

THE THRESHOLDING GREEDY ALGORITHM, GREEDY BASES, AND DUALITY

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ABSTRACT. Some new conditions that arise naturally in the study of the Thresholding Greedy Algorithm are introduced for bases of Banach spaces. We relate these conditions to best n -term approximation and we study their duality theory. In particular, we obtain a complete duality theory for greedy bases.

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1. INTRODUCTION

Let X be a Banach space with a semi-normalized basis (e_n) . An approximation algorithm $(F_n)_{n=1}^\infty$ is a sequence of maps $F_n : X \rightarrow X$ such that for each $x \in X$, $F_n(x)$ is a linear combination of at most n of the basis elements (e_j) . The most natural algorithm is the *linear algorithm* $(S_n)_{n=1}^\infty$ given by the partial sum operators.

Recently, Konyagin and Temlyakov [6] introduced the *Thresholding Greedy Algorithm* (TGA) $(G_n)_{n=1}^\infty$, where $G_n(x)$ is obtained by taking the largest n coefficients (precise definitions are given in Section 2). The TGA provides a theoretical model for the thresholding procedure that is used in image compression and other applications.

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They defined the basis (e_n) to be *greedy* if the TGA is optimal in the sense that $G_n(x)$ is essentially the best n -term approximation to x using the basis vectors, i.e. there exists a constant C such that for all $x \in X$ and $n \in \mathbb{N}$ we have

$$(1.1) \quad \|x - G_n(x)\| \leq C \inf \left\{ \left\| x - \sum_{j \in A} \alpha_j e_j \right\| : |A| = n, \alpha_j \in \mathbb{R}, j \in A \right\}.$$

They then showed that greedy bases can be simply characterized as unconditional bases with the additional property of being *democratic*, i.e. for some $\Delta > 0$ we have $\left\| \sum_{j \in A} e_j \right\| \leq \Delta \left\| \sum_{j \in B} e_j \right\|$ whenever $|A| \leq |B|$.

They also defined a basis to be *quasi-greedy* if there exists a constant C such that $\|G_m(x)\| \leq C\|x\|$ for all $x \in X$ and $n \in \mathbb{N}$. Subsequently, Wojtaszczyk [11] proved that these are precisely the bases for which the TGA merely converges, i.e. $\lim_{n \rightarrow \infty} G_n(x) = x$ for $x \in X$.

In this paper we introduce two natural intermediate conditions. Let us denote the biorthogonal sequence by (e_n^*) . We say (e_n) is *almost greedy* if there is a constant C such that

$$(1.2) \quad \|x - G_n(x)\| \leq C \inf \left\{ \left\| x - \sum_{j \in A} e_j^*(x) e_j \right\| : |A| = n \right\} \quad x \in X, n \in \mathbb{N}.$$

Comparison with (1.1) shows that this is formally a weaker condition; in fact Wojtaszczyk's examples of conditional quasi-greedy bases of ℓ_2 [11] are almost greedy but not greedy. We give two characterizations of almost greedy bases in Theorem 3.3. First, a basis is almost greedy if and only if it is quasi-greedy and democratic. Second, if $\lambda > 1$, then $(e_n)_{n=1}^\infty$ is almost greedy if and only if there exists a constant C such that for all $x \in X$ and $n \in \mathbb{N}$, we have

$$(1.3) \quad \|x - G_{[\lambda n]}(x)\| \leq C \inf \left\{ \left\| x - \sum_{j \in A} \alpha_j e_j \right\| : |A| = n, \alpha_j \in \mathbb{R}, j \in A \right\}.$$

Equation (1.2) is a very natural weakening of (1.1).

We also introduce *partially greedy* bases. These are bases such that for some C we have

$$(1.4) \quad \|x - G_n(x)\| \leq C \left\| \sum_{k=n+1}^\infty e_k^*(x) e_k \right\| \quad x \in X, n \in \mathbb{N}.$$

We give a characterization in Theorem 3.4.

Next we study duality of these conditions. In Theorem 5.1 we show that if (e_n) is a greedy basis of a Banach space X with non-trivial Rademacher type then (e_n^*) is a greedy basis of X^* . However, examples

at the end of the paper show that if X has trivial type then (e_n^*) need not be a greedy basic sequence. Theorem 5.4 concerns duality for almost greedy sequences. It is proved that (e_n) and (e_n^*) are both almost greedy if and only if they are both partially greedy. It is also proved that if (e_n) is almost greedy then (e_n^*) is almost greedy if and only if (e_n) is *bidemocratic*, i.e. for some C we have

$$\left\| \sum_{j \in A} e_j \right\| \left\| \sum_{j \in A} e_j^* \right\| \leq Cn \quad |A| = n, \quad n \in \mathbb{N}.$$

Using this result we extend Theorem 5.1 by showing that if X has non-trivial type and (e_n) is almost greedy then (e_n^*) is an almost greedy basic sequence.

We use standard Banach space notation throughout (see e.g. [8]). For clarity, however, we recall here the notation that is used most heavily. Let X be a Banach space. The *dual space* of X , denoted X^* , is the Banach space of all continuous linear functionals F equipped with the norm:

$$\|F\| = \sup\{F(x) : \|x\| = 1\}.$$

The closed linear span of a set $A \subseteq X$ (resp., a sequence (x_n)) is denoted $[A]$ (resp. $[x_n]$). A *basis* for X is a sequence of vectors (e_n) such that every $x \in X$ has a unique expansion as a norm-convergent series

$$x = \sum_{k=1}^{\infty} e_k^*(x)e_k.$$

Here (e_n^*) is the sequence of *biorthogonal functionals* in X^* defined by $e_n^*(e_m) = \delta_{n,m}$. The basis is said to be *unconditional* if the series expansion converges unconditionally for every $x \in X$. It is said to be *monotone* if

$$\left\| \sum_{k=1}^n e_k^*(x)e_k \right\| \leq \|x\| \quad (x \in X, n \geq 1).$$

Finally, more specialized notions from Banach space theory, such as *type* and *cotype*, will be introduced as needed.

2. GREEDY CONDITIONS FOR BASES

Let $(e_n)_{n \in \mathbb{N}}$ be a semi-normalized basis of a Banach space X (i.e., $1/C \leq \|x_n\| \leq C$ for some C); let $(e_n^*)_{n \in \mathbb{N}}$ be the biorthogonal sequence in X^* . Let us denote by S_m the partial-sum operators:

$$S_m(x) = \sum_{j=1}^m e_j^*(x)e_j.$$

We also define the remainder operators $R_m = I - S_m$. For any $x \in X$ we define the *greedy ordering for x* as the map $\rho : \mathbb{N} \rightarrow \mathbb{N}$ such that $\rho(\mathbb{N}) \supset \{j : e_j^*(x) \neq 0\}$ and so that if $j < k$ then either $|e_{\rho(j)}^*(x)| > |e_{\rho(k)}^*(x)|$ or $|e_{\rho(j)}^*(x)| = |e_{\rho(k)}^*(x)|$ and $\rho(j) < \rho(k)$. The m -th greedy approximation is given by

$$G_m(x) = \sum_{j=1}^m e_{\rho(j)}^*(x) e_{\rho(j)}.$$

We will also introduce the m -th greedy remainder

$$H_m(x) = x - G_m(x).$$

The basis (e_n) is called *quasi-greedy* if $G_m(x) \rightarrow x$ for all $x \in X$. This is equivalent (see [11]) to the condition that for some constant C we have

$$(2.1) \quad \sup_m \|G_m(x)\| \leq C\|x\| \quad x \in X.$$

It will be convenient to define the *quasi-greedy constant K* to be the least constant such that

$$\|G_m(x)\| \leq K\|x\| \quad \text{and} \quad \|H_m(x)\| \leq K\|x\| \quad x \in X.$$

If (e_n) is any basis we denote

$$\sigma_m(x) = \inf \left\{ \left\| x - \sum_{j \in A} \alpha_j e_j \right\| : |A| = m, \alpha_j \in \mathbb{R} \right\}.$$

A basis (e_n) is called *greedy* [6] if there is a constant C such that for any $x \in X$ and $m \in \mathbb{N}$ we have

$$(2.2) \quad \|H_m(x)\| \leq C\sigma_m(x).$$

It is natural to introduce two slightly weaker forms of greediness. For any basis (e_n) let

$$\tilde{\sigma}_m(x) = \inf \left\{ \left\| x - \sum_{k \in A} e_k^*(x) e_k \right\| : |A| \leq m \right\}.$$

Note that

$$\sigma_m(x) \leq \tilde{\sigma}_m(x) \leq \|R_m(x)\| \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Let us say that a basis (e_n) is *almost greedy* if there is a constant C so that:

$$(2.3) \quad \|H_m x\| \leq C\tilde{\sigma}_m(x).$$

We will say that a basis (e_n) is *partially greedy* if there is a constant C so that for any $x \in X$, $m \in \mathbb{N}$,

$$(2.4) \quad \|H_m(x)\| \leq C\|R_m x\|.$$

It is clear that for any basis we have the following implications:

$$\text{greedy} \Rightarrow \text{almost greedy} \Rightarrow \text{partially greedy} \Rightarrow \text{quasi-greedy}.$$

Next we prove two useful lemmas concerning quasi-greedy bases. These are both essentially due to Wojtaszczyk [11]. The first lemma says that every quasi-greedy basis is *unconditional for constant coefficients*.

Lemma 2.1. *Suppose $(e_n)_{n \in \mathbb{N}}$ has quasi-greedy constant K . Suppose A is a finite subset of \mathbb{N} . Then, for every choice of signs $\epsilon_j = \pm 1$, we have*

$$(2.5) \quad \frac{1}{2K} \left\| \sum_{j \in A} e_j \right\| \leq \left\| \sum_{j \in A} \epsilon_j e_j \right\| \leq 2K \left\| \sum_{j \in A} e_j \right\|,$$

and hence for any real numbers $(a_j)_{j \in A}$

$$(2.6) \quad \left\| \sum_{j \in A} a_j e_j \right\| \leq 2K \max_{j \in A} |a_j| \left\| \sum_{j \in A} e_j \right\|.$$

Proof. First note that if $B \subset A$ and $\varepsilon > 0$ then

$$\left\| \sum_{j \in B} (1 + \varepsilon) e_j \right\| \leq K \left\| \sum_{j \in B} (1 + \varepsilon) e_j + \sum_{j \in A \setminus B} e_j \right\|.$$

Letting $\varepsilon \rightarrow 0$, we obtain $\left\| \sum_{j \in B} e_j \right\| \leq K \left\| \sum_{j \in A} e_j \right\|$, and hence for any choice of signs $\epsilon_j = \pm 1$, we have

$$\left\| \sum_{j \in A} \epsilon_j e_j \right\| \leq 2K \left\| \sum_{j \in A} e_j \right\|.$$

This gives the right-hand inequality in (2.5) and the left-hand inequality is similar. By convexity (2.6) follows immediately. \square

Lemma 2.2. *Suppose $(e_n)_{n \in \mathbb{N}}$ has quasi-greedy constant K . Suppose $x \in X$ has greedy ordering ρ . Then*

$$(2.7) \quad |e_{\rho(m)}^*(x)| \left\| \sum_{j=1}^m e_{\rho(j)} \right\| \leq 4K^2 \|x\|$$

and hence if A is any subset of \mathbb{N} and $(a_j)_{j \in A}$ any real numbers,

$$(2.8) \quad \min_{j \in A} |a_j| \left\| \sum_{j \in A} e_j \right\| \leq 4K^2 \left\| \sum_{j \in A} a_j e_j \right\|$$

Proof. We prove (2.7) and then (2.8) is immediate. Let $a_j = e_j^*(x)$. Let $\epsilon_j = \text{sgn } a_j$ and put $1/|a_\rho(0)| = 0$. Then

$$\begin{aligned} |a_{\rho(m)}| \left\| \sum_{j=1}^m \epsilon_{\rho(j)} e_{\rho(j)} \right\| &= |a_{\rho(m)}| \left\| \sum_{j=1}^m \left(\frac{1}{|a_{\rho(j)}|} - \frac{1}{|a_{\rho(j-1)}|} \right) (H_{j-1}(x) - H_m(x)) \right\| \\ &\leq 2K \|x\|. \end{aligned}$$

We then use (2.5). \square

We conclude this section by considering direct and inverse theorems for approximation with regard to almost greedy bases.

We define the *fundamental function* $\varphi(n)$ of a basis (e_n) by

$$\varphi(n) = \sup_{|A| \leq n} \left\| \sum_{k \in A} e_k \right\|.$$

For $x \in X$ with greedy ordering ρ , let

$$a_k(x) := |e_{\rho(k)}^*(x)|.$$

The following theorem was proved in [10].

Theorem 2.3. *Let $1 < p < \infty$ and let (e_n) be a greedy basis with $\varphi(n) \asymp n^{1/p}$. Then for any $0 < r < \infty$ and $0 < q < \infty$, we have the following equivalence:*

$$\sum_n \sigma_n(x)^q n^{rq-1} < \infty \Leftrightarrow \sum_n a_n(x)^q n^{rq-1+q/p} < \infty.$$

We generalize this theorem as follows.

Theorem 2.4. *Let $1 < p < \infty$ and let (e_n) be a democratic quasi-greedy basis with $\varphi(n) \asymp n^{1/p}$. Then for any $0 < r < \infty$ and $0 < q < \infty$, we have the following equivalence:*

$$\sum_n \sigma_n(x)^q n^{rq-1} < \infty \Leftrightarrow \sum_n a_n(x)^q n^{rq-1+q/p} < \infty.$$

The proof of this theorem is similar to the proof of Theorem 2.3 and is based on the following lemmas which are analogous to the corresponding lemmas from [10]. See the Introduction (or the next section) for the definition of a democratic basis.

Lemma 2.5. *Let (e_n) be a democratic quasi-greedy basis with $\varphi(n) \asymp n^{1/p}$. Then there exists a constant C such that for any two positive integers $N < M$ and any $x \in X$, we have*

$$a_M(x) \leq C \|H_N(x)\| (M - N)^{-1/p}.$$

Proof. This lemma follows from (2.8) of Lemma 2.2. \square

Lemma 2.6. *Let (e_n) be a democratic quasi-greedy basis with $\varphi(n) \asymp n^{1/p}$. Then there exists a constant C such that for any sequence $m_0 < m_1 < \dots$ of non-negative integers, we have*

$$\|H_{m_s}(x)\| \leq C \sum_{l=s}^{\infty} a_{m_l}(x)(m_{l+1} - m_l)^{1/p}.$$

Proof. This lemma follows from (2.6) of Lemma 2.1. □

By Theorem 3.3 below we get that a democratic quasi-greedy basis is almost greedy and also has the following property (setting $\lambda = 2$ in (3) of Theorem 3.3):

$$\sigma_{2n}(x) \leq \|H_{2n}(x)\| \leq C\sigma_n(x)$$

This inequality implies that

$$\sum_n \|H_n(x)\|^{qn^{r-1}} < \infty \Leftrightarrow \sum_n \sigma_n(x)^{qn^{r-1}} < \infty.$$

Therefore Theorem 2.3 holds with the assumption that (e_n) is greedy replaced by the assumption that (e_n) is almost greedy, which yields Theorem 2.4.

Remark 2.7. We note that the version of Theorem 2.4 with $\sigma_n(x)$ replaced by $\|H_n(x)\|$ was proved in [4] (it also follows from Lemmas 2.5 and 2.6). Our version of Theorem 2.4 is based on Theorem 3.3 below.

3. DEMOCRATIC AND CONSERVATIVE BASES

We recall that a basis (e_n) in a Banach space X is called *democratic* if there is a constant Δ such that

$$(3.1) \quad \left\| \sum_{k \in A} e_k \right\| \leq \Delta \left\| \sum_{k \in B} e_k \right\| \quad \text{if } |A| \leq |B|.$$

This concept was introduced in [6]. The following characterization of greedy bases was also proved in [6].

Theorem 3.1. *A basis (e_n) is greedy if and only if it is unconditional and democratic.*

Recall that the fundamental function $\varphi(n)$ of (e_n) is defined by

$$\varphi(n) = \sup_{|A| \leq n} \left\| \sum_{k \in A} e_k \right\|.$$

The *dual fundamental function* is given by

$$\varphi^*(n) = \sup_{|A| \leq n} \left\| \sum_{k \in A} e_k^* \right\|.$$

Note that φ (and φ^*) is subadditive (i.e. $\varphi(m+n) \leq \varphi(m) + \varphi(n)$) and increasing. It may also be seen that $\varphi(n)/n$ (and $\varphi^*(n)/n$) is decreasing since for any set A with $|A| = n$ we have

$$\sum_{k \in A} e_k = \frac{1}{n-1} \sum_{k \in A} \sum_{j \neq k} e_j.$$

By convexity, for any set A and any scalars $\{a_j : j \in A\}$ we have

$$\left\| \sum_{j \in A} a_j e_j \right\| \leq \max_{j \in A} |a_j| \max_{\pm} \left\| \sum_A \pm e_j \right\|.$$

Hence

$$(3.2) \quad \left\| \sum_{j \in A} a_j e_j \right\| \leq 2\varphi(|A|) \max_{j \in A} |a_j|.$$

It is clear that (e_k) is democratic with constant Δ in (3.1) if and only if

$$(3.3) \quad \Delta^{-1}\varphi(|A|) \leq \left\| \sum_{k \in A} e_k \right\| \leq \varphi(|A|), \quad |A| < \infty.$$

Lemma 3.2. *Let (e_n) be a democratic quasi-greedy basis. Let K be the quasi-greedy constant and Δ the democratic constant. Then for $x \in X$ if ρ is the quasi-greedy ordering,*

$$(3.4) \quad |e_{\rho(m)}^*(x)| \leq \frac{4K^2\Delta}{\varphi(m)} \|x\|,$$

and

$$(3.5) \quad \sup_{k \in \mathbb{N}} |e_k^*(H_m x)| \leq \frac{4K^2\Delta}{\varphi(m+1)} \|x\|.$$

Proof. This follows directly from (3.3) and Lemma 2.2 (2.7). \square

Next we compare almost greedy bases with greedy bases. It follows from (3) of the theorem below that in an almost greedy basis the convergence of the TGA is ‘almost’ optimal. It follows from (2) of the theorem below and [11] that any conditional quasi-greedy basis of a Hilbert space is actually almost greedy. See also [3] for a conditional almost greedy basis of ℓ_1 .

Theorem 3.3. *Suppose (e_n) is a basis of a Banach space. The following are equivalent:*

- (1) (e_n) is almost greedy.
- (2) (e_n) is quasi-greedy and democratic.

(3) For any (respectively, every) $\lambda > 1$ there is a constant $C = C_\lambda$ such that

$$\|H_{[\lambda m]}x\| \leq C_\lambda \sigma_m(x).$$

Proof. We start by showing (1) implies (2). It is immediate that (e_n) is quasi-greedy. Now suppose $|A| \leq |B|$. Suppose $\delta > 0$ and define

$$x = \sum_{j \in A} e_j + \sum_{j \in B \setminus A} (1 + \delta)e_j.$$

Then if $r = |B \setminus A|$ we have $H_r(x) = \sum_{j \in A} e_j$. However

$$\tilde{\sigma}_r(x) \leq \left\| \sum_{j \in B} e_j^*(x)e_j \right\| \leq \left\| \sum_{j \in B} e_j \right\| + \delta \left\| \sum_{j \in B \setminus A} e_j \right\|.$$

Letting $\delta \rightarrow 0$, it follows from (2.3) that (e_n) is democratic.

Next we show that (2) implies (1) so that (1) and (2) are equivalent. Suppose $x \in X$ and $m \in \mathbb{N}$. Let

$$G_m(x) = \sum_{j \in A} e_j^*(x)e_j$$

where $|A| = m$. Suppose $|B| = r \leq m$. Then

$$H_m(x) = \left(x - \sum_{j \in B} e_j^*(x)e_j\right) + \sum_{j \in B \setminus A} e_j^*(x)e_j - \sum_{j \in A \setminus B} e_j^*(x)e_j.$$

Then $|B \setminus A| \leq s := |A \setminus B|$. Thus

$$\left\| \sum_{j \in B \setminus A} e_j^*(x)e_j \right\| \leq 2K \left(\max_{j \in B \setminus A} |e_j^*(x)| \right) \varphi(s)$$

(by (2.6))

$$\leq 2K \left(\min_{j \in A \setminus B} |e_j^*(x)| \right) \varphi(s)$$

$$\leq 8K^3 \Delta \left\| \sum_{j \in A \setminus B} e_j^*(x)e_j \right\|$$

(by (3.4))

$$= 8K^3 \Delta \left\| G_s \left(x - \sum_{j \in B} e_j^*(x)e_j \right) \right\|$$

$$\leq 8K^4 \Delta \left\| \left(x - \sum_{j \in B} e_j^*(x)e_j \right) \right\|.$$

We also have:

$$\left\| \sum_{j \in A \setminus B} e_j^*(x)e_j \right\| = \left\| G_s \left(x - \sum_{j \in B} e_j^*(x)e_j \right) \right\|.$$

Thus it follows that

$$\|H_m(x)\| \leq (8K^4\Delta + K + 1)\|x - \sum_{j \in B} e_j^*(x)e_j\|$$

and so, optimizing over B with $|B| \leq m$,

$$\|H_m(x)\| \leq (8K^4\Delta + K + 1)\tilde{\sigma}_m(x).$$

Let us prove that (2) implies (3) for every $\lambda > 1$. Assume K is the quasi-greedy constant and Δ is the democratic constant. Assume $m, r \in \mathbb{N}$. For $x \in X$ and A a finite subset of cardinality m , let $v = \sum_{j \notin A} e_j^*(x)e_j$. Now suppose y is such that $e_j^*(y) \neq e_j^*(x)$ only if $j \in A$. Then

$$G_r(y) = \sum_{j \in B} e_j^*(y)e_j$$

where $|B| = r$. Let $|A \cap B| = s$ where $0 \leq s \leq \min(r, m)$. Then

$$H_r(y) - H_{r-s}(v) = H_s(y - v) = \sum_{j \in A \setminus B} e_j^*(y)e_j.$$

Now by (3.5)

$$\max_{j \in A \setminus B} |e_j^*(y)| \leq \frac{4K^2\Delta}{\varphi(r+1)}\|y\|.$$

Hence by (2.6)

$$(3.6) \quad \|H_r(y) - H_{r-s}(v)\| \leq \frac{8K^3\Delta\varphi(m)}{\varphi(r+1)}\|y\|.$$

For $\epsilon > 0$ we can choose y so that $\|y\| < \sigma_m(x) + \epsilon$ and $\{j : e_j^*(y) \neq e_j^*(x)\}$ is contained in a set A of cardinality m as above. Note that

$$\tilde{\sigma}_{m+r}(x) \leq \tilde{\sigma}_{m+r-s}(x) \leq \|H_{r-s}(v)\|$$

and hence (3.6) and the triangle inequality yield

$$\begin{aligned} \tilde{\sigma}_{m+r}(x) &\leq \|H_{r-s}(v)\| \\ &\leq \|H_r(y)\| + \|H_r(y) - H_{r-s}(v)\| \\ &\leq K\|y\| + \frac{8K^3\Delta\varphi(m)}{\varphi(r+1)}\|y\|. \end{aligned}$$

Since $\|y\| \leq \sigma_m(x) + \epsilon$ and ϵ is arbitrary, we obtain

$$\tilde{\sigma}_{m+r}(x) \leq \left(\frac{8K^3\Delta\varphi(m)}{\varphi(r+1)} + K\right)\sigma_m(x).$$

Next suppose $\lambda > 1$ and $r = \lceil \lambda m \rceil - m$. Now $\varphi(m)/\varphi(r+1) \leq m/(r+1)$, so we have

$$\tilde{\sigma}_{\lceil \lambda m \rceil}(x) \leq \left(\frac{8K^3\Delta}{\lambda-1} + K \right) \sigma_m(x)$$

This implies (3) with $C_\lambda \asymp (\lambda-1)^{-1}$.

It remains to show (3) (for some fixed $\lambda > 1$) implies (2). That (e_n) is quasi-greedy is immediate. Note that if $|D| = \lceil \lambda m \rceil$, then

$$\left\| \sum_{j \in D} e_j \right\| \leq \varphi(\lambda m) \leq \lambda \varphi(m).$$

So to prove that (e_n) is democratic it is enough to show that

$$\left\| \sum_{j \in D} e_j \right\| \geq \varphi(m)/C_\lambda.$$

Suppose $|A| \leq m < \infty$. For any set B of cardinality $\lceil \lambda m \rceil$ disjoint from A we have (by a similar argument as in the case (1) implies (2))

$$\left\| \sum_{j \in A} e_j \right\| \leq C_\lambda \sigma_m \left(\sum_{j \in A \cup B} e_j \right) \leq C_\lambda \left\| \sum_{j \in D} e_j \right\|$$

whenever $D \subset A \cup B$ with $|D| \geq \lceil \lambda m \rceil$. Thus, maximizing over all A with $|A| \leq m$,

$$\inf_{|D|=\lceil \lambda m \rceil} \left\| \sum_{j \in D} e_j \right\| \geq \varphi(m)/C_\lambda$$

and so (e_j) is democratic. \square

If A, B are subsets of \mathbb{N} we use the notation $A < B$ to mean that $m \in A, n \in B$ implies $m < n$. We write $n < A$ for $\{n\} < A$. Let us define a basis (e_n) to be *conservative* if there is a constant Γ such that

$$(3.7) \quad \left\| \sum_{k \in A} e_k \right\| \leq \Gamma \left\| \sum_{k \in B} e_k \right\| \quad \text{if } |A| \leq |B| \text{ and } A < B.$$

The analogue of Theorem 3.1 and Theorem 3.3 is:

Theorem 3.4. *A basis (e_n) is partially greedy if and only if it is quasi-greedy and conservative.*

Proof. Clearly a partially greedy basis is also quasi-greedy. Suppose (e_n) is partially greedy (with constant C in (2.4)) and $A < B$ with $|A| = |B| = m$. Let $r = \max A$. Let $D = [1, r] \setminus A$ and then for $\delta > 0$ let

$$x = \sum_{k \in A} e_k + (1 + \delta) \sum_{k \in D \cup B} e_k.$$

Then

$$\|H_r(x)\| = \left\| \sum_{k \in A} e_k \right\|$$

and

$$\|R_r(x)\| = (1 + \delta) \left\| \sum_{k \in B} e_k \right\|$$

so that letting $\delta \rightarrow 0$ gives (3.7) with $\Gamma = C$. \square

Conversely, let us suppose (e_n) is quasi-greedy with constant K and conservative with constant Γ . Suppose $x \in X$ and $m \in \mathbb{N}$. Let ρ be the greedy ordering for x . Then let $D = \{\rho(j) : j \leq m, \rho(j) \leq m\}$, and $B = \{\rho(j) : j \leq m, \rho(j) > m\}$. Let $A = [1, m] \setminus D$. Then $|A| = |B| = r$, say, and $A < B$. Now

$$\left\| \sum_{k \in B} e_k^*(x) e_k \right\| = \|G_r(R_m x)\| \leq K \|R_m x\|.$$

Also

$$\begin{aligned} \left\| \sum_{k \in A} e_k^*(x) e_k \right\| &\leq 2K (\max_{k \in A} |e_k^*(x)|) \left\| \sum_{k \in A} e_k \right\| \\ &\leq 2K\Gamma (\min_{k \in B} |e_k^*(x)|) \left\| \sum_{k \in B} e_k \right\| \\ &\leq 8K^3\Gamma \left\| \sum_{k \in B} e_k^*(x) e_k \right\| \end{aligned}$$

(by (2.8))

$$\leq 8K^4\Gamma \|R_m x\|.$$

Combining gives us

$$\begin{aligned} \|H_m x\| &\leq \|R_m x\| + \left\| \sum_{k \in A} e_k^*(x) e_k \right\| + \left\| \sum_{k \in B} e_k^*(x) e_k \right\| \\ &\leq (8K^4\Gamma + K + 1) \|R_m x\|. \end{aligned}$$

\square

4. BIDEMOCRATIC BASES

Suppose (e_n) is a democratic basis. We shall say that (e_n) has the *upper regularity property (URP)* if there exists an integer $r > 2$ so that

$$(4.1) \quad \varphi(rn) \leq \frac{1}{2} r \varphi(n) \quad n \in \mathbb{N}$$

This of course implies $\varphi(r^k n) \leq 2^{-k} r^k \varphi(n)$ and is therefore easily equivalent to the existence of $0 < \beta < 1$ and a constant C so that if $m > n$,

$$(4.2) \quad \varphi(m) \leq C \left(\frac{m}{n}\right)^\beta \varphi(n).$$

We say (e_n) has the *lower regularity property (LRP)* if there exists $r > 1$ so that for all $n \in \mathbb{N}$ we have

$$(4.3) \quad \varphi(rn) \geq 2\varphi(n) \quad n \in \mathbb{N}.$$

This is similarly equivalent to the existence of $0 < \alpha < 1$ and $c > 0$ so that if $m > n$

$$(4.4) \quad \varphi(m) \geq c \left(\frac{m}{n}\right)^\alpha \varphi(n).$$

Let us recall that a Banach space X has (Rademacher) type $1 < p \leq 2$ if there is a constant C so that

$$\left(\text{Ave}_{\epsilon_j = \pm 1} \left\| \sum_{j=1}^n \epsilon_j x_j \right\|^p\right)^{\frac{1}{p}} \leq C \left(\sum_{j=1}^n \|x_j\|^p\right)^{\frac{1}{p}} \quad x_1, \dots, x_n \in X, n \in \mathbb{N}.$$

The least such constant C is called the type p -constant $T_p(X)$. We say that X has non-trivial (resp. trivial) type if X has (resp. does not have) type p for some (resp. any) $p > 1$. Recall also that X has (Rademacher) cotype $2 \leq q < \infty$ if there exists a constant C such that

$$\left(\sum_{j=1}^n \|x_j\|^q\right)^{\frac{1}{q}} \leq C \left(\text{Ave}_{\epsilon_j = \pm 1} \left\| \sum_{j=1}^n \epsilon_j x_j \right\|^q\right)^{\frac{1}{q}} \quad x_1, \dots, x_n \in X, n \in \mathbb{N}.$$

The least such constant C is called the cotype q -constant $C_q(X)$. We say that X has non-trivial (resp. trivial) cotype if X has (resp. does not have) cotype q for some (resp. any) $q < \infty$.

Proposition 4.1. (1) If (e_n) is an almost greedy basis of a Banach space with non-trivial cotype then (e_n) has (LRP).

(2) If (e_n) is an almost greedy basis of a Banach space with non-trivial type then (e_n) has (LRP) and (URP).

Proof. (1) Suppose K is the quasi-greedy constant of (e_n) and Δ is the democratic constant. Suppose X has cotype $q < \infty$ with constant $C_q(X)$. Let B_1, \dots, B_m be disjoint sets with $|B_k| = n$ and let $A =$

$\cup_{k=1}^m B_k$. Then, using Lemma 2.1, (2.5), and (3.3)

$$\begin{aligned}
m^{\frac{1}{q}}\varphi(n) &\leq \Delta \left(\sum_{k=1}^m \left\| \sum_{j \in B_k} e_j \right\|^q \right)^{\frac{1}{q}} \\
&\leq 2K\Delta \left(\sum_{k=1}^m \text{Ave}_{\epsilon_j = \pm 1} \left\| \sum_{j \in B_k} \epsilon_j e_j \right\|^q \right)^{\frac{1}{q}} \\
&\leq 2K\Delta C_q(X) \left(\text{Ave}_{\epsilon_j = \pm 1} \left\| \sum_{j \in A} \epsilon_j e_j \right\|^q \right)^{\frac{1}{q}} \\
&\leq 4K\Delta C_q(X)\varphi(mn).
\end{aligned}$$

It is clear this implies (4.4) for some suitable constant $c > 0$ and $\alpha = \frac{1}{q}$.

(2) Since non-trivial type implies non-trivial cotype we obtain (LRP) immediately. The proof of (URP) (with $\beta = \frac{1}{p}$ when X has type p) is very similar. Using the same notation and assuming X has type $p > 1$ with constant $T_p(X)$ we have:

$$\begin{aligned}
\varphi(mn) &\leq 2K\Delta \left(\text{Ave}_{\epsilon_j = \pm 1} \left\| \sum_{j \in A} \epsilon_j e_j \right\|^p \right)^{\frac{1}{p}} \\
&\leq 2K\Delta T_p(X) \left(\sum_{k=1}^m \text{Ave}_{\epsilon_j = \pm 1} \left\| \sum_{j \in B_k} \epsilon_j e_j \right\|^p \right)^{\frac{1}{p}} \\
&\leq 4K\Delta T_p(X) m^{\frac{1}{p}} \varphi(n).
\end{aligned}$$

This implies (4.2) for suitable constants. \square

We now say that a basis (e_n) is *bidemocratic* if there is a constant Δ so that

$$(4.5) \quad \varphi(n)\varphi^*(n) \leq \Delta n.$$

Proposition 4.2. *If (e_n) is bidemocratic (with constant Δ) then (e_n) and (e_n^*) are both democratic (with constant Δ) and are both unconditional for constant coefficients.*

Proof. If A is any finite set we have

$$|A| \leq \left\| \sum_{j \in A} e_j^* \right\| \left\| \sum_{j \in A} e_j \right\| \leq \varphi^*(|A|) \left\| \sum_{j \in A} e_j \right\|.$$

Hence

$$\Delta^{-1}\varphi(|A|) \leq \left\| \sum_{j \in A} e_j \right\|$$

and so (e_n) is democratic with constant Δ . Let $(\epsilon_j)_{j \in A}$ be any choice of signs ± 1 . Then

$$|A| \leq \left\| \sum_{j \in A} \epsilon_j e_j^* \right\| \left\| \sum_{j \in A} \epsilon_j e_j \right\| \leq 2\varphi^*(|A|) \left\| \sum_{j \in A} \epsilon_j e_j \right\|.$$

Hence

$$\frac{1}{2\Delta} \varphi(|A|) \leq \left\| \sum_{j \in A} \epsilon_j e_j \right\| \leq 2\varphi(|A|).$$

Hence (e_n) is unconditional for constant coefficients. Similar calculations work for (e_j^*) to obtain the theorem. \square

Proposition 4.3. *A basis (e_n) is bidemocratic if and only if there is a constant C so that for any finite set $A \subset \mathbb{N}$,*

$$(4.6) \quad \left\| \sum_{k \in A} e_k \right\| \left\| \sum_{k \in A} e_k^* \right\| \leq C|A|.$$

Proof. One direction is trivial. Assume (4.6) holds with $C \geq 1$. Suppose $n \in \mathbb{N}$. By passing to an equivalent norm on X , if necessary, we may assume that (e_n) and (e_n^*) are both monotone. There exist $A, B \subset \mathbb{N}$ with $|A| \leq n, |B| \leq n$ and

$$\left\| \sum_{j \in A} e_j \right\| \geq \frac{1}{2} \varphi(n), \quad \left\| \sum_{j \in B} e_j^* \right\| \geq \frac{1}{2} \varphi^*(n).$$

By monotonicity of (e_n) and (e_n^*) we may assume that $|A| = |B| = n$. Let $D = A \cup B, E = D \setminus A$.

If $\left\| \sum_{j \in D} e_j \right\| \geq \frac{1}{8C} \varphi(n)$ and $\left\| \sum_{j \in D} e_j^* \right\| \geq \frac{1}{8C} \varphi^*(n)$ we obtain immediately that

$$\varphi(n) \varphi^*(n) \leq 2^6 C^3 |D| \leq 2^7 C^3 n.$$

Consider when one of these inequalities fails; we need only treat the case $\left\| \sum_{j \in D} e_j \right\| < \frac{1}{8C} \varphi(n)$. Then

$$\left\| \sum_{j \in E} e_j \right\| \geq \left\| \sum_{j \in A} e_j \right\| - \left\| \sum_{j \in D} e_j \right\| > \frac{\varphi(n)}{2} - \frac{\varphi(n)}{8C} > \frac{\varphi(n)}{4}$$

and thus, as $|E| \leq n$, (4.6) gives

$$\left\| \sum_{j \in E} e_j^* \right\| \leq 4Cn \varphi(n)^{-1}.$$

We also have from (4.6) that

$$\left\| \sum_{j \in A} e_j^* \right\| \leq 2Cn \varphi(n)^{-1}.$$

Hence

$$\left\| \sum_{j \in D} e_j^* \right\| \leq 6Cn\varphi(n)^{-1}$$

and so

$$n \leq |D| \leq \left\| \sum_{j \in D} e_j \right\| \left\| \sum_{j \in D} e_j^* \right\| \leq \left(\frac{6Cn}{\varphi(n)} \right) \left(\frac{\varphi(n)}{8C} \right) = \frac{3n}{4}$$

which is a contradiction. \square

Proposition 4.4. *If (e_n) is a democratic quasi-greedy basis with (URP) then (e_n) is bidemocratic.*

Proof. We assume (4.2) holds, that (e_n) is quasi-greedy with constant K and democratic with constant Δ . Suppose A is a finite subset of \mathbb{N} . Pick $x \in X$ so that $\|x\| = 1$ and $\sum_{j \in A} e_j^*(x) > \frac{1}{2} \left\| \sum_{j \in A} e_j^* \right\|$. Let ρ be the greedy ordering for x . Then by (3.5), if $|A| = n$,

$$\begin{aligned} \varphi(n) \left\| \sum_{j \in A} e_j^* \right\| &\leq 2\varphi(n) \sum_{j \in A} |e_j^*(x)| \\ &\leq 2\varphi(n) \sum_{k=1}^n |e_{\rho(k)}^*(x)| \\ &\leq 8K^2 \Delta \sum_{k=1}^n \frac{\varphi(n)}{\varphi(k)} \\ &\leq 8K^2 \Delta C n^\beta \sum_{k=1}^n k^{-\beta} \\ &\leq C_1 n \end{aligned}$$

for a suitable constant C_1 . This implies $\varphi(n)\varphi^*(n) \leq C_1 n$. \square

Corollary 4.5. *Let (e_n) be a quasi-greedy basis for a Hilbert space. Then (e_n) is bidemocratic.*

Proof. Wojtaszczyk [11] proved that (e_n) is democratic and that $\varphi(n) \asymp \sqrt{n}$. So the result follows from Proposition 4.4. \square

Remark 4.6. Proposition 4.4 breaks down for bases that are not quasi-greedy. To see this, let (e_n^p) be the unit vector basis of ℓ_p . We define a normalized basis (f_n) of $\ell_2 \oplus_2 \ell_p$ as follows:

$$f_{2n-1} = \frac{1}{\sqrt{2}}(e_n^2 + e_n^p); \quad f_{2n} = \frac{1}{2}e_n^2 + \frac{\sqrt{3}}{2}e_n^p.$$

Suppose that $1 < p < 2$. It is easy to check that (f_n) and (f_n^*) are both democratic and unconditional for constant coefficients, that $\varphi(n) \asymp$

$n^{1/p}$, and that $\varphi^*(n) \asymp \sqrt{n}$. So both (f_n) and (f_n^*) have (URP) but (f_n) is not bidemocratic.

5. DUALITY OF ALMOST GREEDY BASES

Theorem 5.1. *Let (e_n) be a greedy basis with (URP). Then (e_n^*) is a greedy basic sequence. In particular, if (e_n) is a greedy basis of a Banach space X with non-trivial type then (e_n^*) is a greedy basis of X^* .*

Proof. Since (e_n^*) is automatically unconditional this follows from Proposition 4.4 and Theorem 3.1. The second part follows from Proposition 4.1; note that any space with non-trivial type and an unconditional basis is reflexive by James's theorem [5]. \square

Remark 5.2. The Haar system is a greedy basis of H_1 . However, Oswald [9] proved that the Haar system is not a greedy basic sequence in BMO (i.e. H_1^*). This provides a natural illustration of the fact that the assumption of non-trivial type in Theorem 5.1 cannot be eliminated.

Corollary 5.3. *For $1 < p < \infty$ the Banach space $L_p[0, 1]$ has a greedy basis not equivalent to a rearranged subsequence of the Haar system.*

Proof. For $p > 2$ Wojtaszczyk [11] constructed such a basis with $\varphi(n) \asymp n^{1/p}$, hence with (URP). The case $p < 2$ follows by duality using Theorem 5.1. \square

Theorem 5.4. *Let (e_n) be a quasi-greedy basis of a Banach space X . Then the following are equivalent:*

- (1) (e_n) is bidemocratic.
- (2) (e_n) and (e_n^*) are both almost greedy.
- (3) (e_n) and (e_n^*) are both partially greedy.

Proof. We first prove (1) implies (2). Let Δ denote the bidemocratic constant. In fact by Theorem 3.3 and Proposition 4.2 we only need show that (e_n^*) is quasi-greedy. Let us denote by G_m^* and H_m^* the greedy operator and greedy remainder operators associated to the dual basic sequence (e_n^*) . Suppose $x^* \in X^*$ and $x \in X$.

First note that if $|A| = m$ then

$$\begin{aligned} \sum_{j \in A} |x^*(e_j)| &\leq \|x^*\| \sup_{\epsilon_j = \pm 1} \left\| \sum_{j \in A} \epsilon_j e_j \right\| \\ &\leq 2\varphi(m) \|x^*\|. \end{aligned}$$

Hence

$$(5.1) \quad \sup_{j \in \mathbb{N}} |(H_m^* x^*)(e_j)| \leq 2 \frac{\varphi(m+1)}{m+1} \|x^*\|.$$

On the other hand (3.5) implies that

$$(5.2) \quad \sup_{j \in \mathbb{N}} |e_j^*(H_m(x))| \leq \frac{4K^2\Delta}{\varphi(m+1)} \|x\|.$$

Suppose $G_m(x) = \sum_{j \in A} e_j^*(x)e_j$ and $G_m^*(x^*) = \sum_{j \in B} x^*(e_j)e_j^*$ where $|A| = |B| = m$. Then

$$\begin{aligned} |(H_m^*x^*)(G_m(x))| &= \left| \left(\sum_{j \in A \setminus B} x^*(e_j)e_j^* \right)(x) \right| \\ &\leq \left\| \sum_{j \in A \setminus B} x^*(e_j)e_j^* \right\| \|x\| \\ &\leq 4 \frac{\varphi(m+1)\varphi^*(m)}{m+1} \|x\| \|x^*\| \end{aligned}$$

(by (3.2) and (5.1))

$$\leq 4\Delta \|x\| \|x^*\|.$$

Also,

$$\begin{aligned} |(G_m^*x^*)(H_m(x))| &= |x^* \left(\sum_{j \in B \setminus A} e_j^*(x)e_j \right)| \\ &\leq \|x^*\| \frac{4K^2\Delta \|x\|}{\varphi(m+1)} (2\varphi(m)) \end{aligned}$$

(by (5.2))

$$\leq 8K^2\Delta \|x\| \|x^*\|.$$

Now

$$G_m^*x^*(x) = x^*(G_mx) - (H_m^*x^*)(G_mx) + G_m^*(x^*)(H_mx).$$

Hence

$$|G_m^*x^*(x)| \leq (K + 4\Delta + 8K^2\Delta) \|x\| \|x^*\|$$

so that

$$\|G_m^*x^*\| \leq (K + 4\Delta + 8K^2\Delta) \|x^*\|.$$

This implies (e_n^*) is a quasi-greedy basic sequence, and proves (1) implies (2).

Of course (2) implies (3) so it remains to prove (3) implies (1). By Theorem 3.4 (3) implies that both (e_n) and (e_n^*) are quasi-greedy and conservative. Let us assume that K is a quasi-greedy constant for both (e_n) and (e_n^*) , and that Γ is a conservative constant for both (e_n) and (e_n^*) .

Suppose A is any finite subset of \mathbb{N} . For $x \in [e_j]_{j \notin A}$, let $y = \sum_{j \in A} e_j + x$. First suppose that $|e_j^*(x)| \neq 1$ for all j . Then

$$\begin{aligned} \left\| \sum_{j \in A} e_j \right\| &\leq \left\| \sum_{|e_j^*(y)| \leq 1} e_j^*(y) e_j \right\| + \left\| \sum_{|e_j^*(y)| < 1} e_j^*(y) e_j \right\| \\ &\leq 2K \|y\|. \end{aligned}$$

By continuity, $\left\| \sum_{j \in A} e_j \right\| \leq 2K \|y\|$ for all $x \in [e_j]_{j \notin A}$. Thus, by Nikol'skii's Duality Theorem (see e.g. [7]), there exists $x^* \in [e_j^*]_{j \in A}$ with $\|x^*\| = 1$ and

$$(5.3) \quad |x^*(\sum_{j \in A} e_j)| \geq \frac{1}{2K} \left\| \sum_{j \in A} e_j \right\|.$$

Now suppose $m \in \mathbb{N}$. Choose A_0, B_0 with $|A_0|, |B_0| \leq m$ and

$$\left\| \sum_{j \in A_0} e_j \right\| \geq \frac{1}{2} \varphi(m), \quad \left\| \sum_{j \in B_0} e_j^* \right\| \geq \frac{1}{2} \varphi^*(m).$$

Now let A be any subset of \mathbb{N} with $|A| = 2m$ and $A \supset \max(A_0, B_0)$.

Note that if $D \subset A$ and $|D| \geq m$, then since (e_n) and (e_n^*) are conservative with constant Γ ,

$$(5.4) \quad \left\| \sum_{j \in D} e_j \right\| \geq \frac{1}{2\Gamma} \varphi(m), \quad \left\| \sum_{j \in D} e_j^* \right\| \geq \frac{1}{2\Gamma} \varphi^*(m).$$

Let us choose $u^* \in [e_j^*]_{j \in A}$ such that $\sum_{j \in A} |u_j^*(e_j)|^2$ is minimized subject to $\|u^*\| \leq 1$ and

$$(5.5) \quad \sum_{j \in A} u^*(e_j) \geq \frac{\varphi(m)}{4\Gamma K}.$$

This is possible by (5.3) and (5.4).

Now let $G_m^*(u^*) = \sum_{j \in B} u^*(e_j) e_j^*$ where $B \subset A$ and $|B| = m$. Let $D = A \setminus B$. We observe that by (2.7) we have

$$\min_{j \in B} |u^*(e_j)| \left\| \sum_{j \in B} e_j^* \right\| \leq 4K^2$$

and hence by (5.4)

$$(5.6) \quad \min_{j \in B} |u^*(e_j)| \leq \frac{8K^2\Gamma}{\varphi^*(m)}.$$

We then again use (5.3) to find $v^* \in [e_j^*]_{j \in D}$ with $\|v^*\| = 1$ and

$$\sum_{j \in D} v^*(e_j) \geq \frac{\varphi(m)}{4\Gamma K}.$$

It follows from the minimality assumption on u^* that

$$\sum_{j \in A} ((1-t)u^*(e_j) + tv^*(e_j))^2 \geq \sum_{j \in A} (u^*(e_j))^2$$

for $0 \leq t \leq 1$ and so using (2.8) and (5.6),

$$\begin{aligned} \sum_{j \in A} u^*(e_j)^2 &\leq \sum_{j \in A} u^*(e_j)v^*(e_j) \\ &\leq \min_{j \in B} |u^*(e_j)| \sum_{j \in D} |v^*(e_j)| \\ &\leq \frac{8K^2\Gamma}{\varphi^*(m)} \max_{\epsilon_j = \pm 1} \left\| \sum_{j \in D} \epsilon_j e_j \right\| \\ &\leq \frac{16K^2\Gamma\varphi(m)}{\varphi^*(m)}. \end{aligned}$$

Thus from (5.5)

$$\begin{aligned} (\varphi(m))^2 &\leq 2^4\Gamma^2 K^2 \left(\sum_{j \in A} |u^*(e_j)| \right)^2 \\ &\leq 2^4\Gamma^2 K^2 m \sum_{j \in A} u^*(e_j)^2 \\ &\leq \frac{2^8\Gamma^3 K^4 m \varphi(m)}{\varphi^*(m)} \end{aligned}$$

which gives the estimate

$$\varphi(m)\varphi^*(m) \leq 2^8\Gamma^3 K^4 m,$$

so that (e_n) is bidemocratic. \square

Corollary 5.5. *Let X be a Banach space with non-trivial type. If (e_n) is an almost greedy basis of X then (e_n^*) is an almost greedy basic sequence in X^* .*

Proof. This follows directly from Theorem 5.4 and Proposition 4.4. \square

Remark 5.6. In [3] there is an example of an almost greedy basis (e_n) of ℓ_1 such that (e_n^*) is not unconditional for constant coefficients, thus not quasi-greedy. The example localizes to give a quasi-greedy basis of the reflexive space $(\sum \oplus \ell_1^n)_2$ whose dual basis is not quasi-greedy. On the other hand, it follows from Corollary 4.5 and Theorem 5.4 above that in a Hilbert space the dual basis of a quasi-greedy basis is always quasi-greedy (in fact, both the basis and its dual are almost greedy).

Corollary 5.7. *Suppose that (e_n) and (e_n^*) are both partially greedy and that $\varphi(n) \asymp n$. Then (e_n) is equivalent to the unit vector basis of ℓ_1 .*

Proof. By Theorem 5.4, (e_n) is bide democratic. Hence

$$\varphi^*(n) \asymp n/\varphi(n) \asymp 1.$$

But this implies that (e_n^*) is equivalent to the unit vector basis of c_0 , which gives the result. \square

Example 5.8. Let us conclude this section by showing that if $\varphi : \mathbb{N} \rightarrow (0, \infty)$ is an increasing function satisfying $\varphi(1) = 1$ and $\varphi(n)/n$ is decreasing, but failing (4.1) then it is possible to construct a Banach space with a greedy basis (e_n) with a fundamental function equivalent to $\varphi(n)$ and such that the dual basic sequence (e_n^*) is not greedy. This will show that the preceding theorem is, in some sense, sharp. In Example 5.10, we will show under very mild additional conditions on φ how to make a reflexive example.

Let us define the sequence space X_φ to be the completion of c_{00} for the norm

$$\|\xi\|_\varphi = \sup_n \sup_{\substack{|A|=n \\ n < A}} \frac{\varphi(n)}{n} \sum_{k \in A} |\xi_k|.$$

It is clear that the canonical basis is unconditional. It is also democratic since if $|A| = n$,

$$(5.7) \quad \frac{1}{2}\varphi(n) \leq \left\| \sum_{k \in A} e_k \right\| \leq \varphi(n)$$

Let us suppose the dual basic sequence (e_n^*) is democratic with democratic constant Δ . We note that if $A > n$ then

$$\left\| \sum_{k \in A} e_k^* \right\| \leq n/\varphi(n).$$

It follows from the democratic assumption that

$$\left\| \sum_{k=1}^n e_k^* \right\| \leq \Delta n/\varphi(n).$$

Now consider

$$\xi = \sum_{k=1}^n \frac{1}{\varphi(k)} e_k.$$

Clearly $\|\xi\| \leq 1$ and so

$$\left\| \sum_{k=1}^n e_k^* \right\| \geq \sum_{k=1}^n \frac{1}{\varphi(k)}.$$

We deduce that

$$\sum_{k=1}^n \frac{1}{\varphi(k)} \leq \Delta \frac{n}{\varphi(n)}.$$

Now any m, n with $m \geq 2$ we have

$$\begin{aligned} \frac{n}{\varphi(n)} \log m &\leq \frac{n}{\varphi(n)} \sum_{k=n}^{mn} \frac{1}{k} \\ &\leq \sum_{k=1}^{mn} \frac{1}{\varphi(k)} \\ &\leq \Delta \frac{mn}{\varphi(mn)}. \end{aligned}$$

Hence

$$\varphi(mn) \leq \frac{\Delta}{\log m} m \varphi(n).$$

For large m this shows that (4.1) holds.

Remark 5.9. The end of the proof of Example 5.8 actually establishes one direction of the following equivalence (the other direction is easier): $(\varphi(n))$ satisfies (URP) if and only if $(1/\varphi(n))$ is *regular*, i.e., if and only if there exists $C > 0$ such that

$$\frac{1}{\varphi(n)} \geq \frac{C}{n} \sum_{j=1}^n \frac{1}{\varphi(j)}.$$

Regular weight sequences arise also in the theory of Lorentz spaces.

Example 5.10. Now let us suppose, in addition, that $\varphi(n)/n^\delta$ is increasing for some choice of $\delta > 0$. We show how to make the preceding example reflexive.

Let $\psi(n) = \varphi(n)^{1+\delta} n^{-\delta}$. Then $\psi(n)/n$ is decreasing and $\psi(n)$ is increasing. Define X_ψ as in Example 5.8 for the function ψ . Let $\theta = (1 + \delta)^{-1}$.

Let T denote Tsirelson space (cf. [2]). For our purposes it is only necessary to know that this space is reflexive,

$$\frac{1}{2}n \leq \left\| \sum_{j \in A} e_j \right\|_T \leq n \quad \text{if } |A| = n$$

and

$$\left\| \sum_{j \in A} e_j^* \right\|_{T^*} \leq 2 \quad \text{if } |A| = n \text{ and } n < A.$$

Now let $Y = [T, X_\psi]_\theta$ be the space obtained by complex interpolation. Since T is reflexive it follows from a result of Calderón [1] that Y is reflexive. Note that $Y^* = [T^*, X_\psi^*]_\theta$.

Now suppose $A \subset \mathbb{N}$ and $|A| = n$. Then

$$\left\| \sum_{j \in A} e_j \right\|_Y \leq n^{1-\theta} \left\| \sum_{j \in A} e_j \right\|_{X_\psi}^\theta \leq \varphi(n).$$

On the other hand if $n < A$ we have

$$\left\| \sum_{j \in A} e_j^* \right\|_{Y^*} \leq \left\| \sum_{j \in A} e_j^* \right\|_{T^*}^{1-\theta} \left\| \sum_{j \in A} e_j^* \right\|_{X_{\psi^*}}^\theta \leq 2 \left(\frac{n}{\psi(n)} \right)^\theta = 2 \frac{n}{\varphi(n)}.$$

Hence for any A with $|A| = 2n$ we have

$$\left\| \sum_{j \in A} e_j \right\|_Y \geq \frac{\varphi(n)}{2}.$$

Thus (e_j) is democratic with fundamental function equivalent to φ . Now suppose (e_j^*) is democratic with constant Δ . Then

$$\left\| \sum_{j=1}^n e_j^* \right\|_Y \leq 2\Delta \frac{n}{\varphi(n)}$$

Now $Y^* = (T^*)^{1-\theta} (X_\psi^*)^\theta \subset Z := (\ell_\infty)^{1-\theta} (X_\psi^*)^\theta$ and so we have

$$\left\| \sum_{j=1}^n e_j^* \right\|_{Y^*} \geq \left\| \sum_{j=1}^n e_j^* \right\|_Z = \left\| \sum_{j=1}^n e_j^* \right\|_\infty^{1-\theta} \left\| \sum_{j=1}^n e_j^* \right\|_{X_\psi^*}^\theta = \left\| \sum_{j=1}^n e_j^* \right\|_{X_\psi^*}^\theta.$$

We deduce that

$$\left\| \sum_{j=1}^n e_j^* \right\|_{X_\psi^*} \leq \left(2\Delta \frac{n}{\varphi(n)} \right)^{1/\theta} = (2\Delta)^{1/\theta} \frac{n}{\psi(n)}.$$

Hence by the argument presented in Example 5.8, we have that

$$\psi(m) \leq C_1 \left(\frac{m}{n} \right)^\beta \psi(n) \quad m > n$$

for some $\beta < 1$ and C_1 . Now

$$\varphi(m) \leq C_1^{\frac{1}{1+\delta}} \left(\frac{m}{n} \right)^{\frac{\beta+\delta}{1+\delta}} \varphi(n) \quad m > n.$$

This implies φ satisfies (4.1).

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