Answer the questions in the spaces provided on the question sheets. If you need more scratch paper, there is an extra sheet at the back of the exam.

Name: ________________________________

<table>
<thead>
<tr>
<th>Question</th>
<th>Points</th>
<th>Score</th>
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<tr>
<td>1</td>
<td>5</td>
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<td>5</td>
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<td>3</td>
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<td>Total:</td>
<td>100</td>
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</table>
Multiple Choice Questions

1. (5 points) There are three patients who have been waiting for an organ transplant.

<table>
<thead>
<tr>
<th>Potential Recipient</th>
<th>Months Waiting</th>
<th>Antigens Matched</th>
<th>Percent Sensitized Against</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bob</td>
<td>6</td>
<td>2</td>
<td>20</td>
</tr>
<tr>
<td>Ted</td>
<td>3</td>
<td>4</td>
<td>20</td>
</tr>
<tr>
<td>Carol</td>
<td>2</td>
<td>3</td>
<td>20</td>
</tr>
</tbody>
</table>

A potential recipient receives (fraction of people at or behind their position in line) × 10 points for their waiting time, 2 points for each antigen matched, and 1 point for every 10% of the population they are sensitized against. Who receives the first organ that becomes available?

A. Bob
B. Ted
C. Carol

2. (5 points) Bob, Carol, Ted, and Alice view a cake as below. Suppose Bob cuts the cake into the pieces $X, Y, Z,$ and $W$ below.

If Bob is given $X$, Carol is given $Z$, Ted is given $W$, and Alice is given $Y$, then this assignment is

A. Not proportional and not envy-free
B. Proportional but not envy-free
C. Both proportional and envy-free
Short Answer Questions

3. Bob and Tom have shared an apartment for several years. Now that they are both leaving, they are dividing these items between the two of them using the adjusted winner procedure.

<table>
<thead>
<tr>
<th>Item</th>
<th>Bob</th>
<th>Tom</th>
</tr>
</thead>
<tbody>
<tr>
<td>Encyclopedia</td>
<td>10</td>
<td>30</td>
</tr>
<tr>
<td>Easy Chair</td>
<td>15</td>
<td>10</td>
</tr>
<tr>
<td>Painting</td>
<td>30</td>
<td>25</td>
</tr>
<tr>
<td>Rug</td>
<td>20</td>
<td>15</td>
</tr>
<tr>
<td>Kitchen Set</td>
<td>25</td>
<td>20</td>
</tr>
</tbody>
</table>

(a) (3 points) Make the initial assignment of items. Who is the initial winner and who is the initial loser?

**Solution:** Bob: Easy chair, painting, rug, kitchen set (90 points, initial winner)
Tom: Encyclopedia (30 points, initial loser)

(b) (4 points) Assign point ratios to the items owned by the initial winner.

**Solution:**
- Easy chair: $15/10 = 1.5$
- Painting: $30/25 = 6/5 = 1.2$
- Rug: $20/15 = 4/3 = 1.333...$
- Kitchen Set: $25/20 = 5/4 = 1.25$

(c) (2 points) There is one item which will be transferred from the initial winner to the initial loser. Which item is this?

**Solution:** The item to be transferred is the painting, since this has the lowest point ratio.

(d) (2 points) How many points does each person have after this item has been transferred?

**Solution:** After this item has been transferred, Bob has 60 points and Tom has 55 points.

(e) (2 points) What is the shared item?

**Solution:** The shared item is the kitchen set, since it has the next lowest point ratio.

Problem is continued on the next page
(f) (3 points) Write down algebraic expressions in \( x \) for the points of the initial winner and loser after \( x \) (proportionally) of the shared item has been transferred to the initial loser.

**Solution:** Bob: \( 60 - 25x \)  
Tom: \( 55 + 20x \)

(g) (3 points) Set the expressions in the previous step equal and solve for \( x \). Leave your answer as a fraction.

**Solution:**

\[
60 - 25x = 55 + 20x
\]

\[
5 = 45x
\]

\[
x = \frac{5}{45} = \frac{1}{9}
\]

(h) (2 points) What is the final assignment of items?

**Solution:** Bob: Easy chair, rug, \( 8/9 \) of kitchen set  
Tom: Encyclopedia, painting, \( 1/9 \) of kitchen set

4. Hawkeye (Clint Barton) and the Black Widow (Natasha Romanoff) are dividing up espionage items, which we will denote by the letters A through E. Here are their preference lists:

<table>
<thead>
<tr>
<th>Black Widow</th>
<th>Hawkeye</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>B</td>
<td>C</td>
</tr>
<tr>
<td>C</td>
<td>D</td>
</tr>
<tr>
<td>D</td>
<td>A</td>
</tr>
<tr>
<td>E</td>
<td>E</td>
</tr>
</tbody>
</table>

(a) (3 points) If the Black Widow goes first and both choose sincerely each time, what is the final allocation of items?

**Solution:** Black Widow: A, C, E  
Hawkeye: B, D

(b) (4 points) If the Black Widow goes first and both use the bottom-up strategy, fill in the blanks below indicating the order in which they make their choices (remember to fill in the choices right to left!).

Black Widow: ___ ___ ___ ___
Hawkeye: ___ ___ ___ ___
(c) (1 point) Which method will the Black Widow prefer to use?

Solution: The Black Widow is better off if the bottom-up strategy is used.

(d) (1 point) Which method would Hawkeye prefer to use?

Solution: Hawkeye is better off if both choose sincerely.
5. The three brothers Zeus, Poseidon, and Hades are inheriting the world after defeating their power hungry father, Cronos. They find the Knaster inheritance procedure to be an acceptable method for doing this. Their bids on the various items are recorded on the chart below in billions of dollars (that’s what B stands for).

<table>
<thead>
<tr>
<th>Item</th>
<th>Bids</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Zeus</td>
</tr>
<tr>
<td>The Underworld</td>
<td>10 B</td>
</tr>
<tr>
<td>The Sky and Air</td>
<td>20 B</td>
</tr>
<tr>
<td>The Waters</td>
<td>15 B</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>45 B</strong></td>
</tr>
<tr>
<td><strong>Fair Share</strong></td>
<td><strong>15 B</strong></td>
</tr>
</tbody>
</table>

(a) (4 points) Fill in the total each heir bids and their fair share in the table above.

**Solution:**

<table>
<thead>
<tr>
<th>Item</th>
<th>Bids</th>
</tr>
</thead>
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<tr>
<td></td>
<td>Zeus</td>
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<td><strong>Fair Share</strong></td>
<td><strong>15 B</strong></td>
</tr>
</tbody>
</table>

(b) (4 points) What is the initial assignment of items?

**Solution:** Zeus: The Sky and Air
Poseidon: The Waters
Hades: The Underworld
(c) (4 points) Calculate how much each heir must give to the estate or receive from the estate.

**Solution:**
- Zeus: Gives $20 - 15 = 5$ B to the estate.
- Poseidon: Gives $20 - 12 = 8$ B to the estate.
- Hades: Gives $15 - 8 = 7$ B to the estate.

(d) (4 points) How much money is left with the estate? Calculate how much more goes to each heir (leave your answer as a fraction).

**Solution:** There is $20$ B left with the estate. $20/3$ B goes to each heir.
6. Bob and Carol view a cake as below and they are dividing it between them using the method of divide-and-choose. They are only allowed to use vertical cuts.

(a) (3 points) If Bob cuts the cake, draw a vertical line on his diagram above indicating the cut he should make and label the pieces $A$ and $B$ from left to right. Draw the same vertical line on Carol’s diagram.

\[ \text{Solution:} \]

\[ \begin{array}{c}
\text{Bob} \\
A \quad B \\
\text{Carol} \\
A \quad B
\end{array} \]

(b) (2 points) In terms of the boxes, how much does Carol think each of Bob’s pieces is worth?

\[ \text{Solution:} \text{ Carol thinks A is only worth 2 units and B is worth 6 units.} \]

(c) (2 points) Which piece will Carol choose?

\[ \text{Solution:} \text{ She will choose B.} \]

(d) (3 points) On the diagram below, repeat (a) if Carol cuts first.

\[ \begin{array}{c}
\text{Bob} \\
\text{Carol}
\end{array} \]
(e) (2 points) In terms of the boxes, how much does Bob think each of Carol’s pieces is worth?

**Solution:** Bob thinks A is worth $\frac{16}{3}$ boxes and B is worth $\frac{8}{3}$ boxes.

(f) (2 points) Which piece will Bob choose?

**Solution:** Bob will choose A.
7. Suppose Bob, Carol, Ted, and Alice are dividing a cake using the Last-diminisher method. Their views of the cake are shown below.

(a) (4 points) If Bob cuts off a piece from the right end of the cake that he views as one quarter of the cake, draw a vertical line indicating where he will cut. Draw the corresponding line in each of Carol’s, Ted’s, and Alice’s diagrams.

Solution:

(b) (2 points) Assume the piece gets handed first to Carol, then to Ted, and finally to Alice. Who is the first person to trim the piece that Bob cut?
Solution: Carol thinks the piece Bob cut is 1/4 of the cake, so she passes it unaltered to Ted. Ted thinks this piece is 3/8 of the cake, which is more than 1/4, so he is the first to trim the piece.

(c) (2 points) Shade in the trimmings (the part this person puts back on the cake) on this person’s diagram above.

Solution: Ted thinks this piece is 3/8 of the cake, which is more than 1/4, so he trims off one box and puts that back on the cake. The trimmings are shaded light green below.

(d) (3 points) According to the Last-Diminisher method, who gets this first piece?

Solution: Ted is the only person who trims this piece, so he gets it.

(e) (3 points) What proportion of the cake does this person think he/she is receiving?

Solution: Ted thinks he is receiving exactly 1/4 of the cake.
8. Suppose Bob, Carol, and Ted view the cake as below and they divide it using the Lone-Divider method. Bob starts by cutting the cake into three pieces X,Y, and Z which he thinks are all the same size.

(a) (2 points) Which piece of X,Y,Z do both Carol and Ted want?

**Solution:** Carol and Ted both want Z.

(b) (2 points) Which of the pieces X,Y, and Z are Carol and Ted both willing to give to Bob? (remember this means that both Carol and Ted agree that the piece is at most 1/3 of the cake). There is more than one.

**Solution:** Both are willing to give either X or Y to Bob.

(c) (4 points) Assume that piece X has been given to Bob. Redraw Carol and Ted’s boxes without piece X below.

**Solution:**

(d) (4 points) Carol and Ted use divide-and-choose to distribute the remainder of the cake between the two of them. If Carol cuts, draw the line representing her cut on your diagram in part (c). Label the two pieces A and B.
(e) (2 points) Which piece will Ted choose?

**Solution:** A is worth 2.5 boxes to Ted and B is worth 3.5 boxes, so Ted will choose B.

(f) (2 points) Who does Bob envy?

**Solution:** Bob envies Carol, because she received a piece which is worth more than 1/3 of the cake according to him.