ON THE BOUNDEDNESS OF THRESHOLD OPERATORS IN $L_1[0,1]$ WITH RESPECT TO THE HAAR BASIS

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Abstract. We prove a near-unconditionality property for the normalized Haar basis of $L_1[0,1]$.

1. Introduction

Let (e_i) be a semi-normalized basis for a Banach space X. For a finite subset $A \subset \mathbb{N}$, let $P_A(\sum_i a_i e_i) := \sum_{i \in A} a_i e_i$ denote the projection from X onto the span of basis vectors indexed by A. Recall that (e_i) is an unconditional basis if there exists a constant C such that, for all finite $A \subset \mathbb{N}$, $\|P_A\| \leq C$.

We say that (e_i) is near-unconditional if for all $0 < \delta \leq 1$ there exists a constant $C(\delta)$ such that for all $x = \sum a_i e_i$ satisfying the normalization condition $\sup |a_i| \leq 1$, and for all finite $A \subseteq \{i \colon |a_i| \geq \delta\}$,

$$\|P_A(x)\| \leq C(\delta)\|x\|.$$

Every unconditional basis is near-unconditional, and it is easy to check that a near-unconditional basis is unconditional if and only if $C(\delta)$ can be chosen to be independent of δ.

It was proved in [1] that a basis is near-unconditional if and only if the thresholding operators $\mathcal{G}_\delta(x) := \sum_{|a_i| \geq \delta} a_i e_i$ satisfy, for some constant $C_1(\delta)$,

$$\|\mathcal{G}_\delta(x)\| \leq C_1(\delta)\|x\|,$$

and that the class of near-unconditional bases strictly contains the important class of quasi-greedy bases, defined by Konyagin and Temlyakov

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[5] as the class of bases for which $C_1(\delta)$ may be chosen to be independent of δ.

Elton [2] proved that every semi-normalized weakly null sequence contains a subsequence which is a near-unconditional basis for its closed linear span. On the other hand, Maurey and Rosenthal [6] gave an example of a semi-normalized weakly null sequence with no unconditional subsequence.

By a theorem of Paley [7], the Haar system is an unconditional basis of $L_p[0,1]$ for $1<p<\infty$. For $p=1$, on the other hand, a well-known example (see e.g. [4]) shows that the normalized Haar basis is not unconditional. The same example, which we now recall, also shows that the Haar basis fails to be near-unconditional.

Define $h_0 = 1_{[0,1]}$, and for $k\in\mathbb{N}$, set

$$h_1^{(k)} = 2^{k-1}\big(1_{[0,2^{-k})} - 1_{[2^{-k},2^{1-k})}\big).$$

Observe that for any $n\in\mathbb{N}$

$$\Big\|h_0 + \sum_{k=1}^{2n} h_1^{(k)}\Big\| = 1,$$

and for some constant $c>0$

$$\Big\|h_0 + \sum_{k=1}^{2n} h_1^{(2k)}\Big\| > cn.$$

So, setting $f_n = h_0 + \sum_{k=1}^{2n} h_1^{(k)}$, and $A_n = \{0,2,4,\dots,2n\}$, we have $\|f_n\| = 1$ and $\|P_{A_n}(f_n)\| \ge cn$, which witnesses the failure of near-unconditionality with $\delta = 1$. In this example the nonzero coefficients of f_n are equal and they lie along the left branch of the Haar system. Our main result shows that in a certain sense every example of the failure of near-unconditionality must be of this type.

We state our main result precisely below but the idea is as follows. Suppose that the Haar coefficients of $f\in L_1[0,1]$, δ, and A are as stated in the definition of near-unconditionality. We show that there is an enlargement $B\supseteq A$ such that $\|P_B(f)\| \le C(\delta)\|f\|$ and we provide an explicit construction of B. Roughly speaking, the 'added' coefficients in $B\setminus A$ are those which lie along a segment of a branch of the Haar system such that the coefficient of the maximal element of the segment (with respect to the usual tree ordering) belongs to A and all the coefficients of f along the segment are approximately equal to each other (to within some prescribed multiplicative factor of $1+\varepsilon$). For f_n and the sets A_n, the enlargements are $B_n = \{0,1,2,\dots,2n\}$, and so $P_{B_n}(f_n) = f_n$, which renders the example harmless. Here the enlargement is as large as possible. The interest of our result, however, resides in the fact that, for certain f and A, the enlargement will often be trivial, i.e., $B = A$, or quite small.

The normalized Haar basis is not a quasi-greedy basis of $L_1[0,1]$, i.e., the Thresholding Greedy Algorithm fails to converge for certain initial vectors. In a remarkable paper [3] Gogyan exhibited a weak thresholding algorithm which produces uniformly bounded approximants converging to f for all $f \in L_1[0,1]$. The proof of our main theorem uses results and techniques from [3]. We have chosen to reprove some of these results to achieve what we hope is a self-contained and accessible presentation.

2. Notation and basic facts

We denote the dyadic subintervals of $[0,1]$ by $\mathcal{D}$, and put $\overline{\mathcal{D}} = \mathcal{D} \cup \{[0,2]\}$. We think of $\mathcal{D}$ and $\overline{\mathcal{D}}$ being partially ordered by "$\subset$". We denote by I^+ and I^- the left and the right half subinterval of $I \in \mathcal{D}$, respectively. I^+ and I^- are then the *direct successors* of I, while the set

$$\mathrm{succ}(I) = \{J \in \mathcal{D} : J \subsetneq I\}$$

is called the *successors of* $I \in \mathcal{D}$. The *predecessors of an* $I \in \overline{\mathcal{D}}$ is the set

$$\mathrm{pred}(I) = \{J \in \overline{\mathcal{D}} : J \supsetneq I\}.$$

It follows that the $\mathrm{pred}(I)$ is a linearly ordered set. If $I \subset J$ are in $\overline{\mathcal{D}}$ we put

$$[I, J] = \{K \in \overline{\mathcal{D}} : I \subseteq K \subseteq J\}.$$

Let $\mathcal{S} \subset \overline{\mathcal{D}}$ be finite and not empty. Then $\mathcal{S}$ contains elements I which are *minimal in* $\mathcal{S}$, *i.e.,* there is no $J \in \mathcal{S}$ for which $J \subsetneq I$. We put in this case

$$\mathcal{S}' = \mathcal{S} \setminus \{I \in \mathcal{S} : I \text{ is minimal in } \mathcal{S}\}.$$

Inductively we define $S^{(n)}$ for $n \in \mathbb{N}_0$, by $S^{(0)} = \mathcal{S}$, and, assuming $S^{(n)}$ has been defined, we put $\mathcal{S}^{(n+1)} = \big(S^{(n)}\big)'$. Since $\mathcal{S}$ was assumed to be finite, there is an $n \in \mathbb{N}$, for which $\mathcal{S}^{(n)} = \emptyset$, and we define *the order of* $\mathcal{S}$ by

$$\mathrm{ord}(\mathcal{S}) = \min\{n \in \mathbb{N} : \mathcal{S}^{(n)} = \emptyset\} - 1 = \max\{n \in \mathbb{N} : \mathcal{S}^{(n)} \neq \emptyset\}$$

and for $I \in \mathcal{S}$ we define the *order of* I *in* $\mathcal{S}$ to be the (unique) natural number $m \in [0, \mathrm{ord}(\mathcal{S})]$, for which $I \in \mathcal{S}^{(m)} \setminus \mathcal{S}^{(m+1)}$, and we denote it by $\mathrm{ord}(I, \mathcal{S})$.

$(h_I : I \in \overline{\mathcal{D}})$ denotes the L_1*-normalized Haar basis, i.e.,*

$$h_{[0,2]} = 1_{[0,1]} \text{ and } h_I = 2^n(1_{I^+} - 1_{I^-}), \text{ if } I \in \mathcal{D}, \text{ with } m(I) = 2^{-n},$$

m denoting the Lebesgues measure. If $f \in L_1[0,1]$ we denote the coefficients of f with respect to (h_I) by $c_I(f)$, and thus

$$f = \sum_{I \in \mathcal{D}} c_I(f) h_I \text{ for } f \in L_1[0,1]. \tag{1}$$

From the normalization of (h_I) it follows that

$$c_I(f) = \int_{I^+} f\,dx - \int_{I^-} f\,dx. \tag{2}$$

For $f \in L_1[0,1]$ the *support of f with respect to the Haar basis* is the set

$$\operatorname{supp}_H(f) = \{I \in \overline{\mathcal{D}} : c_I(f) \neq 0\}.$$

We will use the following easy inequalities for $f \in L_1[0,1]$ and $I, J \in \overline{\mathcal{D}}$, with $J \subseteq I$, *i.e.,*

$$\|f\|_I := \int_I |f|\,dx \geq \int_J |f|\,dx \geq \left| \int_{J^+} f\,dx - \int_{J^-} f\,dx \right| = |c_J(f)|. \tag{3}$$

For a finite set $\mathcal{S} \subset \overline{\mathcal{D}}$ we denote by $P_{\mathcal{S}}$ the canonical projection from $L_1[0,1]$ onto the span of $(h_I : I \in \mathcal{S})$:

$$P_{\mathcal{S}} : L_1 \to L_1,\ f \mapsto \sum_{I \in \mathcal{S}} c_I(f) h_I.$$

If $\mathcal{S}$ is cofinite $P_{\mathcal{S}}$ is defined by $\mathrm{Id} - P_{\overline{\mathcal{D}} \setminus \mathcal{S}}$. We will use the fact that the Haar system is a monotone basis with respect to any order, which is consistent with the partial order "$\subset$". It follows therefore that the projections

$$S_I : L_1[0,1] \to L_1[0,1],\ f \mapsto f - \sum_{J \subseteq I} c_J(f) h_J,$$

are bounded linear projections with $\|\mathcal{S}_I\| \leq 1$, for all $I \in \overline{\mathcal{D}}$. Moreover we observe that

$$\begin{aligned} \|S_I(f)\|_I &= \Big\| \sum_{J \in \mathrm{pred}(I)} c_J(f) h_J \Big\|_I \\ &\leq \sum_{J \in \mathrm{pred}(I)} |c_J(f)| \|h_J\|_I \\ &= \sum_{J \in \mathrm{pred}(I)} |c_J(f)| \tfrac{m(I)}{m(J)} \leq \sup_{J \in \mathrm{pred}(I)} |c_J(f)|. \end{aligned} \tag{4}$$

For $f \in L_1[0,1]$, $\varepsilon > 0$, and $\mathcal{A} \subset \operatorname{supp}_H(x)$ we define

(5)
$$\mathcal{A}_\varepsilon = \left\{ J \in \overline{\mathcal{D}} : \exists I \in \mathcal{A},\ I \subseteq J \text{ and } \left| \frac{c_K(f) - c_I(f)}{c_I(f)} \right| < \varepsilon, \text{ for all } K \in [I, J] \right\}.$$

Since $\mathcal{A}_\varepsilon$ depends on ε and the family $\big(c_I(f) : I \in \overline{\mathcal{D}}\big)$, we also write $\mathcal{A}_\varepsilon(f)$ instead of only $\mathcal{A}_\varepsilon$ to emphasize the dependence on f.

We are now ready to state our main result;

Theorem 1. *There is a universal constant C so that for $f \in L_1$, $\delta, \varepsilon > 0$ and*
$$\mathcal{A} \subset \big\{ I \in \overline{\mathcal{D}} : |c_I(f)| \geq \delta \big\},$$
there is an $\mathcal{E} \subset \overline{\mathcal{D}}$, with $\mathcal{A} \subset \mathcal{E} \subset \mathcal{A}_\varepsilon(f)$, so that

(6)
$$\big\| P_{\mathcal{E}}(f) \big\| \leq C \frac{\log^2(1/\delta)}{\varepsilon^2} \|f\|.$$

Remark 2. The proof of Theorem 1 yields an explicit, albeit laborious, description of $\mathcal{E}$.

3. Proof of the main Result

We will first state and prove several Lemmas.

Lemma 3. *Let $f \in L_1[0,1]$ and let K, J and I be elements of $\overline{\mathcal{D}}$, and assume that K is a direct successor of J, which is a direct successor of I. Then*
$$\|f\|_{I \setminus K} \geq \left| \frac{|c_I(f)| - |c_J(f)|}{2} \right|.$$

Proof. We first note that from the monotonicity property of the Haar basis we deduce that

(7)
$$\begin{aligned} \|f\|_{I \setminus J} &= \Big\| \sum_{L \in \operatorname{pred}(I \setminus J)} c_L(f) h_L|_{I \setminus J} + c_{I \setminus J}(f) h_{I \setminus J} + \sum_{L \in \operatorname{succ}(I \setminus J)} c_L(f) h_L \Big\| \\ &\geq \Big\| \sum_{L \in \operatorname{pred}(I \setminus J)} c_L(f) h_L|_{I \setminus J} \Big\| = \|S_{I \setminus J}(f)\|_{I \setminus J} \end{aligned}$$

and similarly we obtain

(8)
$$\|f\|_{J \setminus K} \geq \|S_{J \setminus K}(f)\|_{J \setminus K}.$$

$S_I(f)$ takes a constant value H on I. Denote by a the value of $c_I(f) h_I$ on $I \setminus J$ and denote by b the value of $c_J(f) h_J$ on $J \setminus K$, and let $\delta = m(I)$. Then we compute

$$
\begin{aligned}
\|f\|_{I\setminus K} &= \|f\|_{I\setminus J} + \|f\|_{J\setminus K} \\
&\geq \|S_{I\setminus J}(f)\|_{I\setminus J} + \|S_{J\setminus K}(f)\|_{J\setminus K} \\
&= \|S_I(f)+c_I(f)h_I\|_{I\setminus J} + \|S_I(f)+c_I(f)h_I+c_J(f)h_J\|_{J\setminus K} \\
&= \frac{\delta}{2}|H+a| + \frac{\delta}{4}|H-a+b| \\
&\geq \frac{\delta}{4}|H+a| + \frac{\delta}{4}|-H+a-b| \\
&\geq \frac{\delta}{4}|2a-b| \geq \frac{\delta}{4}\big|2|a|-|b|\big|.
\end{aligned} \tag{9}
$$

Our claim follows then if we notice that $|a|\delta = |c_I(f)|$ and $\delta|b| = 2|c_J(f)|$. ☐

We iterate Lemma 3 to obtain the following result.

Lemma 4. *Let $f \in L_1[0,1]$ and let K, J and I be elements of $\overline{\mathcal{D}}$, and assume that K is a successor of J, which is a successor of I. Then*

$$\|f\|_{I\setminus K} \geq \Big|\frac{|c_I(f)| - |c_J(f)|}{4}\Big|.$$

Proof. First we can, without loss of generality, assume that K is a direct successor of J, we write $[K, I]$ as $[K, I] = \{I_{n+1}, I_n, I_{n-1}, \dots I_0\}$, with $K = I_{n+1} \subsetneq I_n = J \subsetneq I_{n-1} \subsetneq \dots I_0 = I$, so that I_{m+1} is a direct successor of I_m for $m = 0, 1, 2, \dots, n$. From Lemma 3 we obtain

$$
\begin{aligned}
\|f\|_{I\setminus K} &= \sum_{j=0}^{n} \|f\|_{I_j\setminus I_{j+1}} \\
&\geq \frac{1}{2}\big(\|f\|_{I_0\setminus I_1} + \|f\|_{I_1\setminus I_2}\big) + \frac{1}{2}\big(\|f\|_{I_1\setminus I_2} + \|f\|_{I_2\setminus I_3}\big) \\
&\qquad\qquad \dots + \frac{1}{2}\big(\|f\|_{I_{n-1}\setminus I_n} + \|f\|_{I_n\setminus I_{n+1}}\big) \\
&= \frac{1}{2}\sum_{j=0}^{n-1} \|f\|_{I_j\setminus I_{j+2}} \\
&\geq \sum_{j=0}^{n-1}\Big|\frac{|c_{I_j}(f)| - |c_{I_{j+1}(f)}|}{4}\Big| \geq \Big|\frac{|c_I(f)| - |c_J(f)||}{4}\Big|
\end{aligned}
$$

which finishes the proof of our assertion. ☐

Lemma 5. *Assume that $f, g \in L_1[0,1]$ and $\mathcal{F} \subset \overline{\mathcal{D}}$ and that the following properties hold for some $\alpha, \varepsilon \in (0,1)$*

a) $\mathrm{supp}_H(f) \cap \mathrm{supp}_H(g) = \emptyset$,

b) *For every* $I \in \mathcal{F}$ *there is a* $J \in \operatorname{succ}(I)$*, so that*

$$[J,I] \cap \mathcal{F} = \{I\}$$
$$|c_J(f)| \geq \alpha$$
$$|c_I(g) - c_J(f)| \geq \varepsilon |c_J(f)|.$$

Then

$$\|f+g\| \geq \frac{\alpha\varepsilon}{6}|\mathcal{F}|. \tag{10}$$

In order to prove Lemma 5 we will first show the following observation.

Proposition 6. *Let* $\mathcal{F} \subset \overline{\mathcal{D}}$*, and define the following partition of* $\mathcal{F}$ *into sets* $\mathcal{F}_0$*,* $\mathcal{F}_1$ *and* $\mathcal{F}_2$

$$\mathcal{F}_0 = \{I \in \mathcal{F} : I \text{ is minimal in } \mathcal{F}\} = \{I \in \mathcal{F} : \operatorname{succ}(I) \cap \mathcal{F} = \emptyset\},$$
$$\mathcal{F}_1 = \{I \in \mathcal{F} : \operatorname{succ}(I) \cap \mathcal{F} \text{ has exactly one maximal element}\}, \text{ and}$$
$$\mathcal{F}_2 = \{I \in \mathcal{F} : \operatorname{succ}(I) \cap \mathcal{F} \text{ has at least two maximal element}\}.$$

Then

$$|\mathcal{F}_2| < |\mathcal{F}_0|. \tag{11}$$

Proof. In order to verify (11) we first show for $I \in \mathcal{F}_2$ that

$$\big|\{J \in \mathcal{F}_2 : J \subseteq I\}\big| < \big|\{J \in \mathcal{F}_0 : J \subset I\}\big|. \tag{12}$$

Assuming that (12) is true for all $I \in \mathcal{F}_2$, we let $I_1, I_2, \dots I_l$ be the maximal elements of $\mathcal{F}_2$. Since the I_j's are pairwise disjoint, observe that

$$|\mathcal{F}_2| = \sum_{j=1}^{l} \big|\{I \in \mathcal{F}_2 : I \subseteq I_j\}\big| < \sum_{j=1}^{l} \big|\{J \in \mathcal{F}_0 : J \subset I_j\}\big| \leq |\mathcal{F}_0|.$$

We now prove (12) by induction on $n = \big|\{J \in \mathcal{F}_2 : J \subseteq I\}\big|$. If $n = 1$ then I must have at least two successor, say J_1 and J_2 in $\mathcal{F}$ which are incomparable, and thus there are elements $I_1, I_2 \in \mathcal{F}_0$ so that $I_1 \subset J_1$ and $I_2 \subset J_2$. Assume that our claim is true for n, and assume that $\big|\{J \in \mathcal{F}_2 : J \subseteq I\}\big| = n+1 \geq 2$. We denote the maximal elements of $\{J \in \mathcal{F}_2 : J \subsetneq I\}$, by $I_1, I_2, \dots I_m$. Either $m \geq 2$, then it follows from the induction hypothesis, and the fact that I_1,$I_2, \dots I_m$ are incomparable, that

$$|\{J \in \mathcal{F}_2 : J \subseteq I\}| = 1 + \sum_{j=1}^{m} \big|\{J \in \mathcal{F}_2 : J \subseteq I_j\}\big|$$

$$\leq 1 + \sum_{j=1}^{m} \big(\big| \{ J \in \mathcal{F}_0 : J \subseteq I_j \} \big| - 1 \big)$$
$$\leq \big| \{ J \in \mathcal{F}_0 : J \subseteq I \} \big| - 1 < \big| \{ J \in \mathcal{F}_0 : J \subseteq I \} \big|.$$

Or $m = 1$, and if $\tilde{I}$ is the only maximal element of $\{J \in \mathcal{F}_2 : J \subsetneq I\}$, then by the definition of $\mathcal{F}_2$ there must be a $J_0 \in \mathcal{F}_0$ with $J_0 \subset I \setminus \tilde{I}$, and we deduce from our induction hypothesis that

$$\begin{aligned} |\{J \in \mathcal{F}_2 : J \subseteq I\}| &= 1 + \big|\{J \in \mathcal{F}_2 : J \subseteq \tilde{I}\}\big| \\ &< 1 + |\{J \in \mathcal{F}_0 : J \subseteq \tilde{I}\}| \leq \big|\{J \in \mathcal{F}_0 : J \subseteq I\}\big|, \end{aligned}$$

which finishes the proof of the induction step, and the proof of (12). □

Proof of Lemma 5. Assume now that $\alpha, \varepsilon > 0$ and $f, g \in L_1[0,1]$, and $\mathcal{F} \subset \mathcal{D}$ are given satisfying (a), (b). Let $\mathcal{F}_0$, $\mathcal{F}_1$, and $\mathcal{F}_2$ the subsets of $\mathcal{F}$ introduced in Proposition 6. We distinguish between two cases.

Case 1. $|\mathcal{F}_0| \geq \frac{1}{6}|\mathcal{F}|$.

Fix $I \in \mathcal{F}_0$, and let $J \in \mathrm{succ}(I)$ be chosen so that condition (b) is satisfied. It follows then from condition (a) and (3)

$$\|f + g\|_I \geq \|f + g\|_J \geq |c_J(f)| \geq \alpha.$$

Since all the elements in $\mathcal{F}_0$ are disjoint it follows that

$$\|f + g\| \geq \sum_{I \in \mathcal{F}_0} \|f + g\|_I \geq \big|\mathcal{F}_0\big|\alpha \geq \big|\mathcal{F}\big| \frac{\alpha}{6}.$$

Case 2. $|\mathcal{F}_0| < \frac{1}{6}|\mathcal{F}|$.

Applying (11) we obtain that

$$|\mathcal{F}_1| = |\mathcal{F}| - |\mathcal{F}_0| - |\mathcal{F}_2| > |\mathcal{F}| - 2|\mathcal{F}_0| > \frac{2}{3}|\mathcal{F}|.$$

Fix $I \in \mathcal{F}_1$, and let $J \in \mathrm{succ}(I)$ satisfy the conditions in (c), and let $\tilde{I}$, be the unique maximal element of $\mathrm{succ}(I) \cap \mathcal{F}$. It follows that $J \subsetneq I$ and, since by condition (b) $\tilde{I} \notin [J, I]$, we deduce that $J \not\subset \tilde{I}$ which implies that either $\tilde{I} \subsetneq J$ or $\tilde{I} \cap J = \emptyset$. In the first case we deduce from Lemma 4, and condition (b) that

$$\|f + g\|_{I \setminus \tilde{I}} \geq \Big| \frac{|c_I(g)| - |c_J(f)|}{4} \Big| \geq \varepsilon \frac{|c_J(f)|}{4} \geq \frac{\varepsilon\alpha}{4}.$$

In the second case we deduce from (3) and condition (c) that

$$\|f + g\|_{I \setminus \tilde{I}} \geq \|f + g\|_J \geq |c_J(f)| \geq \alpha.$$

We conclude therefore from the fact that the sets $I \setminus \tilde{I}$, with $I \in \mathcal{F}_1$, are pairwise disjoint and therefore

$$\|f+g\| \geq \sum_{I\in\mathcal{F}_1} \|f+g\|_{I\setminus\tilde{I}} \geq |\mathcal{F}_1|\frac{\varepsilon\alpha}{4} \geq \frac{2}{3}|\mathcal{F}|\frac{\varepsilon\alpha}{4} = \frac{\alpha\varepsilon}{6}|\mathcal{F}|.$$

□

In order to formulate our next step we introduce the following *Symmetrization Operators* $\mathcal{L}_1$ and $\mathcal{L}_2$. For that assume that $f \in L_1[0,1]$ and $I \in \mathcal{D}$. We define the following two functions $\mathcal{L}_1(f,I)$ and $\mathcal{L}_2(f,I)$ in $L_1[0,1]$. For $\xi \in [0,1]$ we put

$$\mathcal{L}_1(f,I)(\xi) = \begin{cases} f(\xi) & \text{if } \xi \notin I^- \\ f\big(\xi - \frac{m(I)}{2}\big) & \text{if } \xi \in I^- \end{cases}$$

$$\mathcal{L}_2(f,I)(\xi) = \begin{cases} f(\xi) & \text{if } \xi \notin I^+ \\ f\big(\xi + \frac{m(I)}{2}\big) & \text{if } \xi \in I^+ \end{cases}$$

Note that $\mathcal{L}_1(f,I)$ restricted to I^- is a shift of f restricted to I^+, and vice versa $\mathcal{L}_2(f,I)$ restricted to I^+ is a shift of f restricted to I^-.

We will use this symmetrization only for $f \in L_1[0,1]$ and $I \in \mathcal{D}$, for which $c_I(f) = 0$, We observe in that case that letting $f' = \mathcal{L}_1(f,I)$ or $f' = \mathcal{L}_2(f,I)$, and any $J \in \mathcal{D}$

$$(13) \quad c_J(f') = \begin{cases} c_J(f) & \text{if } J \supsetneq I \text{ (here we use that } c_I(f)=0) \\ c_J(f) & \text{if } J \cap I = \emptyset \\ 0 & \text{if } J = I \\ c_J(f) & \text{if } J \subset I^+ \text{ and } f' = L_1(f,I), \text{ or} \\ & \text{if } J \subset I^- \text{ and } f' = L_2(f,I), \\ c_{J-m(I)/2}(f) & \text{if } J \subset I^- \text{ and } f' = L_1(f,I) \\ c_{J+m(I)/2}(f) & \text{if } J \subset I^+ \text{ and } f' = L_2(f,I). \end{cases}$$

Moreover it follows that

$$(14) \quad \big\|L_1(f,I)\big\| = \|f\| + \Delta(f,I) \text{ and } \big\|L_2(f,I)\big\| = \|f\| - \Delta(f,I), \\ \text{with } \Delta(f,I) = \|f\|_{I^+} - \|f\|_{I^-}.$$

Lemma 7. *Assume that* $f, g \in L_1[0,1]$, *so that* $\text{supp}_H(f), \text{supp}_H(g) \subset \mathcal{D}$ *and* $I \in \mathcal{D}$ *are given. Suppose that* $c_I(f) = c_I(g) = 0$ *and that the following properties hold:*

a) $\text{supp}_H(f) \cap \text{supp}_H(g) = \emptyset$,
b) $\|f\| > 0$, *and thus, by (a), also* $\|f+g\| > 0$.

Then for either $f' = L_1(f, I)$ and $g' = L_1(g, I)$, or $f' = L_2(f, I)$ and $g' = L_2(g, I)$ it follows that

$$\text{supp}_H(f') \cap \text{supp}_H(g') = \emptyset \text{ and } I \notin \text{supp}_H(f') \cup \text{supp}_H(g'); \tag{15}$$

(16) *for any $J \in \mathcal{D}$ it follows that*

$$c_J(f') \in \{c_I(f') : I \in \text{supp}(f)\} \cup \{0\} \text{ and}$$
$$c_J(g') \in \{c_I(g') : I \in \text{supp}(g)\} \cup \{0\},$$

$$\|f' + g'\| > 0 \text{ and } \frac{\|f\|}{\|f + g\|} \le \frac{\|f'\|}{\|f' + g'\|}. \tag{17}$$

Proof. It follows immediately from (13) that (16) and (15) are satisfied for either of the possible choices of f' and g'.

To satisfy (17) we will first consider the case that $(f+g)|_{[0,1]\setminus I^+} \equiv 0$. In this case it follows from (a) that $c_J(f) = c_J(g) = 0$ for all $J \in \mathcal{D}$, with $J \subseteq [0, 1] \setminus I^+$, and thus by (b) $f|_{I^+} \not\equiv 0$ and $(g + f)|_{I^+} \not\equiv 0$. If we choose $f' = L_1(f, I)$ and $g' = L_1(g, I)$ we obtain that $\|f' + g'\| > 0$, $\|f'\| = 2\|f\|$ and $\|f' + g'\| = 2\|f + g\|$.

A similar argument can be made if $(f + g)|_{[0,1]\setminus I^-} \equiv 0$.

If neither of the two previously discussed cases occurs we conclude that

$$\|f+g\| = \|f+g\|_{[0,1]\setminus I} + \|f+g\|_{I^+} + \|f+g\|_{I^-} > \|f+g\|_{I^+} - \|f+g\|_{I^-}$$

and

$$\|f+g\| = \|f+g\|_{[0,1]\setminus I} + \|f+g\|_{I^+} + \|f+g\|_{I^-} > \|f+g\|_{I^-} - \|f+g\|_{I^+}$$

which implies by (14) that in either of the two possible choices for f' and g' it follows that $\|f' + g'\| > 0$.

Finally, if $\Delta(f, I)\|f + g\| \ge \Delta(f + g, I)\|f\|$, we choose $f' = L_1(f, I)$ and $g' = L_1(f, I)$ and note that since in this case we have

$$(\|f\| + \Delta(f, I))\|f + g\| \ge (\|f + g\| + \Delta(f + g, I))\|f\|,$$

it follows that

$$\frac{\|f'\|}{\|f' + g'\|} = \frac{\|f\| + \Delta(f, I)}{\|f + g\| + \Delta(f + g, I)} \ge \frac{\|f\|}{\|f + g\|}.$$

If $\Delta(f, I)\|f + g\| < \Delta(f + g)\|f\|$ and thus $-\Delta(f, I)\|f + g\| > -\Delta(f + g)\|f\|$, we choose $f' = L_2(f, I)$ and $g' = L_2(f, I)$ and note that since in this case we have

$$\big(\|f\| - \Delta(f, I)\big)\|f + g\| > \big(\|f + g\| - \Delta(f + g, I)\big)\|f\|,$$

and it follows that

$$\frac{\|f'\|}{\|f'+g'\|} = \frac{\|f\| - \Delta(f,I)}{\|f+g\| - \Delta(f+g,I)} > \frac{\|f\|}{\|f+g\|},$$

which finishes the verification of (17) and the proof of our claim. □

Assume now that $f,\ g \subset L_1[0,1]$, $\|f\| > 0$, are such that $\mathrm{supp}_H(f)$ and $\mathrm{supp}_H(g)$ are finite and disjoint subsets of $\mathcal{D}$. We also assume that

$$c_{[0,1]}(f) = \int_0^{1/2} f(x)\,dx - \int_{1/2}^1 f(x)\,dx = 0 \text{ and} \tag{18}$$
$$c_{[0,1]}(g) = \int_0^{1/2} g(x)\,dx - \int_{1/2}^1 g(x)\,dx = 0.$$

Define:

$$\mathcal{F}(f) = \{I \in \mathcal{D} : c_I(f) = 0 \text{ but } c_{I^+}(f) \neq 0 \text{ or } c_{I^-}(f) \neq 0\} \tag{19}$$

and make the following assumption

$$\mathrm{supp}_H(g) \cap \mathcal{F}(f) = \emptyset. \tag{20}$$

Let $\mathcal{F}(f) = (I_i)_{i=1}^n$, where $m(I_1) \leq m(I_2) \leq \cdots \leq m(I_n)$. First 'symmetrize' the pair (f,g) on I_1 to obtain a pair (f_1, g_1) satisfying $\|f\|/\|f+g\| \leq \|f_1\|/\|f_1+g_1\|$. Note that $\mathcal{F}(f_1) = \mathcal{F}(f)$. Now symmetrize (f_1, g_1) on I_2 to obtain (f_2, g_2) satisfying $\|f_1\|/\|f_1+g_1\| \leq \|f_2\|/\|f_2+g_2\|$. Note that if $I \in \mathcal{F}(f_2)$ satisfies $m(I) < m(I_2)$, then (f_2, g_2) on I is a 'copy' of (f_1, g_1) on I_1. Hence, f_2 and g_2 are automatically symmetric on I. On the other hand, if $I \in \mathcal{F}(f_2)$ satisfies $m(I) \geq m(I_2)$ then $I = I_j$ for some $j \geq 2$. Now symmetrize (f_2, g_2) on I_2 to obtain (f_3, g_3) satisfying $\|f_2\|/\|f_2+g_2\| \leq \|f_3\|/\|f_3+g_3\|$. Note that if $I \in \mathcal{F}(f_3)$ satisfies $m(I) < m(I_3)$ then (f_3, g_3) on I is a copy of (f_2, g_2) on I_1 or I_2. Hence f_3 and g_3 are automatically symmetric on I. Continuing in this way, we finally obtain, after symmetrizing on I_n, a pair (f_n, g_n) such that f_n and g_n are symmetric on each $I \in \mathcal{F}(f_n)$ and $\|f\|/\|f+g\| \leq \|f_n\|/\|f_n+g_n\|$.

Setting $\tilde{f} = f_n$ and $\tilde{g} = f_n$, the following conditions hold:

(21) $c_{[0,2]}(\tilde{f}) = c_{[0,2]}(\tilde{g}) = c_{[0,1]}(\tilde{f}) = c_{[0,1]}(\tilde{g}) = 0$,

(22) For all $I \in \mathcal{F}(\tilde{f})$ it follows that

$$c_I(\tilde{g}) = 0 \text{ and}$$
$$\tilde{f}(x) = \tilde{f}(x - m(I^+)) \text{ and } \tilde{g}(x) = \tilde{g}(x - m(I^+)), \text{ if } x \in I^-,$$

(23) $\mathrm{supp}_H(\tilde{f}) \cap \mathrm{supp}_H(\tilde{g}) = \emptyset$,

(24) for any $J \in \mathcal{D}$ it follows that

$$c_J(\tilde f) \in \{c_I(f) : I \in \mathrm{supp}(f)\} \cup \{0\} \text{ and}$$
$$c_J(\tilde g) \in \{c_I(g) : I \in \mathrm{supp}(g)\} \cup \{0\},$$

(25) $\|\tilde f + \tilde g\| > 0$ and $\dfrac{\|f\|}{\|f+g\|} \le \dfrac{\|\tilde f\|}{\|\tilde f + \tilde g\|}$.

Lemma 8. *Assume that $f, g \in L_1[0,1]$ are such that $\mathrm{supp}_H(f)$ and $\mathrm{supp}_H(g)$ are finite disjoint subsets of $\mathcal{D} \setminus \{[0,1]\}$, and that $\mathrm{supp}_H(g) \cap \mathcal{F}(f) = \emptyset$, where $\mathcal{F}(f) \subset \mathcal{D}$, was defined above. Assume moreover that for some $\alpha > 0$, we have*

$$|c_J(f)| \ge \alpha, \text{ for all } J \in \mathrm{supp}_H(f), \text{ and} \qquad (26)$$
$$|c_J(g)| \le 1, \text{ for all } J \in \mathrm{supp}_H(g).$$

Then

$$\|f\| \le \Big(\frac{5}{\alpha} + 1\Big)\|f+g\|. \qquad (27)$$

Proof. Let $\tilde f$ and $\tilde g$ be the elements in $L_1[0,1]$ constructed from f and g as before satisfying the conditions (21), (22), (23), (24), and (25). Note also that (24) implies that $|c_J(\tilde f)| \ge \alpha$, for all $J \in \mathrm{supp}_H(\tilde f)$, and $|c_J(\tilde g)| \le 1$, for all $J \in \mathrm{supp}_H(\tilde g)$.

By (25) it is enough to show (27) for $\tilde f$ and $\tilde g$ instead of f and g.

We will deduce our statement from the following

Main Claim. For all $I \in \mathcal{F}(\tilde f)$ it follows that

$$\|\tilde f\|_I \le \Big(\frac{5}{\alpha} + 1\Big)\|\tilde f + \tilde g\|_I - 2\alpha - 8. \qquad (28)$$

Assuming the Main Claim we can argue as follows. Using (21) it follows that out side of $J = \bigcup_{I \in \mathcal{F}(\tilde f)} I$ $\tilde f$ is vanishing. Thus, we can choose disjoint sets $I_1, I_2, \ldots I_n$, in $\mathcal{F}(\tilde f)$ so that $\tilde f$ vanishes outside of $\bigcup_{j=1}^n I_j$, and (28) yields

$$\|\tilde f\| = \sum_{j=1}^n \|\tilde f\|_{I_j} \le \Big(\frac{5}{\alpha} + 1\Big) \sum_{j=1}^n \|\tilde f + \tilde g\|_{I_j} \le \Big(\frac{5}{\alpha} + 1\Big)\|\tilde f + \tilde g\|,$$

which proofs our wanted statement.

In order to show the Main Claim let $I \in \mathcal{F}(\tilde f)$ and denote $k = \mathrm{ord}(I, \mathcal{F}(\tilde f))$. We will show the inequality (28) by induction for all k.

First assume that $k = 0$. From (22) and (3) we obtain

$$\|\tilde f + \tilde g\|_I = 2\|\tilde f + \tilde g\|_{I^+} \ge 2\big|c_{I^+}(\tilde f + \tilde g)\big| = \big|2c_{I^+}(\tilde f)\big| \ge 2\alpha. \qquad (29)$$

Using (4), (23) and (27) we obtain

$$\|S_I(\tilde{g})\|_I \leq 1. \tag{30}$$

From the definition of $\mathcal{F}(\tilde{f})$, and the assumption that $\operatorname{ord}\big(I, \mathcal{F}(\tilde{f})\big) = 0$, (23) we deduce that if $J \in \operatorname{supp}_H(\tilde{f})$ with $J \subset I^+$ or $J \subset I^-$, then $[J, I^+] \subset \operatorname{supp}_H(\tilde{f})$, or $[J, I^-] \subset \operatorname{supp}_H(\tilde{f})$, respectively. But, using (23), this implies that if $J \in \operatorname{supp}_H(\tilde{g})$, with $J \subset I$, then $\operatorname{succ}(J) \cap \operatorname{supp}_H(\tilde{f}) = \emptyset$. We deduce therefore from the monotonicity properties of the Haar basis that

$$\|\tilde{f} + \tilde{g}\|_I \geq \Big\| f + \sum_{J \in \mathcal{D}, J \not\subset I} c_J(\tilde{g}) h_J \Big\|_I = \|\tilde{f} + S_I(\tilde{g})\|_I. \tag{31}$$

We therefore conclude

$$\begin{aligned}\left(\frac{5}{\alpha} + 2\right) \|\tilde{f} + \tilde{g}\|_I &\geq \left(\frac{5}{\alpha} + 1\right) \|\tilde{f} + \tilde{g}\|_I + \|\tilde{f} + S_I(\tilde{g})\|_I \qquad \text{(by (31))} \\ &\geq 10 + 2\alpha + \|\tilde{f}\|_I - \|S_I(\tilde{g})\| \qquad \text{(by (29))} \\ &\geq \|f\|_I + 9 + 2\alpha \text{ (by (30))},\end{aligned}$$

which proves our claim in the case that $\operatorname{ord}\big(I, \mathcal{F}(\tilde{f})\big) = 0$.

Assume that (28) holds for all $I \in \mathcal{F}$ with $\operatorname{ord}\big(I, \mathcal{F}(\tilde{f})\big) < k$, for some $k \in \mathbb{N}$, and assume that $I \in \mathcal{F}(\tilde{f})$ with $\operatorname{ord}(I, \mathcal{F}(\tilde{f})) = k$. By the symmetry condition in (22) the number of elements J of $\mathcal{F}(\tilde{f})$ for which $J \subset I$ and $\operatorname{ord}(J, \mathcal{F}) = k - 1$ is even, half of them being subsets of I^+, the other being subsets of I^-. We order therefore these sets into $J_1, J_2, \ldots J_{2s}$, for some $s \in \mathbb{N}$, with $J_i \subset I^+$ and $J_{s+i} \subset I^-$, for $i = 1, 2, \ldots, s$. We note that the J_i, $i = 1, 2 \ldots$, are pairwise disjoint and that all the $J \in \mathcal{F}(\tilde{f})$, with $F \subset I$, and $\operatorname{ord}(J, \mathcal{F}(\tilde{f})) \leq k - 2$, are subset of some of the J_i, $i = 1, 2 \ldots 2s$.

From our induction hypothesis we deduce that

$$\|\tilde{f}\|_{J_i} \leq \Big(\frac{5}{\alpha} + 2\Big) \|\tilde{f} + \tilde{g}\|_{J_i} - 2\alpha - 8 \text{ for } i = 1, 2, \ldots, 2s. \tag{32}$$

We define $D = I^+ \setminus \bigcup_{i=1}^s J_i$ and

$$\phi = S_{J_1}(S_{J_2}(\ldots S_{J_s}(\tilde{f}) \ldots)) = \sum_{J \in \mathcal{J}} c_J(\tilde{f}) h_J$$

$$\gamma = S_{J_1}(S_{J_2}(\ldots S_{J_s}(\tilde{g}) \ldots)) = \sum_{J \in \mathcal{J}} c_J(\tilde{g}) h_J$$

with

$$\mathcal{J} = \big\{ J \in \mathcal{D} : \forall\, j = 1, 2 \ldots, s \quad J \not\subset I_j \big\}.$$

It follows that

$$\phi|_D = \tilde{f}|_D \text{ and } \gamma|_D = \tilde{g}|_D, \tag{33}$$

(4) implies that

$$\|\gamma\|_{J_i} \leq 1, \text{ for } i = 1, 2 \dots s, \tag{34}$$

and since for any $J \in \mathcal{D}$, with $J \subseteq D$, for which $c_J(\phi) \neq 0$, we have $[J, I^+] \subset \text{supp}_H(\phi)$ (otherwise there would be an $K \in \mathcal{F}(\tilde{f})$ with $K \subset I^+$ and $K \supsetneq J_i$, for some $i \in \{1, 2 \dots s\}$, or $K \subset D$) it follows from the monotonicity property of the Haar system and (4) that

$$\|\phi + \gamma\|_{I^+} \geq \|\phi + S_{I^+}(\gamma)\|_{I^+} \geq \|\phi\|_{I^+} - 1. \tag{35}$$

It follows that

$$\begin{aligned}
\|\phi + \gamma\|_{I^+ \setminus D} &\leq \|\phi\|_{I^+ \setminus D} + \|\gamma\|_{I^+ \setminus D} \\
&= \|\phi\|_{I^+ \setminus D} + \sum_{i=1}^{s} \|\gamma\|_{J_i} \\
&\leq \|\phi\|_{I^+ \setminus D} + s
\end{aligned}$$

and

$$\begin{aligned}
\|\tilde{f} + \tilde{g}\|_D &= \|\phi + \gamma\|_D \\
&= \|\phi + \gamma\|_{I^+} - \|\phi + \gamma\|_{I^+ \setminus D} \\
&\geq \|\phi\|_{I^+} - 1 - \|\phi\|_{I^+ \setminus D} - s = \|\phi\|_D - s - 1.
\end{aligned}$$

This implies together with (32) that

$$\begin{aligned}
\|\tilde{f}\|_{I^+} &= \|\tilde{f}\|_D + \sum_{i=1}^{s} \|\tilde{f}\|_{J_i} \\
&= \|\phi\|_D + \sum_{i=1}^{s} \|\tilde{f}\|_{J_i} \\
&\leq \|\tilde{f} + \tilde{g}\|_D + s + 1 + \Big(\frac{5}{\alpha} + 2\Big) \sum_{i=1}^{s} \|\tilde{f} + \tilde{g}\|_{J_i} - s(2\alpha + 8) \\
&\leq \Big(\frac{5}{\alpha} + 2\Big) \|\tilde{f} + \tilde{g}\|_{I^+} - 7s - 2\alpha s + 1.
\end{aligned}$$

By the symmetry condition (22) we also obtain that

$$\|\tilde{f}\|_{I^-} \leq \Big(\frac{5}{\alpha} + 2\Big) \|\tilde{f} + \tilde{g}\|_{I^+} - 7s - 2\alpha s + 1.$$

Adding these two inequalities yields our Main Claim since $s \geq 1$. $\square$

Theorem 9. *Let $h \in L_1[0,1]$, with $\operatorname{supp}_H(h) \subset \operatorname{succ}([0,1])$, and let $0<\varepsilon<1$, $0<\alpha\le 1$ and $b\in\mathbb{R}^+$. Assume that $\mathcal{S}\subset\mathcal{D}$, is such that*

$$|c_I(h)| \ge \alpha b, \text{ if } I\in\mathcal{S}, \text{ and } |c_I(h)|\le b, \text{ if } I\notin\mathcal{S}.$$

Then

$$\big\|P_{\mathcal{S}_\varepsilon}(h)\big\| \le \frac{42}{\alpha^2\varepsilon}\|h\|.$$

Proof. After rescaling we can assume that $b=1$. Put $f=P_{\mathcal{S}_\varepsilon}(h)$ and $g=h-P_{\mathcal{S}_\varepsilon}(h)$. We note that f and g satisfy the assumptions of Lemma 5 with

$$\mathcal{F}=\big\{I\in\mathcal{D}: I\notin\mathcal{S}_\varepsilon,\ \text{but } I^+\in\mathcal{S}_\varepsilon \text{ or } I^-\in\mathcal{S}_\varepsilon\big\}.$$

Indeed, condition (a) of Lemma 5 is clearly satisfied, and in order to verify (b) let $I\in\mathcal{F}$. Without loss of generality we can assume that $I^+\in\mathcal{S}_\varepsilon$. Thus there is a $J\in\mathcal{S}$, with $J\subset I$, and so that J is maximal with that property. It follows therefore from the definition of $\mathcal{S}_\varepsilon$ that $[J,I^+]\subset\mathcal{S}_\varepsilon$, and thus $[J,I]\cap\mathcal{F}=\{I\}$, $|c_J(f)|=|c_J(h)|\ge\alpha$, and

$$|c_I(g)-c_J(f)|=|c_I(h)-c_J(h)|\ge\varepsilon|c_J(f)|.$$

Lemma 5 yields that

$$\|f+g\|\ge\frac{\alpha\varepsilon}{6}|\mathcal{F}|.$$

Setting

$$\overline{g}=g-\sum_{I\in\mathcal{F}}c_I(g)h_I,$$

then, by our assumption on h,

$$\begin{aligned}\|f+\overline{g}\| &\le \|f+g\|+\|\overline{g}-g\|\\ &\le \|f+g\|+\Big\|\sum_{I\in\mathcal{F}}c_I(h)h_I\Big\|\\ &\le \|f+g\|+|\mathcal{F}|\\ &\le \Big(1+\frac{6}{\alpha\varepsilon}\Big)\|f+g\|.\end{aligned}$$

Note that since $\mathcal{F}=\mathcal{F}(f)$ (where $\mathcal{F}(f)$ was defined in (19)) the pair f and $\overline{g}$ satisfies the assumption of Lemma 8 and we deduce that

$$\begin{aligned}\|h\| &= \|f+g\|\\ &\ge \frac{\alpha\varepsilon}{\alpha\varepsilon+6}\|f+\overline{g}\|\\ &\ge \frac{\alpha\varepsilon}{\alpha\varepsilon+6}\frac{\alpha}{\alpha+5}\|f\|\\ &\ge \frac{\alpha^2\varepsilon}{42}\|f\| = \frac{\alpha^2\varepsilon}{42}\|P_{\mathcal{S}_\varepsilon}(h)\|\end{aligned}$$

which implies our claim. $\square$

Corollary 10. *Let* $f \in L_1[0,1]$, *with* $\mathrm{supp}_H(f) \subset \mathrm{succ}([0,1])$, $\mathcal{A} \subset \mathcal{D}$, $0 < \varepsilon < 1$, $\rho \in \mathbb{R}^+$. *Put* $\mathcal{B} = \mathcal{A} \cap \{I \in \mathcal{D} : \rho < |c_I(f)| \leq 2\rho\}$.

Then there exists $\mathcal{C} \subset \mathcal{D}$, *with* $\mathcal{B} \subseteq \mathcal{C} \subseteq \mathcal{B}_\varepsilon(f)$, *so that*

$$\|P_{\mathcal{C}}(f)\| \leq \frac{45738}{\varepsilon}\|f\|.$$

Proof. We first apply Theorem 9 to the set $\mathcal{S} = \{J \in \mathcal{D} : |c_J(f)| > 3\rho\}$, the numbers $b = 3\rho$, $\alpha = 1$ and $\varepsilon = \frac{1}{3}$. It follows that

$$\big\|P_{\mathcal{S}_\varepsilon}(f)\big\| \leq 120\|f\|. \tag{36}$$

Note that

$$\mathcal{S}_\varepsilon = \left\{ J \in \mathcal{D} : \begin{matrix} \exists I \in \mathcal{S}, I \subset J \,\forall K \in [I,J] \\ |c_I(f) - c_K(f)| \leq \frac{1}{3}|c_I(f)| \end{matrix} \right\} \subset \big\{ J \in \mathcal{D} : |c_J(f)| > 2\rho \big\}$$

Put $\mathcal{B}^{(1)} := \mathcal{D} \setminus \mathcal{S}_\varepsilon$, and $g = P_{\mathcal{B}^{(1)}}(f)$ then,

$$\|g\| \leq 121\|f\| \tag{37}$$

and

$$\big\{J \in \mathcal{D} : |c_J(f)| \leq 2\rho\big\} \subseteq \mathcal{B}^{(1)} \subseteq \big\{J \in \mathcal{D} : |c_J(f)| \leq 3\rho\big\}. \tag{38}$$

Then we apply Theorem 9 again, namely to the function g, the set

$$\mathcal{B}^{(2)} = \big\{I \in \mathcal{D} : I \in \mathcal{A},\ \rho < |c_I(g)| \leq 2\rho\big\},$$

and the numbers $b = 3\rho$, $\alpha = \frac{1}{3}$. We deduce that for each $\varepsilon \in (0,1)$)

$$\big\|P_{\mathcal{B}^{(2)}_\varepsilon}(g)\big\| \leq \frac{378}{\varepsilon}\|g\|. \tag{39}$$

Here we mean by $\mathcal{B}^{(2)}_\varepsilon$, to be precise, the set $\mathcal{B}^{(2)}_\varepsilon(g)$. Since for every $I \in \mathcal{D}$, with $c_I(g) \neq 0$, it follows that $c_I(g) = c_I(f)$, we deduce that

$$\begin{aligned}
\mathcal{B}^{(2)}_\varepsilon(g) &= \left\{ J \in \mathcal{D} : \begin{matrix} \exists I \in \mathcal{A}, I \subset J \quad \rho < |c_I(g)| \leq 2\rho \text{ s.th.} \\ \forall K \in [I,J] |c_I(f) - c_K(f)| \leq \varepsilon |c_I(f)| \end{matrix} \right\} \\
&\subseteq \left\{ J \in \mathcal{D} : \begin{matrix} \exists I \in \mathcal{A}, I \subset J \quad \rho < |c_I(f)| \leq 2\rho \text{ s.th.} \\ \forall K \in [I,J] |c_I(f) - c_K(f)| \leq \varepsilon |c_I(f)| \end{matrix} \right\} = \mathcal{B}^{(2)}_\varepsilon(f).
\end{aligned}$$

Letting therefore $\mathcal{C} = \mathcal{B}^{(2)}_\varepsilon(y)$, we deduce our claim from (37), (39) and the fact that $\mathcal{B} \subset \mathcal{B}^{(1)}$. $\square$

We are now in the position to prove Theorem 1.

Proof of Theorem 1. Let $f \in L_1$, and $\varepsilon, \delta > 0$. We can assume that $\varepsilon < \frac{1}{3}$ and that $\mathrm{supp}_H(f) \subset \mathrm{succ}([0,1])$, with $|c_I(f)| \le 1$, for all $I \in \mathrm{supp}_H(f)$. We choose $m_0 \in \mathbb{N}$, so that $2^{-m_0} < \delta \le 2^{1-m_0}$, which implies that $m_0 \le \log_2(2/\delta)$.

For each $m = 1, 2, \dots m_0$, we apply Corollary 10 to the function f, $\rho = 2^{-m}$, the set $\mathcal{A} \cap \{I \in \mathcal{D} : \delta < |c_H(f)|\}$. We put $C_\varepsilon = 45738/\varepsilon$ and

$$\mathcal{B}^{(m)} = \mathcal{A} \cap \big\{I \in \mathcal{D} : 2^m \vee \delta < |c_I(f)| \le 2^{1-m}\big\} \text{ for } m = 1, 2 \dots m_0$$

and deduce that there are sets $\mathcal{C}_m$, $\mathcal{B}^{(m)} \subseteq \mathcal{C}^{(m)} \subseteq \mathcal{B}^{(m)}_\varepsilon(f)$, so that

$$\big\|P_{\mathcal{C}^{(m)}}(f)\big\| \le C_\varepsilon \|f\| \text{ for } m = 1, 2 \dots m_0. \tag{40}$$

Since $\varepsilon \le \frac{1}{3}$ it follows for $i, j \in \{1, 2, 3 \dots m_0\}$, with $|i - j| \ge 2$, that $\mathcal{B}^{(i)}_\varepsilon(f) \cap \mathcal{B}^{(j)}_\varepsilon(f) = \emptyset$, and, thus, that $\mathcal{C}^{(i)} \cap \mathcal{C}^{(j)} = \emptyset$.

We let $\mathcal{F} = \bigcup_{m=1, m \text{ odd}}^{m_0} \mathcal{C}^{(m)}$. It follows from (40) that

$$\big\|\mathcal{P}_{\mathcal{F}}(f)\big\| \sum_{m=1,\, m \text{ odd}}^{m_0} \|P_{\mathcal{C}^{(m)}}(f)\| \le C_\varepsilon \Big\lceil \frac{m_0}{2} \Big\rceil \le C_\varepsilon \log\Big(\frac{1}{\delta}\Big). \tag{41}$$

We are now applying again Corollary 10 to the function $g = f - \mathcal{P}_{\mathcal{F}}(f)$ and the set $\tilde{\mathcal{A}} = (\mathcal{A} \cap \{I \in \mathcal{D} : \delta < |c_H(f)|\}) \setminus \mathcal{F}$, and find sets $\tilde{\mathcal{C}}^{(j)}$, with $\tilde{\mathcal{B}}^{(j)} \subset \tilde{\mathcal{C}}^{(j)} \subset \tilde{\mathcal{B}}^{(j)}_\varepsilon(g)$, where

$$\tilde{\mathcal{B}}^{(m)} = \tilde{\mathcal{A}} \cap \big\{I \in \mathcal{D} : 2^m \vee \delta < |c_I(f)| \le 2^{1-m}\big\} \text{ for } m = 1, 2 \dots m_0,$$

so that

$$\big\|P_{\tilde{\mathcal{C}}^{(m)}}(g)\big\| \le C_\varepsilon \|g\| \text{ for } m = 1, 2 \dots m_0. \tag{42}$$

We note that for every odd m in $\{1, 2 \dots m_0\}$ the set $\tilde{\mathcal{C}}^{(m)}$ is empty and that therefore the $\tilde{\mathcal{C}}^{(m)}$'s are pairwise disjoint. We also note that $\tilde{\mathcal{B}}^{(m)}_\varepsilon(g) \cap \mathcal{F} = \emptyset$ (since $\tilde{\mathcal{B}}^{(m)}_\varepsilon(g) \subseteq \mathrm{supp}_H(g)$ which is disjoint from $\mathcal{F}$) and thus that $\tilde{\mathcal{C}}^{(m)} \cap \mathcal{F} = \emptyset$, for all $m = 1, 2 \dots m_0$. Putting now $\mathcal{E} = \mathcal{F} \cup \bigcup_{m=1, m \text{ even}}^{m_0} \tilde{C}^{(m)}$ we obtain

$$\begin{aligned}
\big\|P_{\mathcal{E}}(f)\big\| &\le \big\|P_{\mathcal{F}}(f)\big\| + \big\|P_{\mathcal{E}}\big(f - P_{\mathcal{F}}(f)\big)\big\| \\
&\le \big\|P_{\mathcal{F}}(f)\big\| + \sum_{m=1}^{m_0} \big\|P_{\tilde{\mathcal{C}}_m}(g)\big\| \\
&\le C_\varepsilon \log\Big(\frac{1}{\delta}\Big) + \Big\lfloor \frac{m_0}{2} \Big\rfloor C_\varepsilon \|g\| \\
&\le C_\varepsilon \log\Big(\frac{1}{\delta}\Big) + C_\varepsilon \log\Big(\frac{1}{\delta}\Big)\Big(1 + C_\varepsilon \log\Big(\frac{1}{\delta}\Big)\Big)
\end{aligned}$$

which proves our claim. □

Our next example provides a lower bound for the constant on the right side of (6).

Example 11. For $n \in \mathbb{N}$ and $\delta = 2^{-2n}$ we claim that there is a function $f \in L_1$ and an $\mathcal{A} \subset \mathrm{supp}_H(f)$, so that for any $0 < \varepsilon < 1$ it follows that $\mathcal{A}_\varepsilon(f) = \mathcal{A}$ and

$$\|\mathcal{P}_\varepsilon(f)\| \geq \log\Big(\frac{1}{\delta}\Big)\|f\|.$$

Indeed, we define $h_0 = 1_{[0,1]}$, and for $k \in \mathbb{N}$ and $j = 1, 2\ldots, 2^{k-1}$ we put

$$h_j^{(k)} = 2^{k-1}\big(1_{[(2j-2)2^{-k},(2j-1)2^{-k})} - 1_{[(2j-1)2^{-k},2j2^{-k})}\big).$$

We observe that for any $n \in \mathbb{N}$

$$\Big\|h_0 + \sum_{k=1}^{2n} h_1^{(k)}\Big\| = 1$$

and for some universal constant $c > 0$.

$$\Big\|h_0 + \sum_{k=1}^{2n} h_1^{(2k)}\Big\| > cn.$$

We secondly observe that the joint distribution of the sequence

$$h_0, \frac{1}{2}(h_1^{(2)}+h_2^{(2)}), \frac{1}{4}(h_1^{(4)}+h_2^{(4)}+h_3^{(4)}+h_4^{(4)}), \frac{1}{8}(h_1^{(6)}+h_2^{(6)}+\ldots+h_8^{(6)}), \ldots,$$

is equal to the joint distribution of h_0, $h_1^{(1)}$,$h_1^{(2)}, \ldots$

It follows therefore that

$$\Big\|h_0 + \sum_{k=1}^{2n} 2^{-k}\sum_{j=1}^{2^k} h_j^{2k}\Big\| = \Big\|h_0 + \sum_{k=1}^{2n} h_1^{(k)}\Big\| = 1, \text{ and}$$

$$\Big\|h_0 + \sum_{k=1}^{n} 2^{-2k}\sum_{j=1}^{2^{2k}} h_j^{4k}\Big\| \geq cn.$$

Therefore if we choose

$$f = h_0 + \sum_{k=1}^{2n} 2^{-k}\sum_{j=1}^{2^k} h_j^{2k}$$

and $\mathcal{A} = \{h^{(0)}\} \cup \{h_j^{4k} : k = 1, 2\ldots n \text{ and } j = 1, 2\ldots 2^{2k}\}$, we obtain for $\delta = 2^{-2n}$, and any $0 < \varepsilon < 1$ that $\mathcal{A}_\varepsilon(f) = \mathcal{A}$ and $\|P_{\mathcal{A}}(f)\| \geq cn \sim \log(1/\delta)$.

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