Research Statement

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My research interests are in applied algebraic geometry and commutative algebra. I am especially interested in problems which can be studied from a computational standpoint. My main research project involves splines, which are piecewise polynomial functions on a polytopal complex $\mathcal{P} \subseteq \mathbb{R}^n$ satisfying certain smoothness conditions. Splines play a central role in approximation theory, geometric modelling, and PDEs (via the finite element method) \cite{10}. Recently continuous splines have also made an appearance in toric geometry; Payne \cite{31} shows that the equivariant cohomology ring of the toric variety $X_\Sigma$ is the ring $C^0(\Sigma)$ of continuous splines on the fan $\Sigma$.

Although splines were initially studied from an analytic point of view, Billera introduced tools from homological and commutative algebra in \cite{6}. It is this algebraic perspective which I use in my research. In terms of commutative algebra the object of interest is the set of splines of smoothness $r$, denoted $C^r(\mathcal{P})$, which is a finitely generated algebra over the polynomial ring $\mathbb{R}[x_1, \ldots, x_n]$ (more on this in § 1). To study these objects, I began by writing a package in the computer algebra system Macaulay2 and performing calculations. Since then I have used Macaulay2 in conjunction with Mathematica to generate examples and formulate conjectures. I describe my results in § 2, and give directions for future research in § 3.

A broad theme in my research is the interplay of algebraic properties with the combinatorics and geometry of the underlying objects. This is present everywhere in algebraic splines - in investigations of freeness, Hilbert functions, regularity bounds, etc. I am interested in pursuing this interplay in the related area of toric varieties. A proposed project is described in § 3.

1 Background

Multivariate Splines. Let $\mathcal{P} \subseteq \mathbb{R}^n$ denote a polytopal complex; for our purposes this means $\mathcal{P}$ is a subdivision by convex polytopes of an $n$-dimensional topological manifold with boundary. We denote by $C^r(\mathcal{P})$ the set of real-valued functions, continuously dif-
differentiable of order $r$, whose restriction to a maximal face (facet) of $\mathcal{P}$ is a polynomial in $n$ variables. $C^r(\mathcal{P})$ has the structure of a finitely generated algebra over $\mathbb{R}[x_1, \ldots, x_n]$ with multiplication defined pointwise. A spline $F \in C^r(\mathcal{P})$ is described by giving its restriction $F|_{\sigma} \in \mathbb{R}[x_1, \ldots, x_n]$ for each facet $\sigma \in \mathcal{P}$. Given two facets $\sigma_1$ and $\sigma_2$ of $\mathcal{P}$ which intersect in a codimension one face $\tau$, let the affine span of $\tau$ have equation $l_{\tau}$ and let $F_i = F|_{\sigma_i}$ for $i = 1, 2$. The differentiability condition on $F$ across $\tau$ is expressed algebraically as $(l_{\tau})^{r+1} | (F_1 - F_2)$; under mild assumptions on $\mathcal{P}$ this local divisibility requirement in codimension 1 guarantees that $F \in C^r(\mathcal{P})$ (see [8]). In Figure 1 we show a simplicial complex $\Delta$ and a spline $F \in C^0(\Delta)$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Figure 1}
\end{figure}

Let $l_{ij}$ be the form cutting out the edge $\tau_{ij} = \sigma_i \cap \sigma_j$ in Figure 1. Then $l_{12} = x, l_{13} = y, l_{23} = y - x$. Note the divisibility conditions for $F$ are met:

\begin{align*}
F_1 - F_2 &= 3x = 3l_{12} \\
F_1 - F_3 &= 3y = 3l_{13} \\
F_2 - F_3 &= 3(y - x) = 3l_{23}.
\end{align*}

Of particular importance are the vector spaces $C^r_d(\mathcal{P})$, defined as the set of splines whose restriction to each facet is a polynomial of degree $\leq k$. In Figure 1, $F \in C^0_1(\Delta)$. The spaces $C^r_d(\Delta)$ are finite dimensional over $\mathbb{R}$. Two central problems in spline theory are

1. Expressing $\dim_\mathbb{R} C^r_d(\mathcal{P})$ in terms of $d$ and the combinatorial/geometric data of $\mathcal{P}$.

2. Constructing bases of these spaces with particular properties.

When $\mathcal{P} = \Delta$ is a planar simplicial complex, a signature result [2] of Alfeld–Schumaker, proved using analytic techniques, is a formula for $\dim_\mathbb{R} C^r_d(\Delta)$ when $d \geq 3r + 1$ and $\Delta$ generic. Descriptions of local bases (bases supported on the simplices surrounding a vertex) of $C^r_d(\Delta)$ for $d \geq 3r + 2$ appear in [19, 21]. For intermediate values of $d$ ($r < d < 3r + 2$), $\dim C^r_d(\Delta)$ is unknown except in the case $r = 1, d = 4$ [1], due in part to the fact that bases with local support may not exist [5].
Billera initiates an algebraic approach to spline theory in [6], defining a chain complex $\mathcal{R}/I$ whose top homology is the spline algebra and using a result of Whiteley [36] to prove a conjecture of Strang. Billera-Rose make the shift to graded commutative algebra in [8] by forming the homogenization $\hat{P} \subset \mathbb{R}^{n+1}$ of $P \subset \mathbb{R}^n$ by embedding $P$ in the hyperplane $x_0 = 1$ in $\mathbb{R}^{n+1}$ and taking the cone with the origin. $C^\sigma(\hat{P})$ is a graded $\mathbb{R}[x_0, \ldots, x_n]$-module and $C_d^\sigma(P) \cong C_d^\sigma(\hat{P})$. Hence $\dim C_d^\sigma(P) = HF(C^\sigma(\hat{P}), d)$, the Hilbert function of $C^\sigma(\hat{P})$ in degree $d$. For $d \gg 0$, $HF(C^\sigma(\hat{P}), d)$ agrees with the Hilbert polynomial $HP(C^\sigma(\hat{P}), d)$ of $C^\sigma(\hat{P})$.

In [32], Schenck and Stillman replace Billera’s complex with the chain complex $R/P \subset \mathbb{R}^2$ for simplicial complexes $\Delta \subset \mathbb{R}^2$ [33]. McDonald and Schenck extend this to a computation of $HP(C^\sigma(\hat{P}), d)$ for $P \subset \mathbb{R}^n$ a polytopal complex [24]. A subtle question, algebraically cast in terms of Castelnuovo-Mumford regularity, is determining for which values of $d$ we have $\dim C^\sigma(\hat{P}) = HP(C^\sigma(\hat{P}), d)$. We return to this story in §2.2.

**Toric Varieties.** In concrete terms, a (projective) toric variety is the closure of a map $(C^*)^k \to \mathbb{P}^n$ where $C^* = \mathbb{C} \setminus \{0\}$ and $\phi$ is given by monomials. Specifically, suppose $\alpha = (a_1, \ldots, a_k) \in \mathbb{N}^k$ and $t = (t_1, \ldots, t_k) \in \mathbb{C}^k$. Write $t^\alpha = t_1^{\alpha_1} \cdots t_k^{\alpha_k}$. Then for a choice of $A = \{\alpha_1, \ldots, \alpha_n\}$, $\phi_A : (C^*)^k \to \mathbb{P}^n$ is given in coordinates by $t \to (t_1^{\alpha_1}, \ldots, t_n^{\alpha_n})$. The projective closure of this map, $X_A \subset \mathbb{P}^n$, is the projective toric variety of $A$.

More generally, abstract toric varieties $X_\Sigma$ are described by the data of a polyhedral fan $\Sigma \subset \mathbb{R}^n$, which is a collection of convex cones $\sigma$ that fit together in the same fashion as a polytopal complex, and an ambient lattice $N \cong \mathbb{Z}^n \subset \mathbb{R}^n$. The data of an abstract variety is provided by associating to each cone $\sigma$ the affine semigroup algebra $\mathbb{C}[\sigma^\vee \cap M]$, where $\sigma^\vee$ is the dual cone to $\sigma$ and $M$ is the dual lattice to $N$. Virtually all geometric constructions on the toric variety $X_\Sigma$ can be interpreted combinatorially in terms of the fan $\Sigma$ and the lattice $N$.

Toric varieties are simple in the sense that they admit parametrizations by monomials, yet they provide a rich and concrete setting in which to encounter non-trivial algebro-geometric theory. For instance, a complete toric variety $X_\Sigma$ is projective iff $\Sigma$ has a piecewise linear function (spline!) which is strictly convex [11, Theorem 7.2.4], leading to simple constructions of smooth, complete, nonprojective varieties. In the wider world of algebraic geometry such examples are not nearly as transparent, as evidenced by constructions of Nagata [29] and Hironaka [20].

## 2 Past Research on Spline Modules

### 2.1 Freeness

Recall that an $R$-module $M$ is free if $M \cong R^n$ for some $n \in \mathbb{N}$. Computationally, freeness of $C^\sigma(\hat{P})$ is an important consideration since calculations of $\dim C_d^\sigma(P)$ are simplified if
$C^r(\hat{P})$ is free [9]. For simplicial complexes $\Delta \subset \mathbb{R}^n$, Billera [7] shows that $C^0(\Delta) \cong A_{\Delta}$, where $A_{\Delta}$ is the Stanley-Reisner ring of $\Delta$. In algebraic combinatorics it is known that if $\Delta$ is shellable ($\Delta$ can be built up in a nice way from its facets), then $A_{\Delta}$ is free [25, Theorem 13.45]. As a natural extension, Schenck asks in [35] if shellability of a polytopal complex $\mathcal{P} \subset \mathbb{R}^n$ implies freeness of $C^0(\hat{P})$. I prove the following theorem in [12].

**Theorem 2.1** For a pure, shellable, $n$-dimensional polytopal complex $\mathcal{P} \subset \mathbb{R}^n$ with $n \geq 2$, freeness of $C^0(\hat{\mathcal{P}})$ as an $S = \mathbb{R}[x_0, \ldots, x_n]$-module depends on the embedding of $\mathcal{P}$.

I prove this by example. For the shellable polytopal complex $\mathcal{Q}_1 \subset \mathbb{R}^2$ shown in Figure 2, $C^0(\hat{\mathcal{Q}}_1)$ is not free. A slight perturbation yields the complex $\mathcal{Q}_2$, for which $C^0(\hat{\mathcal{Q}}_2)$ is free. Each complex has 5 two dimensional faces, 8 interior edges, and 4 interior vertices.

![Figure 2](image-url)

This shows that there is no purely combinatorial characterization of the freeness of $C^0(\hat{\mathcal{P}})$. Instead there is a subtle interplay between the combinatorics and geometry of $\mathcal{P}$ which determines freeness of $C^0(\hat{\mathcal{P}})$. For instance, the vector space $C^0(\hat{\mathcal{Q}}_1)_1$ is spanned by the linear functions $x, y, z$. On the other hand, $C^0(\hat{\mathcal{Q}}_2)_1$ is four dimensional, with basis $x, y, z$ and the nontrivial piecewise linear function depicted in Figure 2. The existence of this ‘extra’ piecewise linear function forces a certain local cohomology module to be nonzero, which in turn implies $C^0(\hat{\mathcal{Q}}_1)$ is not free.

### 2.2 Local Generators and Regularity

When $\mathcal{P} = \Delta$ is a 2 or 3 dimensional simplicial complex, the vector space $C^r_d(\Delta)$ has a basis for $d \gg 0$ which is ‘local’ in the sense that the basis elements are nonzero only at a vertex $v$ and on the simplices which contain this vertex. Schumaker, Ibrahim, and Hong construct these ‘star-supported’ bases for $C^r_d(\Delta)$, $\Delta \subset \mathbb{R}^2$ for $d \geq 3r + 2$ in [19, 21]. Alfeld, Schumaker, and Sirvent construct a star-supported basis for $C^r_d(\Delta)$, $\Delta \subset \mathbb{R}^3$, for $d > 8r$ in [4].

In [13] I show there is an analogue for a star-supported basis in the polytopal case, with an interesting twist: instead of only being nonzero on facets containing a particular
vertex, the elements of this basis are supported on lattice complexes $P_W$ associated to linear subspaces $W \subset \mathbb{R}^n$ in the intersection lattice of affine hulls of codimension one faces of the polytopal complex $P$. The subcomplexes $P_W$ are closely related to an equivalence relation of Yuzvinsky [38].

**Theorem 2.2** For a polytopal complex $P \subset \mathbb{R}^n$ there are distinguished subcomplexes $P_W \subset P$ so that $C^r_d(P)$ has a basis of splines supported on the $P_W$ for $d \gg 0$.

A spline with support in one of the $P_W$ is called lattice-supported, as is a basis of $C^r_d(P)$ consisting of lattice-supported splines. If $P$ is a simplicial complex, the subcomplexes $P_W$ reduce to unions of stars of vertices. Hence a lattice-supported basis coincides with a star-supported basis in the simplicial case. See §4 for an example of the subcomplexes $P_W$.

More generally, I use lattice-supported splines to build approximating subalgebras $LS^{r,k}(P) \subset C^r(P)$, generated by splines $F \in C^r(P)$ which are supported on subcomplexes $P_W$ with $\text{codim}(W) \leq k$. I prove the following approximation theorem.

**Theorem 2.3** Let $C$ be the cokernel of the inclusion $LS^{r,k}(P) \hookrightarrow C^r(P)$. Then $C$ is supported at primes of codimension $< k$.

It turns out that the approximations $LS^{r,k}(P)$ can be used to bound the Castelnuovo-Mumford regularity of $C^r(\hat{P})$, denoted $\text{reg}(C^r(\hat{P}))$. In the case of splines, regularity is key because it bounds when $d$ is large enough so that $\dim C^r_d(P) = HP(C^r(\hat{P}), d)$. Based on computations I have made the following conjectures in the polytopal case [13].

**Conjecture 2.4** Let $P \subset \mathbb{R}^2$ be a hereditary polytopal complex with $F$ being the maximum number of edges of a facet of $P$. Then $\text{reg}(C^r(\hat{P})) \leq \text{reg}(LS^{r,2}(\hat{P})) \leq F(r+1) - 1$.

In the simplicial case, the above conjecture specializes to the known $3r+1$ bound due to Alfeld-Schumaker [2] (for generic simplicial complexes).

**Conjecture 2.5** Let $P \subset \mathbb{R}^2$ be a hereditary polytopal complex with $F$ being the maximum number of edges of a facet of $P$. Then $\text{reg}(C^r(\hat{P})) \leq (F-1)(r+1)$.

Conjecture 2.5 specializes to a conjecture of Schenck (the $2r+1$ conjecture) in the simplicial case, which remains unproved. An approach to the $2r+1$ conjecture using the cohomology of sheaves on $\mathbb{P}^2$ appears in [34]. I give an example in [13] showing that the bound in Conjecture 2.5 cannot be improved.

### 2.3 Regularity of Mixed Spline Spaces

In [15], I give bounds on the Castelnuovo-Mumford regularity of the algebra $C^r(\hat{P})$ of planar polytopal splines. I prove these bounds in the more general context of mixed splines (differing smoothness conditions are imposed across codimension one faces); for simplicity I state the bounds in the case of uniform smoothness. There are two main results. The first connects to well-known results from the analytic perspective.
Theorem 2.6 Let $\Delta \subset \mathbb{R}^2$ be a simplicial complex. Then $\text{reg}(C^r(\hat{\Delta})) \leq 3(r+1)$.

In [27], Mourrain and Villamizar give an algebraic proof that $HP(C^r(\hat{\Delta}), d)$ (explicitly computed by Alfeld-Schumaker) gives the correct value for $\dim C^r_d(\Delta)$ for $d \geq 4r+1$. Theorem 2.6 improves on their result, providing the first algebraic proof of a fact known from the analytic perspective - that $\dim C^r_d(\Delta) = HP(C^r(\hat{\Delta}), d)$ for $d \geq 3r+2$. This was first proved by Hong [19] and Ibrahim and Schumaker [21] by explicit construction of local bases. The next result is a step toward proving Conjecture 2.4 and provides the first general regularity bound on the algebra of planar polytopal splines.

Theorem 2.7 Let $\mathcal{P} \subset \mathbb{R}^2$ be a polytopal complex and $F$ the maximum number of edges in a polytope of $\mathcal{P}$. Then $\text{reg}(C^r(\hat{\mathcal{P}})) \leq (2F-1)(r+1)$.

As a corollary, $HP(C^r(\hat{\mathcal{P}}), d) = \dim C^r_d(\mathcal{P})$ for $d \geq (2F-1)(r+1)-1$. Since McDonald and Schenck have explicitly computed $HP(C^r(\hat{\mathcal{P}}), d)$, this result is of practical use.

The proof of these theorems relies on the approximation of $C^r(\hat{\mathcal{P}})$ by the subalgebras $LS^{r,k}(\hat{\mathcal{P}})$ constructed in [13]. A technique used in the proof of the Gruson-Lazarsfeld-Peskine theorem bounding the regularity of curves in projective space applies in our situation, yielding the regularity bound

$$\text{reg}(LS^{r,k}(\hat{\mathcal{P}})) \geq \text{reg}(C^r(\hat{\mathcal{P}})),$$

whenever the projective dimension of $C^r(\hat{\mathcal{P}})$ is at most $k$. I apply this for polytopal complexes $\mathcal{P} \subset \mathbb{R}^2$, reducing the problem of bounding $\text{reg}(C^r(\hat{\mathcal{P}}))$ to the problem of bounding $\text{reg}(LS^{r,1}(\hat{\mathcal{P}}))$. This latter problem can ultimately be reduced to bounding the regularity of a certain ideal in the polynomial ring in three variables.

2.4 Associated Primes of Spline Complexes

In [14], I analyze the spline complex $\mathcal{R}/\mathcal{J}$. The technical heart of the paper describes associated primes of the modules $H_i(\mathcal{R}/\mathcal{J})$, $1 \leq i < n$.

Theorem 2.8 Let $\mathcal{P} \subset \mathbb{R}^n$ be an $n$-dimensional polytopal complex, and $S = \mathbb{R}[x_0, \ldots, x_n]$.

1. $\text{Ass}_S(H_i(\mathcal{R}/\mathcal{J})) \subset \{I(W) | \dim(W) \leq i\}$, where $W \subset \mathbb{P}^n$ is linear

2. If $\mathcal{P} = \Delta$ is simplicial, then $\text{Ass}_S(H_i(\mathcal{R}/\mathcal{J})) \subset \{I(\gamma) | \gamma \in \Delta, \dim(\gamma) \leq i-1\}$.

I use Theorem 2.8 to compute the third coefficient of the Hilbert polynomial of the spline algebra in the context of mixed splines over polytopal complexes, generalizing computations in [17, 24]. A second consequence of Theorem 2.8 is a description of the fourth coefficient of the Hilbert polynomial of $HP(C^r(\hat{\Delta}), d)$ for simplicial complexes $\Delta \subset \mathbb{R}^3$. I use this to derive the result of Alfeld, Schumaker, and Whiteley on the generic dimension of $C^1$ tetrahedral splines for $d \gg 0$ [3] and indicate how this description may be applied in nongeneric configurations.
3 Future Research Projects

A. Prove Conjecture 2.5. Significant improvement of regularity bounds is challenging because global geometry must be taken into account for low degree spline computations. A well-known example of such a computation is the construction of a basis for $C^1_3(\Delta)$ by Alfeld, Piper, and Schumaker [1] for $\Delta \subset \mathbb{R}^2$; even dim $C^1_3(\Delta)$ is still unknown in general.

My approach in [15] relies on local geometry, so it will not succeed in decreasing the regularity bound to the level of Conjecture 2.5. One alternate approach, following Schenck-Stiller in [34], is to use vector bundle techniques, specifically semistability. Another method successfully applied by Whiteley [37] is to analyze $C^r_\Delta(\Delta)$ for generic planar $\Delta$ using techniques from rigidity theory.

B. Prove Conjecture 2.4. This involves improving the regularity bound $\text{reg}(C^r(\hat{P})) \leq (2F - 1)(r + 1)$ of Theorem 2.7 to $\text{reg}(C^r(\hat{P})) \leq F(r + 1) - 1$. My approach to reduce the bound in the simplicial case (Theorem 2.6) involves reduction to the local case and bounding the regularity of a specific ideal by building on work of Tohaneanu and Minac [26]. The same approach should work in the nonsimplicial case. The corresponding ideal is more difficult to analyze but yields itself readily to computational experimentation.

C. Improving dimension bounds for tetrahedral splines. There is a long history of refining dimension bounds for spline spaces since these bounds are important for computations. In [28], Mourrain and Villamizar use homological methods to provide bounds on the dimension $\dim C^r(\Delta)$, where $\Delta \subset \mathbb{R}^3$ is a simplicial complex. The upper and lower bounds they find differ for general complexes in large degree by at least the dimension of the homology module $H_2(\mathbb{R}/J)$.

Theorem 2.8 gives a precise description of the associated primes of the complex $\mathbb{R}/J$. Using this, I have precisely described contributions to the homology module $H_2(\mathbb{R}/J)$ in large degree as coming from ‘unexpected’ splines around vertices in low degree. The bounds of Mourrain and Villamizar could be improved by accounting for this constant. This is a delicate process, however, due to the presence of several nonzero local cohomology modules.

D. Eisenbud-Goto for toric surfaces. In [16], Eisenbud and Goto conjecture that if $I_X$ is a prime ideal of an irreducible variety $X$ which contains no linear form, then the regularity of $X$ is bounded by $\text{deg}(X) - \text{codim}(X)$. This is true for curves (Gruson-Lazarsfeld-Peskine [18]), and smooth surfaces (Lazarsfeld [22]), but is still widely open. Using the methods of Gruson-Lazarsfeld-Peskine, L’vovsky[23] derives a combinatorial regularity bound for toric curves (defined by bivariate monomials). Recently, Nitsche derives the Eisenbud-Goto bound for toric curves in an entirely combinatorial fashion, and also shows that the Eisenbud-Goto bound holds for certain simplicial affine semigroup rings [30]. As a future research project, I plan to investigate combinatorial regularity bounds for toric surfaces.
4 Example of locally supported splines

The following is an example to show that star-supported splines are not enough to generate \( C^r_d(P) \) in large degree if \( P \) is not simplicial. Consider the two dimensional polytopal complex \( Q \) in Figure 3 with 5 faces, 8 interior edges, and 4 interior vertices.

\[
\begin{array}{cccc}
(-2,2) & (-1,1) & (1,1) & (2,2) \\
(-1,-1) & (1,-1) & (2,-2) & (-2,-2)
\end{array}
\]

Figure 3: \( Q \)

It is readily verifiable that the unit \( 1 \in C^r_d(Q) \) cannot be written as a sum of splines with support on the stars of the 4 interior vertices, i.e. splines which restrict to 0 outside of the shaded regions in Figure 4, for any \( d > 0 \). However I show in [13] that \( C^r_d(Q) \) does have a basis consisting of splines supported on either the star of an interior vertex or one of the shaded complexes in Figure 5, for \( d \gg 0 \).

References


