6, 48].

Let $\mathbf{E}_x$ and $\mathbf{P}_x$ denote the expectation and probability for a Brownian motion, b_t starting at x. If $\tau_R = \inf\{t > 0 | b_t \notin B_{R_n}\}$ denotes the exit time from the ball B_R, then

$$(D_n f)(x) = \mathbf{E}_x\left[e^{-iS_A(b)}\left(e^{-\int_0^1 (V+u)(b_s)ds} - e^{-\int_0^1 V(b_s)ds}\right)\chi_{\{\tau_n \leq 1\}}(b) f(b_1)\right]$$

where S_A^t is real valued stochastic process corresponding to the purely magnetic part of the Schrödinger operator. To be precise, one has to fix a suitable gauge, e.g., Coulomb gauge, i.e., $\operatorname{div} A = 0$, for this and then use gauge invariance for the general case, see [38].

By taking the modulus and using the triangle inequality, one sees that the magnetic vector potential drops out:

$$|D_n f|(x) \leq \mathbf{E}_x\big[e^{-\int_0^1 V(b_s)ds}\big|e^{-\int_0^1 u(b_s)ds} - 1\big|\chi_{\{\tau_n \leq 1\}}(b)|f(b_1)|\big].$$

Moreover, only Brownian paths which both visit $\operatorname{supp} u$ and leave B_R within one unit of time contribute to the expectation. Thus if τ_u is the hitting time for $\operatorname{supp}(u)$ and $\mathcal{B} = \{\tau_R \leq 1, \tau_u \leq 1\}$,

$$|D_n f|(x) \leq \mathbf{E}_x\big[e^{-\int_0^1 V(b_s)ds}\big|e^{-\int_0^1 u(b_s)ds} - 1\big|\chi_{\mathcal{B}}(b)|f(b_1)|\big],$$

so, applying Hölder's inequality,

$$\begin{aligned}|D_n f|(x) \leq & \Big(\mathbf{E}_x\big[e^{-8\int_0^1 V(b_s)ds}\big]\Big)^{1/8}\Big(\mathbf{E}_x\big[\big|e^{-\int_0^1 u(b_s)ds} - 1\big|^8\Big)^{1/8} \\ & \Big(\mathbf{E}_x\big[\chi_{\mathcal{B}}(b)\big]\Big)^{1/4}\Big(\mathbf{E}_x\big[|f(b_1)|^2\big]\Big)^{1/2}.\end{aligned}$$

By Kashminskii's lemma, the Kato-class condition on V_- and u_- implies that the first two terms are bounded uniformly in x, [2, 49].

Levy's inequality combined with elementary estimates imply $\mathbf{P}_{x=0}\{\tau_R \leq 1\} \leq 2\mathbf{P}_{x=0}\{|b_1| \geq R\} \leq Ce^{-R^2/4}$. As any path in $\mathcal{B}$ must cover the distance r between supp u and the complement of the ball B_R, we can deduce that $\mathbf{P}_x(\mathcal{B}) \leq Ce^{-r^2/4} \leq Ce^{-R^2/8}$ where we chose without loss of generality $r \geq R/\sqrt{2}$. Thus

$$|D_n f|(x) \leq Ce^{-R^2/32}\big\{\mathbf{E}_x|f(b_1)|^2\big\}^{1/2} = Ce^{-R^2/32}\big\{(e^{\Delta}|f|^2)(x)\big\}^{1/2},$$

in particular, using the fact that e^{Δ} is an L^1 contraction,

$$\|D_n f\|_2 \le Ce^{-R^2/32}\big\|(e^{\Delta}|f|^2)\big\|_1^{1/2} \le Ce^{-R^2/32}\|f^2\|_1^{1/2} = Ce^{-R^2/32}\|f\|_2.$$

To balance the two bounds obtained for $\mu_n(A_R)$ and $\|D_n\|$ one chooses $R := n^{1/2d}$, which leads to (2). □

2.3. **Singular value decay implies SSF estimate.**

Proof of Theorem 2. Let F_t and the two Schrödinger operators $H_2 = H_1 + u$ be as in the Theorem. Using the invariance principle and a change of variables, we have

$$\begin{aligned}\int_{-\infty}^{T} F(|\xi(\lambda, H_2, H_1)|)\, d\lambda &= \int_{-\infty}^{T} F(|\xi(e^{-\lambda}, e^{-H_2}, e^{-H_1})|)\, d\lambda \\ &\le e^T \int_{e^{-T}}^{\infty} F(|\xi(s, e^{-H_2}, e^{-H_1})|)\, ds\end{aligned}$$

By an estimate of Hundertmark and Simon [25], the integral on the RHS is bounded by

$$\begin{aligned}\int_{-\infty}^{\infty} F(|\xi(s, e^{-H_2}, e^{-H_1})|)\, ds &\le \sum_{n=1}^{\infty} \mu_n(V_{\text{eff}})(F(n) - F(n-1)) \\ &\le \sum_{n=1}^{\infty} \mu_n(V_{\text{eff}}) \int_{n-1}^{n} (e^{ts^{1/d}} - 1) ds \le C \sum_{n=1}^{\infty} e^{(t-c)n^{1/d}}\end{aligned}$$

which is finite, if we chose t smaller than the constant c from Theorem 1. This proves (4).

To prove (5), we dualize the bound (4) with the help of Young's inequality for an appropriate pair of functions. Note that F_t is non-negative, convex with $F_t'(0) = 0$ and hence its Legendre transform G is well defined and satisfies

$$G(y) := \sup_{x \ge 0}\{xy - F(x)\} \le y\Big(\frac{\log(1+y)}{t}\Big)^d \text{ for all } y \ge 0$$

Thus, by the very definition of G, Young's inequality holds: $yx \le F(x) + G(y)$. So, with $b = \sup\operatorname{supp}(f)$,

$$\int f(\lambda)\xi(\lambda)\, d\lambda \le \int_{-\infty}^{b} F(|\xi(\lambda)|)\, d\lambda + \int G(|f(\lambda)|)\, d\lambda \tag{17}$$

Using the estimate (4), the first integral is bounded by $K_1 e^b$. For the second integral in (17), we estimate

$$\int G(|f(\lambda)|)\, d\lambda \le \int |f(\lambda)|\Big(\frac{\log(1+|f(\lambda)|)}{t}\Big)^d d\lambda \le t^{-d}|\log(1 + \|f\|_\infty)|^d\, \|f\|_1.$$

This finishes the proof of Theorem 2. □

3. Proof of the Wegner estimate

3.1. A partial integration formula for singular distributions. The main new idea to deal with single site distributions that are not absolutely continuous, is the following simple

Lemma 6. *Let μ be a probability measure with support in $(a,b))$ (or (a,∞), if its support is unbounded from above) and $\phi \in C^1(\mathbb{R})$ be non-decreasing and bounded. Then for any $\varepsilon > 0$,*

$$\int_{\mathbb{R}} [\phi(\lambda+\varepsilon) - \phi(\lambda)]\, d\mu(\lambda) \leq s(\mu,\varepsilon)\cdot[\phi(b+\varepsilon)-\phi(a)]$$

where $s(\mu,\varepsilon)$, the modulus of continuity of μ, is defined in (9).(If $b=\infty$, $\phi(b+\epsilon)$ means $\lim_{x\to\infty}\phi(x)$, which exists by the properties of ϕ.)

Proof. The proof of this lemma follows immediately from the well-known integration-by-parts formula for Stieltjes integrals. We include it for the convenience of the reader. We write $d\mu = dM$, where M is the distribution function of μ. In the following, all integrals are defined as Stieltjes integrals. Shifting variables and using that M is constant outside of $[a,b]$ gives

$$\int [\phi(\lambda+\varepsilon)-\phi(\lambda)]\, d\mu(\lambda) = \int_a^{b+\varepsilon} \phi(\lambda)\, d[M(\lambda-\varepsilon)-M(\lambda)].$$

Integrating by parts gives

$$\begin{aligned}\int [\phi(\lambda+\varepsilon)-\phi(\lambda)]\, d\mu(\lambda) = {} & \Big[\phi(\lambda)[M(\lambda-\varepsilon)-M(\lambda)]\Big]_a^{b+\varepsilon} \\ & - \int_a^{b+\varepsilon} \phi'(\lambda)[M(\lambda-\varepsilon)-M(\lambda)]\, d\lambda.\end{aligned}$$

The first term is zero, since M is constant outside of $[a,b]$ (in case $b=\infty$, one uses boundedness of ϕ and $\lim_{\lambda\to\infty}[M(\lambda-\varepsilon)-M(\lambda)]=0$). The second term is bounded by

$$\begin{aligned}\int_a^{b+\varepsilon} \phi'(\lambda)[M(\lambda)-M(\lambda-\varepsilon)]\, d\lambda & \leq \sup_\lambda\, [M(\lambda)-M(\lambda-\varepsilon)]\cdot \int_a^{b+\varepsilon}\phi'(\lambda)\, d\lambda \\ & \leq s(\mu,\varepsilon)\cdot(\phi(b+\varepsilon)-\phi(a)),\end{aligned}$$

since $\phi' \geq 0$. □

3.2. **Proof of the Wegner estimate.** Let ρ be a switch function adapted to the interval $[E-\varepsilon, E+\varepsilon]$; see the discussion preceding (7). Then

$$\chi_{[E-\varepsilon,E+\varepsilon]}(x) \leq \rho(x+2\varepsilon)-\rho(x-2\varepsilon)$$

We may assume without loss of generality $\sum_k u(\cdot - k) \geq 1$. By the mini-max principle for eigenvalues, we conclude

$$\mathrm{Tr}[\rho(H_\omega^l+\varepsilon)] \leq \mathrm{Tr}\Big[\rho(H_\omega^l + \varepsilon\sum_k u(\cdot-k))\Big].$$

Assume without loss of generality that $l \in \mathbb{N}$. Then Λ_l is decomposed in $L := l^d$ unit cubes. We enumerate the lattice sites in Λ_l by $k\colon \{1,\dots,L\} \to \tilde{\Lambda} = \Lambda \cap \mathbb{Z}^d$, $n \mapsto k(n)$ and set

$$W_0 \equiv 0, \quad W_n = \sum_{m=1}^n u(\cdot - k(m)), \qquad n = 1,2,\dots,L$$

Thus

$$\begin{aligned}\mathbb{E}\{\mathrm{Tr}[\chi_{[E-\varepsilon,E+\varepsilon]}(H^l_\omega)]\} &\le \mathbb{E}\{\mathrm{Tr}[\rho(H^l_\omega+2\varepsilon)-\rho(H^l_\omega-2\varepsilon)]\} \\ &\le \mathbb{E}\{\mathrm{Tr}[\rho(H^l_\omega+2\varepsilon)-\rho(H^l_\omega+2\varepsilon-4\varepsilon W_L)]\} \qquad (18)\\ &\le \mathbb{E}\left\{\sum_{n=1}^{L}\mathrm{Tr}[\rho(H^l_\omega+2\varepsilon-4\varepsilon W_{n-1})-\rho(H^l_\omega+2\varepsilon-4\varepsilon W_n)]\right\}\end{aligned}$$

We fix $n\in\{1,\dots,L\}$, denote $k_0=k(n)$,

$$\omega^\perp := \{\omega^\perp_k\}_{k\in\tilde\Lambda}, \qquad \omega^\perp_k := \begin{cases}0 & \text{if } k=k_0,\\ \omega_k & \text{if } k\neq k_0,\end{cases}$$

and set

$$\phi(\omega_{k_0}) = \mathrm{Tr}\big[\rho(H^l_{\omega^\perp}-2\varepsilon+4\varepsilon W_{n-1}+\omega_{k_0}u(\cdot-k_0))\big], \quad \omega_n\in\mathbb{R}.$$

The function ϕ is continuously differentiable, monotone increasing and bounded. By definition of ϕ,

$$\mathbb{E}\{\mathrm{Tr}[\rho(H^l_\omega+2\varepsilon-4\varepsilon W_n))-\rho(H^l_\omega+2\varepsilon-4\varepsilon W_{n+1})]\} \le \mathbb{E}\{\int[\phi(\omega_{k_0}+2\varepsilon)-\phi(\omega_{k_0})]\,d\mu(\omega_{k_0})\}$$

Let $a=\inf\mathrm{supp}(\mu)-1$ and $b=\sup\mathrm{supp}(\mu)+1$. Using Lemma 6 and the Krein trace identity (12) together with the second part of Theorem 2, we have

$$\int[\phi(\omega_{k_0}+2\varepsilon)-\phi(\omega_{k_0})]\,d\mu(\omega_{k_0}) \le s(\mu,2\varepsilon)[\phi(b+2\varepsilon)-\phi(a)] \le C_E\, s(\mu,2\varepsilon)\,(\log(1/\varepsilon))^d$$

which implies that (18) is bounded by

$$C_E\sum_{n=1}^{L} s(\mu,2\varepsilon)\,(\log(1/\varepsilon))^d \le C_E\, s(\mu,2\varepsilon)\,(\log(1/\varepsilon))^d\; l^d$$

Note that we apply Theorem 2 successively L times. However, the constant C_E depends only on the diameter of u and a local norm of the negative part of the background potential. For this local norm exist an uniform estimate independent of Λ_l and the configuration of the coupling constants $\omega_k, k\neq k_0$.

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