ON THE EXISTENCE OF ALMOST GREEDY BASES IN BANACH SPACES

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Abstract. We consider several greedy conditions for bases in Banach spaces that arise naturally in the study of the Thresholding Greedy Algorithm (TGA). In particular, we continue the study of *almost greedy* bases begun in [3]. We show that almost greedy bases are essentially optimal for n-term approximation when the TGA is modified to include a Chebyshev approximation. We prove that if a Banach space X has a basis and contains a complemented subspace with a symmetric basis and finite cotype then X has an almost greedy basis. We show that c_0 is the only $\mathcal{L}_\infty$ space to have a quasi-greedy basis. The Banach spaces which contain almost greedy basic sequences are characterized.

Contents

1. Introduction

Let X be a real Banach space with a semi-normalized basis (e_n). An algorithm for n-term approximation produces a sequence of maps $F_n : X \to X$ such that, for each $x \in X$, $F_n(x)$ is a linear combination

The research of the first author was supported by a summer research grant from the Universtity of South Carolina while on sabbatical as a Visiting Scholar at The University of Texas at Austin. The research of the second author was supported by NSF grant DMS-9800027.

of at most n of the basis elements (e_j). The most natural algorithm is the *linear algorithm* $(S_n)_{n=1}^{\infty}$ given by the partial sum operators.

Recently, Konyagin and Temlyakov [11] introduced the *Thresholding Greedy Algorithm* (TGA) $(G_n)_{n=1}^{\infty}$, where $G_n(x)$ is obtained by taking the largest n coefficients (precise definitions are given in Section 2). The TGA provides a theoretical model for the thresholding procedure that is used in image compression and other applications.

They defined the basis (e_n) to be *greedy* if the TGA is optimal in the sense that $G_n(x)$ is essentially the best n-term approximation to x using the basis vectors, i.e. if there exists a constant C such that for all $x \in X$ and $n \in \mathbb{N}$, we have

$$\|x - G_n(x)\| \leq C \inf\{\|x - \sum_{j\in A} \alpha_j e_j\| : \ |A| = n, \ \alpha_j \in \mathbb{R}\}. \tag{1.1}$$

They then showed (see Theorem 2.3 below) that greedy bases can be simply characterized as unconditional bases with the additional property of being *democratic*, i.e. for some $\Delta > 0$, we have

$$\|\sum_{j\in A} e_j\| \leq \Delta \|\sum_{j\in B} e_j\| \quad \text{whenever } |A| \leq |B|.$$

They also defined a basis to be *quasi-greedy* if there exists a constant C such that $\|G_m(x)\| \leq C\|x\|$ for all $x \in X$ and $n \in \mathbb{N}$. Subsequently, Wojtaszczyk [22] proved that these are precisely the bases for which the TGA merely converges, i.e. $\lim_{n\to\infty} G_n(x) = x$ for $x \in X$.

The class of *almost greedy* bases was introduced in [3]. Let us denote the biorthogonal sequence by (e_n^*). Then (e_n) is almost greedy if there is a constant C such that

$$\|x - G_n(x)\| \leq C \inf\{\|x - \sum_{j\in A} e_j^*(x) e_j\| : \ |A| = n\} \qquad x \in X, \ n \in \mathbb{N}. \tag{1.2}$$

Comparison with (1.1) shows that this is formally a weaker condition: in (1.1) the infimum is taken over all possible n-term approximations, while in (1.2) only *projections* of x onto the basis vectors are considered. It was proved in [3] (see Theorem 2.5 below) that (e_n) is almost greedy if and only if (e_n) is quasi-greedy and democratic.

In this paper we continue the study of almost greedy bases and related greedy conditions for bases. In Section 3 we consider a natural modification of the TGA which improves the rate of convergence. Let $G_n^C(x)$ be an n-term Chebyshev approximation to x using the same basis vectors given by the TGA, i.e., those with the largest n coefficients. We show that if (e_n) is almost greedy, then $G_n^C(x)$ is essentially the best n-term approximation in the sense described above. For Banach spaces with finite cotype, we also show that the latter property characterizes almost greedy bases.

In Section 4 we consider the *thresholding operators*:

$$\mathcal{G}_a(x) = \sum_{|e_i^*(x)|\geq a} e_i^*(x)e_i \qquad (a > 0, x \in X).$$

There are natural boundedness conditions to impose on these operators and a corresponding class of *thresholding-bounded* bases which satisfy these conditions. We show that this class coincides with the class of *nearly unconditional* bases introduced by Elton [5] and that it strictly contains the class of quasi-greedy bases.

In Section 5 we prove existence results for almost greedy basic sequences. In particular, we give necessary and sufficient conditions for a semi-normalized weakly null sequence to have an almost greedy subsequence and we characterize the Banach spaces which contain almost greedy basic sequences.

The rest of the paper concerns the existence (and nonexistence) of quasi-greedy and almost greedy *bases* (as opposed to basic sequences). The results contained in Sections 6-7 extend a theorem of Wojtaszczyk [22]. We prove that if X has a basis and contains a complemented subspace with a symmetric basis and finite cotype, then X has an almost greedy basis. More generally, we show that if X has a basis and contains a complemented 'good' (loosely, 'far from c_0') unconditional basic sequence, then X has a quasi-greedy basis. The fact that there is no corresponding result for c_0 is explained by the last section of the paper.

Section 8 contains the nonexistence results. We prove that c_0 is the only $\mathcal{L}_\infty$ space to have a quasi-greedy basis. Thus, $C[0,1]$ and (by similar reasoning) the disc algebra do not have quasi-greedy bases. Lastly, we deduce from the Lindenstrauss-Zippin theorem [13] that c_0 is the only infinite-dimensional Banach space up to isomorphism to have a unique quasi-greedy basis up to equivalence.

Standard Banach space notation and terminology are used throughout (see [12]). For clarity, however, we record the notation that is used most heavily. We write $X \sim Y$ if X and Y are linearly isomorphic Banach spaces. We say that X and Y are λ-isomorphic if there exists an isomorphism $T : X \to Y$ with $\|T\|\|T^{-1}\| \leq \lambda$. A subspace Z of X is said to be *complemented* if Z is the range of a continuous linear projection on X.

Let (x_n) be a sequence in X. We say that (x_n) is *semi-normalized* (resp. *normalized*) if there exist $C > 0$ such that $1/C \leq \|x_n\| \leq C$ (resp. $\|x_n\| = 1$) for all $n \geq 1$. The closed linear span of (x_n) is denoted

$[x_n]$. We say that a sequence (x_n) of nonzero vectors is *basic* if there exists a positive constant K such that

$$\|\sum_{i=1}^{m} a_i x_i\| \leq K \|\sum_{i=1}^{n} a_i x_i\|$$

for all scalars (a_i) and all $1 \leq m \leq n \in \mathbb{N}$; (x_n) is *monotone* if we can take $K = 1$; (x_n) is λ-*unconditional* if

$$\|\sum_{i=1}^{\infty} \varepsilon_i a_i x_i\| \leq \lambda \|\sum_{i=1}^{\infty} a_i x_i\|$$

for all scalars (a_i), all choices of signs $\varepsilon_i = \pm 1$. We say that (x_n) is λ-*symmetric* if

$$\|\sum_{i=1}^{\infty} a_{\sigma(i)} x_i\| \leq \lambda \|\sum_{i=1}^{\infty} a_i x_i\|$$

for all permutations σ of $\mathbb{N}$. A *basis* for X is a sequence of vectors (e_n) such that every $x \in X$ has a unique expansion as a norm-convergent series

$$x = \sum_{i=1}^{\infty} e_i^*(x) e_i,$$

where (e_i^*) is the sequence of *biorthogonal functionals* in the dual space X^* defined by $e_i^*(e_j) = \delta_{i,j}$. The usual norms of the sequence spaces ℓ_p and ℓ_∞ are denoted $\|\cdot\|_p$ and $\|\cdot\|_\infty$. The sequence space c_{00} consits of all sequences with only finitely many nonzero terms. For a sequence $(X_n, \|\cdot\|_n)$ of Banach spaces, the direct sum $(\sum_{n=1}^{\infty} \oplus X_n)_p$ is the space of all sequence (x_n) $(x_n \in X_n)$ equipped with the norm $\|(x_n)\| = (\sum_{n=1}^{\infty} \|x_n\|_n^p)^{1/p}$. More specialized notions from Banach space theory will be introduced as needed.

Finally, it is worth emphasizing that we consider only *real* Banach spaces in this paper.

2. Preliminaries

Let $(e_n)_{n\in\mathbb{N}}$ be a semi-normalized basis of a Banach space X, and let $(e_n^*)_{n\in\mathbb{N}}$ be the biorthogonal sequence in X^*. For $x \in X$, we define the *greedy ordering for* x as the map $\rho : \mathbb{N} \to \mathbb{N}$ such that $\rho(\mathbb{N}) \supset \{j : e_j^*(x) \neq 0\}$ and such that if $j < k$ then

$$|e_{\rho(j)}^*(x)| > |e_{\rho(k)}^*(x)| \text{ or } |e_{\rho(j)}^*(x)| = |e_{\rho(k)}^*(x)| \text{ and } \rho(j) < \rho(k).$$

The m-th greedy approximation is given by

$$G_m(x) = \sum_{j=1}^{m} e^*_{\rho(j)}(x) e_{\rho(j)}.$$

The basis (e_n) is called *quasi-greedy* if $G_m(x) \to x$ for all $x \in X$. This is equivalent (see [22]) to the condition that for some constant K we have

$$\sup_m \|G_m(x)\| \le K\|x\| \qquad x \in X. \tag{2.3}$$

We define the *quasi-greedy constant* K to be the least such constant.

The following two lemmas are essentially due to Wojtaszczyk [22]. For proofs we refer to [3] or to Section 4 below for slightly more general 'localized' versions of the same results. (2.4) says that a quasi-greedy basis is *unconditional for constant coefficients.*

Lemma 2.1. *Suppose that $(e_n)_{n\in\mathbb{N}}$ has quasi-greedy constant K and that A is a finite subset of $\mathbb{N}$. Then, for every choice of signs $\varepsilon_j = \pm 1$, we have*

$$\frac{1}{2K}\|\sum_{j\in A} e_j\| \le \|\sum_{j\in A} \varepsilon_j e_j\| \le 2K\|\sum_{j\in A} e_j\|, \tag{2.4}$$

and hence for any real numbers $(a_j)_{j\in A}$

$$\|\sum_{j\in A} a_j e_j\| \le 2K \max_{j\in A}|a_j| \|\sum_{j\in A} e_j\|. \tag{2.5}$$

Lemma 2.2. *Suppose $(e_n)_{n\in\mathbb{N}}$ has quasi-greedy constant K. Suppose $x \in X$ has greedy ordering ρ. Then*

$$|e^*_{\rho(m)}(x)| \|\sum_{j=1}^{m} (\operatorname{sgn} e^*_{\rho(j)})(x) e_{\rho(j)}\| \le 2K\|x\|. \tag{2.6}$$

Hence if $A \subset \mathbb{N}$ is finite and $(a_j)_{j\in A}$ are any real numbers,

$$\min_{j\in A}|a_j| \|\sum_{j\in A} \operatorname{sgn} a_j e_j\| \le (1+K)\|\sum_{j\in A} a_j e_j\|. \tag{2.7}$$

If (e_n) is any Schauder basis we define

$$\sigma_m(x) := \inf\{\|x - \sum_{j\in A} \alpha_j e_j\| : \ |A| = m, \ \alpha_j \in \mathbb{R}\}.$$

A basis (e_n) is called *greedy* [11] if there is a constant C such that for any $x \in X$ and $m \in \mathbb{N}$ we have

$$\|x - G_m(x)\| \le C\sigma_m(x). \tag{2.8}$$

A basis (e_n) is called *democratic* [11] if there is a constant Δ such that

$$\|\sum_{k\in A} e_k\| \le \Delta \|\sum_{k\in B} e_k\| \quad \text{if } |A|\le |B|. \tag{2.9}$$

Note that a democratic basis is automatically semi-normalized.

The following characterization of greedy bases was proved in [11].

Theorem 2.3. *A basis (e_n) is greedy if and only if it is unconditional and democratic.*

For a semi-normalized basis (e_n) we define the *fundamental function* $\varphi(n)$ by

$$\varphi(n) = \sup_{|A|\le n} \|\sum_{k\in A} e_k\|.$$

Note that φ is subadditive (i.e. $\varphi(m+n) \le \varphi(m)+\varphi(n)$) and increasing. It may also be seen that $\varphi(n)/n$ is decreasing since for any set A, with $|A| = n$, we have

$$\sum_{k\in A} e_k = \frac{1}{n-1}\sum_{k\in A}\sum_{j\ne k} e_j.$$

It follows that for any finite $A \subset \mathbb{N}$ and any real scalars $(a_j)_{j\in A}$ we have:

$$\|\sum_{j\in A} a_j e_j\| \le 2\varphi(|A|)\max_{j\in A}|a_j|. \tag{2.10}$$

It is clear that if (e_k) is democratic with constant Δ in (2.9) then

$$\Delta^{-1}\varphi(|A|) \le \|\sum_{k\in A} e_k\| \le \varphi(|A|), \qquad |A| < \infty. \tag{2.11}$$

Combining (2.4), (2.6), and (2.9) yields the following estimate (cf. [6]).

Lemma 2.4. *Let (e_n) be a democratic quasi-greedy basis. Let K be the quasi-greedy constant and Δ the democratic constant. Then for $x \in X$, with greedy ordering ρ, we have*

$$|e^*_{\rho(m)}(x)| \le \frac{4K^2\Delta}{\varphi(m)}\|x\|. \tag{2.12}$$

A slightly weaker form of greediness was introduced in [3]. For a basis (e_n), let

$$\tilde\sigma_m(x) := \inf\{\|x - \sum_{k\in A} e^*_k(x)e_k\| : \ |A| \le m\}.$$

Note that

$$\sigma_m(x) \le \tilde\sigma_m(x) \le \|x - S_m(x)\| \to 0 \qquad \text{as } m\to\infty.$$

We say that a basis (e_n) is *almost greedy* if there is a constant C such that for all $x \in X$, we have

$$\|x - G_m(x)\| \leq C\tilde{\sigma}_m(x). \tag{2.13}$$

The following characterization of almost greedy bases was proved in [3].

Theorem 2.5. *Suppose that (e_n) is a basis of a Banach space. The following are equivalent:*
(1) (e_n) is almost greedy.
(2) (e_n) is quasi-greedy and democratic.
(3) For some (respectively, every) $\lambda > 1$ there is a constant C_λ such that

$$\|x - G_{[\lambda m]}x\| \leq C_\lambda \sigma_m(x).$$

Most of the democratic bases which we consider in this paper actually satisfy a stronger property. Following [11], we say that a basis (e_n) is *superdemocratic* if there exists a constant C such that for all finite $A, B \subseteq \mathbb{N}$, and for all choices of signs $(\varepsilon_i)_{i\in A}$ and $(\eta_i)_{i\in B}$, we have

$$\frac{1}{C}\|\sum_{i\in B}\eta_i e_i\| \leq \|\sum_{i\in A}\varepsilon_i e_i\| \leq C\|\sum_{i\in B}\eta_i e_i\|.$$

It is easy to see that a basis is superdemocratic if and only if it is democratic and unconditional for constant coefficients. By (2.4), every almost greedy basis is superdemocratic. An example of a basis that is superdemocratic but not quasi-greedy is given below (Example 4.8).

3. Semi-greedy bases

In this section we consider an obvious enhancement of the TGA which improves the rate of convergence. Suppose that $x \in X$ and let ρ be the greedy ordering for x. Let $G_n^C(x) \in \operatorname{span}\{e_{\rho(i)} \colon 1 \leq i \leq n\}$ be a Chebyshev approximation to x. Thus,

$$\|x - G_n^C(x)\| = \min\{\|x - \sum_{i=1}^n a_i e_{\rho(i)}\| \colon (a_i)_{i=1}^n \in \mathbb{R}^n\}.$$

It is natural to make the following definition. Let (e_n) be a semi-normalized basis for X. We say that (e_n) is *semi-greedy* if there exists a constant C such that for all $n \geq 1$ and for all $x \in X$, we have

$$\|x - G_n^C(x)\| \leq C\sigma_n(x).$$

We prove below that every almost greedy basis is semi-greedy. The proof uses the fact that the norm in a space with a quasi-greedy basis behaves well under 'truncation of coefficients'.

Fix $M > 0$. Define the 'truncation function' $f_M\colon \mathbb{R} \to [-M, M]$ thus:

$$f_M(x) = \begin{cases} M & \text{for } x > M; \\ x & \text{for } x \in [-M, M]; \\ -M & \text{for } x < -M. \end{cases}$$

Proposition 3.1. *Suppose that (e_n) is quasi-greedy with quasi-greedy constant K. Then, for every $M > 0$ and for all real scalars (a_i), we have*

$$\|\sum_{i=1}^{\infty} f_M(a_i)e_i\| \leq (1 + 3K)\|\sum_{i=1}^{\infty} a_i e_i\|.$$

Proof. Let $x = \sum_{i=1}^{\infty} a_i e_i$ and let ρ be the greedy ordering for x. If $M > \max|a_i|$ there is nothing to prove. So suppose that there exists N such that

$$|a_{\rho(N+1)}| < M \leq |a_{\rho(N)}|.$$

Then

$$\begin{aligned} \|\sum_{i=1}^{N} f_M(a_{\rho(i)})e_{\rho(i)} &= M\|\sum_{i=1}^{N} \operatorname{sgn}(a_{\rho(i)})e_{\rho(i)}\| \\ &\leq |a_{\rho(N)}|\|\sum_{i=1}^{N} \operatorname{sgn}(a_{\rho(i)})e_{\rho(i)}\| \\ &\leq 2K\|x\| \end{aligned}$$

by Lemma 2.2. Moreover,

$$\begin{aligned} \|\sum_{i=N+1}^{\infty} f_M(a_{\rho(i)})e_{\rho(i)}\| &= \|\sum_{i=N+1}^{\infty} a_{\rho(i)}e_{\rho(i)}\| \\ &= \|x - G_N(x)\| \leq (1 + K)\|x\|. \end{aligned}$$

Combining, we get

$$\begin{aligned} \|\sum_{i=1}^{\infty} f_M(a_i)e_i\| &\leq \|\sum_{i=1}^{N} f_M(a_{\rho(i)})e_{\rho(i)}\| + \|\sum_{i=N+1}^{\infty} f_M(a_{\rho(i)})e_{\rho(i)}\| \\ &\leq (1 + 3K)\|x\|. \end{aligned}$$

□

Theorem 3.2. *Every almost greedy basis is semi-greedy.*

Proof. Fix $n \geq 1$ and $x = \sum_{i=1}^{\infty} a_i e_i$ in X. Let ρ be the greedy ordering for x. Let $A = \{\rho(1), \dots, \rho(n)\}$ and let $z := \sum_{i \in B} b_i e_i$ be a good n-term approximation to x, with $|B| = n$ and

$$\|x - z\| \leq 2\sigma_n(x).$$

If $A = B$ then there is nothing to prove. So we may assume that $A \setminus B$ is nonempty. Let $k = |A \setminus B|$, so that $1 \le k \le n$, and let $M = |a_{\rho(n)}|$. Then by (2.12)

$$M\phi(k) \le 4K^2\Delta\|x - z\|, \tag{3.14}$$

since $|e_i^*(x - z)| \ge M$ for all $i \in A \setminus B$. Let

$$x - z := \sum_{i=1}^{\infty} y_i e_i.$$

By Proposition 3.1, we have

$$\|\sum_{i=1}^{\infty} f_M(y_i)e_i\| \le (1 + 3K)\|x - z\|. \tag{3.15}$$

Note that

$$\begin{aligned} w :&= \sum_{i \in A} f_M(y_i)e_i + \sum_{i \in \mathbb{N}\setminus A} a_i e_i \\ &= \sum_{i=1}^{\infty} f_M(y_i)e_i + \sum_{i \in B\setminus A} (a_i - f_M(y_i))e_i. \end{aligned}$$

Thus,

$$\begin{aligned} \|w\| &\le \|\sum_{i=1}^{\infty} f_M(y_i)e_i\| + \|\sum_{i \in B\setminus A} (a_i - f_M(y_i))e_i.\| \\ &\le (1 + 3K)\|x - z\| + 4M\phi(k) \end{aligned}$$

(by (3.15) and by (2.10) since $|a_i - f_M(y_i)| \le 2M$ for all $i \in B \setminus A$)

$$\le 2(1 + 3K)\sigma_n(x) + 16K^2\Delta\|x - z\|$$

(by (3.14))

$$\le (2(1 + 3K) + 32K^2\Delta)\sigma_n(x).$$

Thus (e_n) is semi-greedy. □

Next we discuss the converse of Theorem 3.2. It is convenient to introduce the following notation. For finite sets $A, B \subset \mathbb{N}$, we write $A < B$ if $\max\{n \colon n \in A\} < \min\{n \colon n \in B\}$.

Proposition 3.3. *Every semi-normalized semi-greedy basis (e_n) is superdemocratic.*

Proof. Suppose that $|A| = |B| = n$ and let (ε_i) be any choice of signs. Choose $\varepsilon > 0$ and $D \subset \mathbb{N}$, with $A \cup B < D$ and $|D| = n$. Consider

$$x = \sum_{i\in A} \varepsilon_i e_i + (1+\varepsilon) \sum_{i \in D} e_i.$$

Since (e_n) is semi-greedy, we have

$$\| \sum_{i\in A} \varepsilon_i e_i + \sum_{i\in D} c_i e_i \| \leq C\sigma_n(x)$$

for some real scalars (c_i). Hence

$$\begin{aligned}\sigma_n(x) &\leq \| \sum_{i\in A} \varepsilon_i e_i \| \\ &\leq K \| \sum_{i\in A} \varepsilon_i e_i + \sum_{i\in D} c_i e_i \| \\ &\leq CK\sigma_n(x),\end{aligned}$$

where K is the basis constant of (e_n). Now consider

$$y = (1+\varepsilon) \sum_{i\in A} \varepsilon_i e_i + \sum_{i\in D} e_i.$$

A similar argument gives

$$\sigma_n(y) \leq \| \sum_{i\in D} e_i \| \leq C(1+K)\sigma_n(y).$$

Since $\|y - x\| \to 0$ as $\varepsilon \to 0$, we obtain

$$\frac{1}{CK} \| \sum_{i\in A} \varepsilon_i e_i \| \leq \| \sum_{i\in D} e_i \| \leq C(K+1) \| \sum_{i\in A} \varepsilon_i e_i \|.$$

The above inequalities also hold with A replaced by B. Hence (e_n) is superdemocratic. $\square$

Remark 3.4. Let (e_n) be a semi-greedy basis. The previous result shows that (e_n) has a fundamental function $(\varphi(n))$ and that there exists a constant C such that

$$\min |a_i| \phi(|A|) \leq C \| \sum_{i\in A} a_i e_i \|$$

for all finite $A \subset \mathbb{N}$ and all scalars a_i, $i \in A$.

A democratic basis (e_n) is said to have the *lower regularity property* (LRP) if its fundamental function satisfies $C\varphi(mn) \geq m^\alpha \varphi(n)$ for all $m, n \in \mathbb{N}$, where C and $0 < \alpha \leq 1$ are constants.

Let us recall that a Banach space X has cotype q, where $2 \le q < \infty$, if there exists a constant C such that

$$(\sum_{j=1}^{n} \|x_j\|^q)^{\frac{1}{q}} \le C(\operatorname*{Ave}_{\epsilon_j=\pm 1} \|\sum_{j=1}^{n} \epsilon_j x_j\|^q)^{\frac{1}{q}} \tag{3.16}$$

for all $x_1, \cdots, x_n \in X$ and $n \in \mathbb{N}$. The least such constant C is called the cotype q-constant $C_q(X)$. We say that X has *finite cotype* if X has cotype q for some $q < \infty$.

Proposition 3.5. *Let (e_n) be a superdemocratic basis for a Banach space X which has finite cotype. Then (e_n) has (LRP).*

Proof. Suppose that X has cotype q. Since (e_n) is superdemocratic there exists a constant C such that for all $m, n \in \mathbb{N}$

$$C\varphi(mn) \ge \operatorname*{Ave}_{\epsilon_j=\pm 1} \|\sum_{i=0}^{m-1} \epsilon_j \sum_{j=in+1}^{(i+1)n} e_i\| \ge (1/CC_q)m^{1/q}\varphi(n). \tag{3.17}$$

□

The last theorem of this section is a partial converse to Theorem 3.2.

Theorem 3.6. *Suppose that (e_n) is a semi-greedy basis for a Banach space X which has finite cotype. Then (e_n) is almost greedy.*

Proof. We shall not keep track of the constants, so C will denote a constant whose value changes from line to line.

Suppose that $x = \sum_{i \in F} a_i e_i$, $\|x\| = 1$, and $|F| = n$. By Proposition 3.3, it suffices to prove that (e_n) is quasi-greedy, i.e. that $\|G_k(x)\| \le C$ for $1 \le k \le n$. Let ρ be the greedy ordering for x. Since (e_i) is democratic, we have (cf. (2.10))

$$\|x - G_k(x)\| \le 2|a_{\rho(k)}|\varphi(n-k).$$

By Remark 3.4,

$$\|G_k(x)\| \ge \frac{1}{C}|a_{\rho(k)}|\varphi(k).$$

Hence

$$\frac{\|x - G_k(x)\|}{\|G_k(x)\|} \le C\frac{\varphi(n-k)}{\varphi(k)}.$$

The right-hand side tends to zero as $k/n \to 1$ since (e_n) has (LRP). It follows that there exists $\alpha < 1$ such that $\|G_k(x)\| \le C$ for all $k \ge \alpha n$. By iteration m times, where $\alpha^m \le 1/2$, we get

$$\|G_k(x)\| \le C \qquad \text{for all } k \ge n/2. \tag{3.18}$$

Fix $1 \leq k \leq n$. Let $A := \{\rho(1), \ldots, \rho(k)\}$ and let $B := \{\rho(k+1), \ldots, \rho(2k)\}$. Choose $D > F$, with $|D| = k$, and let $\varepsilon > 0$. Consider

$$y := \sum_{i \in F \setminus A} a_i e_i + (|a_{\rho(k)}| + \varepsilon)(\sum_{i \in D} e_i).$$

Then

$$\sigma_k(y) \leq \|x + (|a_{\rho(k)}| + \varepsilon)(\sum_{i \in D} e_i) \leq 1 + (|a_{\rho(k)}| + \varepsilon)\varphi(k).$$

Since (e_i) is semi-greedy there exist scalars c_i $(i \in D)$ such that

$$\| \sum_{i \in F \setminus A} a_i e_i + \sum_{i \in D} c_i e_i \| \leq C \sigma_n(y).$$

Since $F < D$, (e_i) is a Schauder basis, and $\varepsilon > 0$ is arbitrary, we get

$$\| \sum_{i \in F \setminus A} a_i e_i \| \leq C(1 + |a_{\rho(k)}| \varphi(k)),$$

and hence

$$\|G_k(x)\| = \| \sum_{i \in A} a_i e_i \| \leq C(1 + |a_{\rho(k)}| \varphi(k)). \tag{3.19}$$

Let $z := x - \sum_{i \in A} a_i e_i$. Then $\sigma_k(z) \leq \|x\| \leq 1$. Since (e_i) is semi-greedy there exist scalars (c_i) $(i \in B)$ with $\|z - \sum_{i \in B} c_i e_i\| \leq C$. Hence

$$\| \sum_{i \in A} a_i e_i + \sum_{i \in B} c_i e_i \| = \|x - (z - \sum_{i \in B} c_i e_i)\| \leq C. \tag{3.20}$$

Let $E := \{i \in B \colon |c_i| \geq |a_{\rho(k)}|\}$. Then

$$\sum_{i \in A} a_i e_i + \sum_{i \in E} c_i e_i = G_m(\sum_{i \in A} a_i e_i + \sum_{i \in B} c_i e_i)$$

for some $k \leq m \leq 2k$. So (3.18) and (3.20) yield

$$\| \sum_{i \in A} a_i e_i + \sum_{i \in E} c_i e_i \| \leq C. \tag{3.21}$$

On the other hand, Remark 3.4 yields

$$|a_{\rho(k)}| \varphi(k) \leq C \| \sum_{i \in A} a_i e_i + \sum_{i \in E} c_i e_i \|. \tag{3.22}$$

Finally, combining (3.19), (3.21), and (3.22), we get $\|G_k(x)\| \leq C$. $\square$

4. Thresholding-bounded bases

Let (e_n) be a semi-normalized basis for X. For $a \geq 0$, the thresholding operator $\mathcal{G}_a$ is defined as follows:

$$\mathcal{G}_a(x) = \sum_{|e_n^*(x)| \geq a} e_n^*(x) e_n.$$

Suppose that there exists $a_0 > 0$ and K such that $\|\mathcal{G}_{a_0}(x)\| \leq K\|x\|$ for all $x \in X$. This implies, by scaling, that $\|\mathcal{G}_a(x)\| \leq K\|x\|$ for *all* $a > 0$ and for all $x \in X$, i.e. that (e_n) is a quasi-greedy basis with quasi-greedy constant K.

To obtain a *new* class of bases, we consider (following Elton [5, 15]) boundedness of $\mathcal{G}_a$ on the set Q of vectors whose coefficient sequences belong to the unit ball of ℓ_∞ defined thus:

$$Q := \{\sum_{n=1}^{\infty} a_n e_n \colon |a_n| \leq 1\}.$$

Let $0 < a \leq 1$ and suppose that there exists a constant $C < \infty$ such that

$$\|\mathcal{G}_a(x)\| \leq C\|x\| \tag{4.23}$$

for all $x \in Q$. Let $\theta(a)$ be the least constant C such that (4.23) holds. If there is no such constant C, set $\theta(a) = \infty$.

Proposition 4.1. *Let (e_n) be a semi-normalized basis for a Banach space X.*
(i) If (e_n) is a normalized basis, then

$$\max |e_n^*(x)| \leq \theta(1)\|x\| \qquad (x \in X).$$

(ii) If $0 < a \leq b \leq 1$, then $\theta(b) \leq \theta(a)$.
(iii) $\theta(a) < \infty$ for some $a < 1 \Rightarrow \theta(t) < C_1 t^{-b}$ for all $t \leq 1$ for some positive constants C_1 and b.

Proof. (i) By scaling, we may assume that $\max |e_n^*(x)| = 1$. The result is then clear (from the definition of $\theta(1)$) when $\max |e_n^*(x)|$ is attained uniquely. By perturbing the basis coefficients of x slightly we can assume that this is the case.
(ii) This follows from the identity

$$\mathcal{G}_b(x) = (b/a)\mathcal{G}_a((a/b)x)) \qquad (x \in Q).$$

(iii) Note that, for $k \geq 1$, we have

$$\mathcal{G}_{a^{k+1}} x = \mathcal{G}_{a^k} x + a^k \mathcal{G}_a(a^{-k}(x - \mathcal{G}_{a^k} x)).$$

Hence by the triangle inequality

$$\theta(a^{k+1}) \le \theta(a^k) + \theta(a)(1+\theta(a^k)) \le 3\theta(a)\theta(a^k).$$

It follows that $\theta(t) < C_1 t^{-b}$ for some positive constants C_1 and b. □

We say that (e_n) is *thresholding-bounded* if $\theta(a) < \infty$ for all $a > 0$ (equivalently, for some $a < 1$).

The next two propositions are 'localized' versions of Lemmas 2.1 and 2.2. They will be used in Section 8 below.

Proposition 4.2. *Suppose that (e_n) is a thresholding-bounded basis for X. Let σ be a finite subset of $\mathbb{N}$. Then*

$$\|\sum_{n\in\sigma} a_n e_n\| \le 2\theta(1)\max|a_n|\|\sum_{n\in\sigma} e_n\|,$$

for all real scalars (a_n).

Proof.

$$\|\sum_{n\in\sigma} a_n e_n\| \le \max|a_n| \max_{\pm} \|\sum_{n\in\sigma} \pm e_n\|$$

(by convexity)

$$\le 2\max|a_n| \max_{\tau\subseteq\sigma}\|\sum_{n\in\tau} e_n\|$$

$$\le 2\theta(1)\max|a_n|\|\sum_{n\in\sigma} e_n\|.$$

□

Proposition 4.3. *Suppose that (e_n) is a thresholding-bounded basis for X. Then, for every $m \ge 1$ and $x \in Q$ with greedy ordering ρ, we have*

$$|e^*_{\rho(m)}(x)|\|\sum_{k=1}^{m} \operatorname{sgn} e^*_k(x) e^*_k\| \le 2\theta(|e^*_{\rho(m)}(x)|)\|x\|$$

Proof. Let $a_j = e^*_j(x)$, $\varepsilon_j = \operatorname{sgn} a_j$, and put $1/|a_0| = 0$. Then

$$|a_{\rho(m)}|\|\sum_{j=1}^{m} \varepsilon_{\rho(j)} e_{\rho(j)}\| = |a_{\rho(m)}|\|\sum_{j=1}^{m}\left(\frac{1}{|a_{\rho(j)}|} - \frac{1}{|a_{\rho(j-1)}|}\right)\sum_{k=j}^{m} a_{\rho(k)} e_{\rho(k)}\|$$

$$\le \max_{1\le j\le m}\|\sum_{k=j}^{k=m} a_{\rho(k)} e_{\rho k}\|$$

$$\le 2\theta(|a_{\rho(m)}|)\|x\|.$$

□

A similar argument gives the following.

Proposition 4.4. *Suppose that (e_n) is a thresholding-bounded basis for X. Then, for every finite $\sigma \subset \mathbb{N}$ and for all real scalars (a_n), with* $\sup |a_n| \leq 1$, *we have*

$$\min_{n\in\sigma} |a_n| \| \sum_{n\in\sigma} (\operatorname{sgn} a_n) e_n \| \leq (1 + \theta(\min_{n\in\sigma} |a_n|)) \| \sum_{n\in\sigma} a_n e_n \|.$$

Note that every quasi-greedy basis is thresholding-bounded and that a thresholding-bounded basis is quasi-greedy with constant K if and only if $\sup_{a>0} \theta(a) = K$.

Let us recall a notion introduced by Elton (see [5, 15]). A semi-normalized basis (e_n) in a Banach space X is called *nearly unconditional* if, for every $0 < a \leq 1$, there exists a constant $\phi(a)$ such that for every $x = \sum_{n=1}^{\infty} e_n^*(x) e_n \in Q$, and for every $A \subseteq \{n \in \mathbb{N} \colon |e_n^*(x)| \geq a\}$, we have

$$\| \sum_{n\in A} a_n e_n \| \leq \phi(a) \|x\|.$$

Note that (e_n) is unconditional if and only if $\sup_{a>0} \phi(a) < \infty$. Clearly, we have the implication:

$$\text{nearly unconditional} \Rightarrow \text{thresholding-bounded.}$$

Surprisingly, the converse implication also holds.

Proposition 4.5. *Every thresholding-bounded basis is nearly unconditional. Moreover,*

$$\phi(a) \leq \frac{4\theta(1)\theta(a)}{a} \qquad (0 < a \leq 1).$$

Proof. Fix $0 < a \leq 1$ and $x \in Q$. Set $\sigma(a) := \{i \in \mathbb{N} \colon |e_i^*(x)| \geq a\}$, and suppose that $A \subseteq \sigma(a)$. Then

$$\begin{aligned} \| \sum_{i\in A} e_i^*(x) e_i \| &\leq \max_{\pm} \| \sum_{i\in\sigma(a)} \pm e_i \| \\ &\leq 2\theta(1) \| \sum_{i\in\sigma(a)} (\operatorname{sgn} e_i^*(x)) e_i \| \end{aligned}$$

(by the same argument used to prove Proposition 4.2)

$$\leq 4\theta(1) \frac{\theta(a)}{a} \|x\|$$

by Proposition 4.3. The estimate for $\phi(a)$ follows. $\square$

We conclude this section with an example of a thresholding-bounded basis that is not quasi-greedy. The construction uses the following simple sufficient condition for thresholding-boundedness. (Recall the definition (see also (6.34) below) of the weak-ℓ_1 quasi-norm:

$$\|(a_n)\|_{1,\infty} = \sup n a_n^*,$$

where (a_n^*) is the nonincreasing rearrangement of $(|a_n|)$.)

Lemma 4.6. *Let (e_n) be a normalized basis of a Banach space X such that*

$$\|\sum_{i=1}^{\infty} a_i e_i\| \geq c\|(a_i)\|_{1,\infty}$$

for all real scalars (a_i), where c is a constant. Then (e_n) is thresholding-bounded and $\theta(a) \leq (ca)^{-1}$ for $0 < a \leq 1$.

Proof. Suppose that $x \in Q$. Then

$$\|\mathcal{G}_a(x)\| \leq |\{i\colon |e_i^*(x)| \geq a\}| \leq \frac{1}{a}\|(e_i^*(x))\|_{1,\infty} \leq \frac{1}{ca}\|x\|.$$

Hence $\theta(a) \leq (ca)^{-1}$. □

We recall the definition of the dyadic Hardy space H_1. Let $(h_n)_{n=1}^{\infty}$ be the dyadic Haar system on $[0,1]$ normalized in L_1. The norm in H_1 is given as follows:

$$\|\sum_{n=1}^{\infty} a_n h_n\| = \int_0^1 (\sum_{n=1}^{\infty} a_n^2 h_n^2)^{1/2}\, dx.$$

Clearly, (h_n) is a normalized 1-unconditional basis for H_1.

Lemma 4.7. *There exists a constant C such that for every $N \geq 1$ there exist integers $n_1, n_2, \dots, n_{2N}$ and a normalized 1-unconditional basis $(e_i)_{i=1}^{P}$ (where $P := P(N) = \sum_{i=1}^{2N} n_i$) of a finite-dimensional normed space $(\mathbb{R}^P, \|\cdot\|_N)$, satisfying the following:*

(a) *For all real scalars $(a_i)_{i=1}^{P}$, we have*

$$\|\sum_{i=1}^{P} a_i e_i\|_N \geq \frac{1}{C}\|(a_i)\|_{1,\infty}.$$

(b) *For all real scalars $(b_i)_{i=1}^{2N}$, we have*

$$\frac{1}{C}(\sum_{i=1}^{2N} b_i^2)^{1/2} \leq \|\sum_{i=1}^{2N} b_i f_i\|_N \leq C(\sum_{i=1}^{2N} b_i^2)^{1/2},$$

where

$$f_i = \frac{1}{n_i}(e_{n_1+\cdots+n_{i-1}+1} + \cdots + e_{n_1+\cdots+n_i}).$$

(c) $n_1 < n_3 \cdots < n_{2N-1} < n_2 < n_4 < \cdots < n_{2N}$

Proof. We take the vectors (e_j) for

$$n_1 + \cdots + n_{i-1} + 1 < j \leq n_1 + \cdots + n_i$$

to be all the Haar functions on a certain level of the L_1-normalized Haar system on $[0, 1]$. It is known that consecutive levels of the the dyadic Hardy space H_1 satisfy (a) and (b). To ensure that (c) is satisfied, we simply rearrange the levels. Since the Haar system is a 1-unconditional basis of H_1, *every* rearrangement of the levels is a 1-unconditional basis satisfying (a) and (b). □

Example 4.8. There exists a reflexive Banach space X with a thresholding-bounded basis that is not quasi-greedy. Recall the following expression for the norm $\|\cdot\|_J$ of the James space J [8]:

$$\|\sum a_i e_i\|_J = \sup\{(\sum_{j=1}^{k}(\sum_{i=m_{j-1}+1}^{m_j} a_i)^2)^{1/2},$$

where the supremum is taken over all $k \geq 1$ and all $0 = m_0 < m_1 < \cdots < m_k$. Fix $N \geq 1$ and let $(f_i)_{i=1}^{2N}$ be defined as in Lemma 4.7. Note that

$$\|f_1 + f_3 + \ldots f_{2N-1}\|_J = N, \tag{4.24}$$

and that

$$\|f_1 - f_2 + f_3 - f_4 + \cdots + f_{2N-1} - f_{2N}\|_J \leq 2\sqrt{N}. \tag{4.25}$$

Define the norm $|x|_N := \max(\|x\|_N, \|x\|_J)$. Consider

$$x = f_1 - f_2 + f_3 - f_4 + \cdots + f_{2N-1} - f_{2N} \quad \text{and} \quad a = 1/n_{2N-1}.$$

Then $x \in Q$; by (4.25) and condition (b) of Lemma 4.7, we have

$$|x|_N \leq 2C\sqrt{N}$$

By (4.24) and condition (c) of Lemma 4.7, we have

$$|\mathcal{G}_a x|_N = |f_1 + f_3 + \cdots + f_{2N-1}|_N = N.$$

Hence $\theta(a) \geq (2C)^{-1}\sqrt{N}$. Thus, for each N we have constructed a finite-dimensional normed space $(F_N, |\cdot|_N)$ with a monotone basis (e_i) such that

$$|\sum a_i e_i|_N \geq (1/C)\|(a_i)\|_{1,\infty}$$

and

$$\sup_{a>0} \theta(a) \geq \frac{1}{2C}\sqrt{N}.$$

Let $X = (\sum_{N=1}^{\infty} \oplus F_N)_2$ and let (e_n) be the natural basis obtained by concatenating the bases of the F_N's. For $x = (x_N) \in Q$, we have

$$\begin{aligned}\|\mathcal{G}_a(x)\| &= (\sum_{N=1}^{\infty} |\mathcal{G}_a(x_N)|_N^2)^{1/2} \\ &\leq \frac{C}{a}(\sum_{N=1}^{\infty} |x_N|_N^2)^{1/2} = \frac{C}{a}\|x\|.\end{aligned}$$

Hence (e_n) is a thresholding-bounded basis for X. However, $\theta(a) \to \infty$ as $a \to 0$. Thus, (e_n) is not quasi-greedy.

5. Subsequences of weakly null sequences

The motivation for the results of this section is the following theorem of Elton [5] (paraphrased slightly to suit our purposes).

Theorem 5.1. *Let (e_n) be a semi-normalized weakly null sequence in a Banach space X. Then (e_n) has a thresholding-bounded basic subsequence satisfying*

$$\phi(a) \leq C \log(1 + 1/a)$$

for some absolute constant C.

Remark 5.2. The estimate for $\phi(a)$, while not hitherto explicitly stated anywhere, follows from the proof of Elton's theorem presented in [15]. In its usual formulation, Elton's theorem asserts the existence of a subsequence that is nearly unconditional. However, by Proposition 4.5, this is equivalent to the above.

For Banach spaces which are 'far from c_0' (in a sense made precise below) we can improve this result significantly by showing the existence of a quasi-greedy subsequence.

To that end let us recall the notion of *spreading model* (see e.g. [1]). Let (e_i) be a semi-normalized basic sequence in a Banach space X and let (s_i) be a basis for a Banach space $(Y, |\cdot|)$. Then (s_i) is said to be a spreading model for (e_i) if, for all $k \geq 1$ and for all real scalars $a_1, \dots, a_k$, we have

$$\left|\sum_{i=1}^{k} a_i s_i\right| = \lim_{\substack{n_1 \to \infty \\ n_1 < \cdots < n_k}} \left\|\sum_{i=1}^{k} a_i x_{n_i}\right\|.$$

It is known that every normalized basic sequence has a subsequence with a spreading model, and that if the basic sequence is weakly null, then the spreading model is monotone and 2-unconditional.

Proposition 5.3. *Let (e_i) be a semi-normalized basic sequence in a Banach space X which has spreading model (s_i). Suppose that*

$$g(n) := |\sum_{i=1}^{n} s_i| \to \infty \quad as \quad n \to \infty.$$

Then, (e_n) has a democratic subsequence. Moreover, if (e_n) is weakly null then we may take the democratic constant of the subsequence to be $\Delta \leq 1 + \varepsilon$ for any given $\varepsilon > 0$.

Proof. Since (s_i) is (by definition) democratic, by using the condition $g(n) \to \infty$ it is easy to construct the desired democratic subsequence of (e_i). To avoid repetition, we refer to Theorem 5.4 below for a similar, but more complicated, argument. To get $\Delta \leq 1 + \varepsilon$ in the weakly null case, we use the fact that (s_i) is monotone when (e_i) is weakly null. □

To obtain the main result of this section we need the following result which is stated without proof in the Introduction of [14] (see also [1] for the proof). Given $\varepsilon > 0$, every semi-normalized weakly null sequence has a subsequence (e_n) which is $(2+\varepsilon)$-*Schreier-unconditional*, i.e. such that

$$\|P_A x\| \leq (2 + \varepsilon)\|x\|$$

for every $x \in [e_n]$ and for every finite $A \subset \mathbb{N}$ satisfying $|A| \leq \min A$. (Here $P_A x := x\chi_A$ denotes the projection onto A.)

Theorem 5.4. *Let (e_n) be a semi-normlized weakly null basic sequence in a Banach space X with spreading model (s_n). Suppose that*

$$g(n) := |\sum_{i=1}^{n} s_i| \to \infty \quad as \quad n \to \infty.$$

Then, given $\varepsilon > 0$, (e_n) has a quasi-greedy subsequence with quasi-greedy constant $K \leq 3 + \varepsilon$.

Proof. By passing to a subsequence and rescaling, we may assume that (e_n) is a normalized basic sequence, with basis constant at most $1+\varepsilon/4$. Choose an increasing sequence (n_k) such that $g(n_k) > 24k/\varepsilon$. Using Schreier-unconditionality, and by passing to further subsequences, we may assume that (e_n) satisfies the following. For every $k \geq 1$ and for every $A \subset \mathbb{N}$, with $|A| \le n_k$ and $\min A \ge k$, we have

$$\frac{1}{2}\|\sum_{i=1}^{|A|} a_i s_i\| \le \|\sum_{i\in A} a_i e_i\| \le 2\|\sum_{i=1}^{|A|} a_i s_i\| \tag{5.26}$$

and

$$\|P_A x\| \le (2+\varepsilon/4)\|x\|. \tag{5.27}$$

Suppose that $x = \sum_{i=1}^{\infty} x_i e_i$ and that $\|x\| = 1$. Suppose that $a > \varepsilon/k$, where $k \ge 2$. Define

$$D := \{i \ge k \colon |x_i| \ge a\} \quad \text{and} \quad E := \{i < k \colon |x_i| \ge a\}.$$

Suppose, to derive a contradiction, that $|D| \ge n_k$. Choose $A \subseteq D$ with $|A| = n_k$. Then by (5.27)

$$\|P_A x\| \le (2+\varepsilon/4)\|x\| \le 2+\varepsilon/4.$$

On the other hand,

$$\begin{aligned}\|P_A x\| &= \|\sum_{i\in A} e_i^*(x) e_i\| \\ &\ge \frac{1}{2}|\sum_{i=1}^{|A|} e_i^*(x) s_i|\end{aligned}$$

(by (5.26))

$$\ge \frac{1}{4}\min\{|e_i^*(x)| \colon i \in A\} g(|A|)$$

(by 2-unconditionality of (s_i))

$$\ge \frac{1}{4} a g(|A|) > \frac{\varepsilon g(n_k)}{8k} > 3,$$

which is the desired contradiction. Hence $|D| < n_k$; in particular,

$$\|P_D x\| \le (2+\varepsilon/4)\|x\| = 2+\varepsilon/4.$$

First suppose $a > \varepsilon/2$, i.e. $k = 2$. Then, since the basis constant of $e_i)$ is at most $1+\varepsilon/4$,

$$\|\mathcal{G}_a x\| \le |a_1| + \|P_D x\| \le (1+\varepsilon/4) + (2+\varepsilon/4) < 3+\varepsilon. \tag{5.28}$$

Now suppose that $a \le \varepsilon/2$. Choose $k \ge 2$ such that

$$1/k < 2a/\varepsilon \le 1/(k-1).$$

By the triangle inequality,

$$\begin{aligned}\|P_E x\| &\le \|\sum_{i=1}^{k-1} e_i^*(x)e_i\| + (k-1)a \\ &\le (1+\varepsilon/4)\|x\| + \varepsilon/2 = 1 + 3\varepsilon/4.\end{aligned}$$

Combining, we get

$$\|\mathcal{G}_a x\| \le \|P_D x\| + \|P_E x\| \le (2+\varepsilon/4) + (1+3\varepsilon/4) = 3+\varepsilon. \tag{5.29}$$

By homogeneity, (5.28) and (5.29) prove that $\theta(a) \le 3+\varepsilon$ for all $a \in (0,1]$, i.e. that (e_n) is quasi-greedy with quasi-greedy constant $K \le 3+\varepsilon$. □

Theorem 5.5. *Let (e_n) be a semi-normalized democratic weakly null basic sequence in a Banach space. Then (e_n) has an almost greedy subsequence.*

Proof. Let $(\varphi(n))$ be the fundamental function. We may suppose that (e_i) has spreading model (s_i). Suppose that (s_i) is not equivalent to the unit vector basis of c_0. Then, since (s_i) is unconditional, $|\sum_{i=1}^n s_i| \to \infty$ as $n \to \infty$. Theorem 5.4 now show that (e_i) has a quasi-greedy, and hence almost greedy, subsequence.

On the other hand, suppose that (s_i) is equivalent to the unit vector basis of c_0. Then $(\varphi(n))$ is bounded above by K', say. Thus, by (2.10)

$$\|\sum_{i=1}^n a_i e_i\| \le 2\varphi(n) \max |a_i| \le 2K' \max |a_i|.$$

Thus, (e_n) is equivalent to the unit vector basis of c_0, which is greedy. □

Corollary 5.6. *Suppose that X is a Banach space which does not have c_0 as a spreading model (e.g., if X has finite cotype (see* (3.16) *above)). Then every semi-normalized weakly null sequence in X has an almost greedy subsequence.*

Corollary 5.7. *Let X be a Banach space. Then the following are equivalent:*

(i) *X contains a weakly null sequence with spreading model not equivalent to the unit vector basis of c_0 or X contains an isomorphic copy of c_0 or ℓ_1.*

(ii) *X contains an almost greedy basic sequence.*

(iii) *X contains a semi-greedy basic sequence.*

(iv) *X contains a superdemocratic basic sequence.*

Proof. (i) $\Rightarrow$ (ii) is immediate from Theorem 5.5, (ii) $\Rightarrow$ (iii) is Theorem 3.2, and (iii) $\Rightarrow$ (iv) is Proposition 3.3. Suppose that (iv) holds. Let (x_n) be a superdemocratic basic sequence. If X does not contain ℓ_1, then by Rosenthal's ℓ_1 theorem [18], we may assume by passing to a subsequence, setting $y_n := x_{2n} - x_{2n-1}$, that $(y_n)_{n=1}^{\infty}$ is a weakly null basic sequence with spreading model (s_n). Clearly, (y_n) is also superdemocratic, and therefore democratic. If X does not contain c_0, then from the proof of Theorem 5.5 we see that (s_n) is not equivalent to the unit vector basis of c_0. Thus, (i) holds. $\square$

Remark 5.8. The 'original' Tsirelson space [20] does not contain a subspace isomorphic to c_0 or to ℓ_1 and yet all of its spreading models (s_i) are equivalent to the unit vector basis of c_0. Thus, this space does not contain any democratic basic sequence.

6. Existence of democratic bases

Let X be a Banach space with a basis (b_n). By passing to an equivalent norm we may assume that (b_n) is normalized and bimonotone. Let S be a 1-symmetric and 1-unconditional symmetric sequence space with Schauder basis (e_i) (here e_i denotes the sequence $(\delta_{ij})_{j=1}^{\infty}$). Let (e_i^*) be the sequence of biorthogonal functionals in S^*. Define

$$f(n) := \|e_1 + \cdots + e_n\|_S$$

and

$$g(n) := n/f(n) = \|e_1^* + \cdots + e_n^*\|_{S^*}.$$

We shall assume that (e_i) is not equivalent to the unit vector basis of c_0. Thus,

$$f(n) \uparrow \infty \quad \text{as} \quad n \to \infty.$$

For $n \geq 1$, let $\sigma_n = [2^{n-1}, 2^n - 1]$, so that $|\sigma_n| = 2^{n-1}$. Let

$$v_n := \frac{1}{f(2^{n-1})} \sum_{k \in \sigma_n} e_k \quad \text{and} \quad v_n^* := \frac{1}{g(2^{n-1})} \sum_{k \in \sigma_n} e_k^*.$$

Let P be the norm-one projection on S defined by

$$P\xi := \sum_{n=1}^{\infty} \langle \xi, v_n^* \rangle v_n,$$

and let $Q = I - P$.

Define a norm on c_{00} by

$$\|\xi\|_Y := \|Q\xi\|_S + \|\sum_{n=1}^{\infty}\langle \xi, v_n^*\rangle b_n\|_X,$$

and then complete to obtain a sequence space Y.

Proposition 6.1. *Suppose that (b_n) is a bimonotone basis for X. Then (e_n) is a Schauder basis for Y such that*

$$\frac{1}{8}\sup_n \eta_n f(n) \leq \|\xi\|_Y \leq 6\sum_{n=1}^{\infty}\frac{f(n)}{n}\eta_n \tag{6.30}$$

for all real scalars $\xi = (\xi_n)$ in c_{00}, where (η_i) is the nonincreasing rearrangement of $(|\xi_i|)$.

Proof. It is easy to check that the spaces $(F_n) = [e_k : \ k \in \sigma_n]$ form a Schauder decomposition for Y and that each is 3-isomorphic to $[e_k : k \in \sigma_n]$ considered as a subspace of S. Hence $\{e_k : k \in \sigma_n\}$ forms a basis of each F_n with uniformly bounded basis constant. Thus (e_n) is a basis of Y. For the upper estimate, note that

$$\begin{aligned}\|\sum_{n=1}^{\infty}\langle \xi, v_n^*\rangle b_n\|_X &\leq \sum_{n=1}^{\infty}\frac{1}{g(2^{n-1})}\sum_{k\in\sigma_n}|\xi_k| \\ &\leq 2\sum_{n=1}^{\infty}\frac{f(n)}{n}\eta_n.\end{aligned}$$

Also

$$\begin{aligned}\|Q\xi\|_S &\leq 2\|\xi\|_S \\ &\leq 2\sum_{n=1}^{\infty}(\eta_n - \eta_{n+1})f(n)\end{aligned}$$

(by partial summation)

$$\begin{aligned}&= 2\eta_1 f(1) + 2\sum_{n=2}^{\infty}\eta_n(f(n) - f(n-1)) \\ &\leq 4\sum_{n=1}^{\infty}\frac{f(n)}{n}\eta_n\end{aligned}$$

since $f(n) - f(n-1) \leq f(n-1)/(n-1) \leq 2f(n)/n$. Combining, we get

$$\|\xi\|_Y = \|Q\xi\|_S + \|\sum_{n=1}^{\infty}\langle \xi, v_n^*\rangle b_n\|_X \leq 6\sum_{n=1}^{\infty}\frac{f(n)}{n}\eta_n.$$

For the lower estimate, suppose that $\|\xi\|_Y = 1$. For $a > 0$, define

$$A := \{k : |\xi_k| > a\} \quad \text{and} \quad B := \{n : \ |\langle \xi, v_n^* \rangle| < \frac{1}{2} f(2^{n-1})a\}.$$

We assume that $|A| = N \geq 1$. Then

$$1 = \|\xi\|_Y \geq \|Q\xi\|_S \geq \frac{a}{2} f(\sum_{n \in B} |A \cap \sigma_n|). \tag{6.31}$$

Let $D := \mathbb{N} \setminus B$. Then

$$1 = \|\xi\|_Y \geq \| \sum_{n=1}^{\infty} \langle \xi, v_n^* \rangle b_n \|_X \geq \frac{1}{2} \max_{n \in D} f(2^{n-1})a, \tag{6.32}$$

by the bimonotonicity of (b_n). (6.31), (6.32), and the fact that A is nonempty imply that there exists a largest positive integer m such that $f(2^{m-1})a \leq 2$. Moreover, (6.32) implies that $D \subset \{1, 2, \cdots, m\}$. Hence

$$\sum_{n \in D} |A \cap \sigma_n| \leq 2^m.$$

From this and (6.31) we deduce that

$$\frac{a}{2} f(N - 2^m) \leq \frac{a}{2} f(\sum_{n \in B} |A \cap \sigma_n|) \leq 1.$$

Hence $N \leq 2^{m+1}$ from the choice of m. Thus,

$$af(|\{k : |\xi_k| > a\}|) \leq af(2^{m+1}) \leq 4af(2^{m-1}) \leq 8.$$

By homogeneity, we get

$$\|\xi\|_Y \geq \frac{1}{8} \sup_{a>0} af(|\{k : |\xi_k| > a\}|) = \frac{1}{8} \sup_n \eta_n f(n)$$

for all ξ. □

Theorem 6.2. *Suppose that X is a Banach space with a basis which contains a complemented subspace isomorphic to S. Then X has a basis (e_n) satisfying*

$$\frac{1}{C} \sup_n \eta_n f(n) \leq \| \sum_{n=1}^{\infty} \xi_n e_n \|_Y \leq C \sum_{n=1}^{\infty} \frac{f(n)}{n} \eta_n \tag{6.33}$$

for some constant C.

Proof. We may suppose that X has a bimonotone basis (b_n). The space Y constructed above is isomorphic to $X \oplus Q(S)$. By assumption, we have $X \sim Z \oplus S$ for some Banach space Z. Thus,

$$Y \sim X \oplus Q(S) \sim Z \oplus S \oplus Q(S).$$

By [12, p. 117] $S \sim S \oplus P(S)$. Hence

$$Y \sim Z \oplus S \oplus P(S) \oplus Q(S) \sim Z \oplus S \oplus S \sim Z \oplus S \sim X.$$

By Proposition 6.1, Y has a basis (e_n) with the required property, and hence so does X. □

To state our next corollary, let us recall the definition of the quasi-norms $\|\cdot\|_{p,1}$ and $\|\cdot\|_{p,\infty}$ of the Lorentz sequence spaces $\ell_{p,1}$ and $\ell_{p,\infty}$ $(1 \le p < \infty)$. Here (a_n^*) denotes the nonincreasing rearrangement of the sequence $(|a_n|)$.

$$\|(a_n)\|_{p,1} = \sum_{n=1}^{\infty} a_n^* n^{1/p-1} \qquad \text{and} \qquad \|(a_n)\|_{p,\infty} = \sup n^{1/p} a_n^*. \tag{6.34}$$

Corollary 6.3. *Let $1 \le p < \infty$. Suppose that X is a Banach space with a basis which contains a complemented subspace isomorphic to ℓ_p. Then X has a superdemocratic basis (e_n) satisfying*

$$\frac{1}{C}\|(a_i)\|_{p,\infty} \le \|\sum_{i=1}^{\infty} a_i e_i\| \le C\|(a_i)\|_{p,1}$$

for some constant C. In particular,

$$\|\sum_{n\in A} \pm e_n\| \sim |A|^{1/p}$$

for all choices of signs and for all finite $A \subset \mathbb{N}$.

Proof. Apply Theorem 6.2 with $S = \ell_p$, so that $f(n) = n^{1/p}$. □

Remark 6.4. Corollary 6.3 implies a theorem of Wojtaszczyk [21, Theorem 4.5] on the existence of a normalized basis of X that is q-Besselian for all $q > p$. (Recall that a basis is q-Besselian if it satisfies a lower estimate $\|\sum_{i=1}^{\infty} a_i e_i\| \ge c\|(a_i)\|_q$.)

Corollary 6.5. *Suppose that S has finite cotype. Let X be a Banach space with a basis which contains a complemented subspace isomorphic to S. Then X has a superdemocratic basis with fundamental function equivalent to $(f(n))$.*

Proof. Consider the basis (e_i) of X satisfying (6.33). Suppose that S has cotype q with constant C_q. Then by Proposition 3.5

$$f(mn) \ge (1/C_q) m^{1/q} f(n).$$

Hence, for all $A \subset \mathbb{N}$, with $|A| = n$, we have

$$\frac{f(n)}{C} \le \|\sum_{i\in A} \pm e_i\| \le C \sum_{k=1}^{n} \frac{f(k)}{k} \le C C_q \frac{f(n)}{n^{1/q}} \sum_{k=1}^{n} k^{1/q-1} \le C' f(n).$$

So (e_i) is superdemocratic with fundamental function equivalent to $(f(n))$. $\square$

7. Existence of quasi-greedy bases

We continue to use the notation introduced in Section 6. We start with the case $S = \ell_1$, which requires special treatment.

Theorem 7.1. *Suppose that (b_n) is basis for X and that $S = \ell_1$. Then (e_n) is a quasi-greedy basis for Y.*

Proof. We may assume as above that (b_n) is a normalized bimonotone basis. It is convenient to introduce the following notation. Fix $a > 0$. Then, for $x \in \mathbb{R}$, define x^a as follows:

$$x^a := \begin{cases} x & \text{for } |x| \geq a; \\ 0 & \text{for } |x| \leq a. \end{cases}$$

For $\xi = (\xi_n) \in c_{00}$, let $\xi^a = (\xi^a_n)(= \mathcal{G}_a \xi)$. Note that

$$\|\xi\|_Y = \|Q\xi\|_1 + \|\sum_{n=1}^{\infty}(\sum_{k\in\sigma_n} \xi_k)b_n\|_X \tag{7.35}$$

and

$$\|\mathcal{G}_a\xi\|_Y = \|Q\xi^a\|_1 + \|\sum_{n=1}^{\infty}(\sum_{k\in\sigma_n} \xi^a_k)b_n\|_X. \tag{7.36}$$

Suppose that $\|\xi\|_Y \leq 1$. Since (b_n) is bimonotone, we have

$$|\sum_{k\in\sigma_n} \xi_k| \leq 1 \qquad (n \geq 1).$$

Moreover, (7.35) and (7.36) clearly imply that $\mathcal{G}_a\xi = 0$ if $a > 3/2$. Hence we may assume that $a \leq 3/2$. Let N be the smallest positive integer for which $a > 2/|\sigma_N|$. Note that $N \geq 2$. Then, for $n \geq N$, we have

$$a > \frac{2}{|\sigma_n|}|\sum_{k\in\sigma_n} \xi_k|.$$

Let E_n denote the projection onto σ_n, i.e. $E_n\xi = \xi\chi_{\sigma_n}$. Hence

$$|\sum_{k\in\sigma_n} \xi^a_k| \leq 2\sum_{k\in\sigma_n}|\xi_k - \frac{1}{|\sigma_n|}\sum_{k\in\sigma_n}\xi_k| = 2\|E_nQ\xi\|_1 \qquad (n \geq N).$$

Hence, by the triangle inequality,

$$\|\sum_{n=N}^{\infty}(\sum_{k\in\sigma_n} \xi^a_k)b_k\|_X \leq 2\sum_{n=N}^{\infty}\|E_nQ\xi\|_1 \leq 2\|\xi\|_Y. \tag{7.37}$$

Also

(7.38)
$$\begin{aligned}\|\sum_{n=1}^{N-1}(\sum_{k\in\sigma_n}\xi_k^a)b_n\|_X &\le \|\sum_{n=1}^{N-1}(\sum_{k\in\sigma_n}\xi_k)b_n\|_X + \|\sum_{n=1}^{N-1}(\sum_{k\in\sigma_n}(\xi_k^a-\xi_k))b_n\|_X\\ &\le \|\xi\|_Y + (\sum_{n=1}^{N-1}|\sigma_n|)a\end{aligned}$$

(since $|\xi_k^a - \xi_k| \le a$)

$$\begin{aligned}&\le \|\xi\|_Y + 2|\sigma_{N-1}|\frac{2}{|\sigma_{N-1}|}\\ &= \|\xi\|_Y + 4.\end{aligned}$$

Combining (7.37) and (7.38), we get

$$\|\sum_{n=1}^{\infty}(\sum_{k\in\sigma_n}\xi_k^a)b_n\|_X \le 3\|\xi\|_Y + 4.$$

Similarly, for all $n \ge N$, we have $\|E_n\mathcal{G}_a\xi\|_1 \le 2\|E_nQ\xi\|_1$, whence

$$\begin{aligned}\|(\sum_{n=N}^{\infty}E_n)Q\mathcal{G}_a\xi\|_1 &\le \|Q\|\sum_{n=N}^{\infty}\|E_n\mathcal{G}_a\xi\|_1\\ &\le 4\sum_{n=N}^{\infty}\|E_nQ\xi\|_1\\ &\le 4\|\xi\|_Y.\end{aligned}\tag{7.39}$$

Also

$$\begin{aligned}\|(\sum_{n=1}^{N-1}E_n)Q\mathcal{G}_a\xi\|_1 &\le \|(\sum_{n=1}^{N-1}E_n)Q(\xi-\mathcal{G}_a\xi)\|_1 + \|(\sum_{n=1}^{N-1}E_n)Q\xi\|_1\\ &\le 2a(\sum_{n=1}^{N-1}|\sigma_n|) + \|\xi\|_Y\\ &\le 4a|\sigma_{N-1}| + \|\xi\|_Y\\ &\le 8 + \|\xi\|_Y.\end{aligned}\tag{7.40}$$

Combining (7.39) and (7.40), we get

$$\|Q\mathcal{G}_a\xi\|_1 \le 5\|\xi\|_Y + 8$$

Finally,

$$\|\mathcal{G}_a\xi\|_Y = \|Q\mathcal{G}_a\xi\|_1 + \|\sum_{n=1}^{\infty}(\sum_{k\in\sigma_n}\xi_k^a)b_n\|_X \le 8\|\xi\|_Y + 12 = 20\|\xi\|_Y.$$

Hence (e_n) is quasi-greedy with quasi-greedy constant $K \le 20$. □

Now we consider the case $S \ne \ell_1$. Here we we can prove a more general result by dropping the assumption that S is symmetric. Let us say that (e_n) is a *good* unconditional basis if if it satisfies three conditions:

(1) There is a function $f(n) \uparrow \infty$ as $n \to \infty$ such that $\|\sum_{j\in A} e_j\| \ge f(|A|)$ for all finite $A \subset \mathbb{N}$.
(2) There is a normalized block basic sequence (u_n) with biorthogonal sequence (u_n^*) such that
$$\lim_{n\to\infty}\|u_n\|_\infty = \lim_{n\to\infty}\|u_n^*\|_\infty = 0.$$
(3) The projection
$$P\xi = \sum\langle\xi, u_n^*\rangle u_n$$
is bounded on S.

Note that if S has a symmetric basis and if S is neither c_0 nor ℓ_1, then the basis is good. Both c_0 and ℓ_1 fail to satisfy condition (2). In particular, the argument of Theorem 7.2 does not work for $S = \ell_1$.

Theorem 7.2. *If S has a good unconditional basis and X has a basis then $S \oplus X$ has a quasi-greedy basis.*

Proof. We may assume that X has a bimonotone normalized basis (b_n). We suppose S is given as a 1-unconditional sequence space (no longer symmetric) and that

$$\|\sum_{j\in A} e_j\| \ge f(|A|),$$

where $f(n) \uparrow \infty$. We suppose further the existence of a normalized block basic sequence (u_n) with dual functionals (u_n^*) (also blocked on the same blocks) so that

$$\delta_n := \max(\|u_n\|_\infty, \|u_n^*\|_\infty) \to 0$$

as $n \to \infty$. We also assume that the projection $P\xi = \sum\langle\xi, u_n^*\rangle u_n$ is bounded with $\|P\| = \Lambda$. By passing to a subsequence we may suppose that there is an increasing sequence $(M_n)_{n=0}^{\infty}$, with $M_0 = 0$, such that

(i) u_n, u_n^* are supported on $\sigma_n = (M_{n-1}, M_n]$.
(ii) $f(M_n) > M_{n-1}$.

(iii) $\delta_n M_{n-1} < 1$.
(iv) $\delta_n \le \frac{1}{2}\delta_{n-1}$ for $n \ge 2$.

Let $Q = I - P$. As before we introduce Y as the completion of the norm on c_{00} defined by

$$\|\xi\|_Y = \|Q\xi\|_S + \|\sum_{n=1}^{\infty}\langle \xi, u_n^*\rangle b_n\|_X.$$

Let $E_n\xi = \xi\chi_{\sigma_n}$. Since $\|E_1 + \cdots + E_n\|_Y = 1$, it follows as before that (e_n) is a basis for Y. Let C be the basis constant of (e_n). Note that if ξ is supported on some σ_n, then

$$\|\xi\|_Y = \|\xi - \langle \xi, u_n^*\rangle u_n\|_S + |\langle \xi, u_n^*\rangle| \le \|\xi\|_S + 2\|P\xi\|_S$$

so that

$$\|\xi\|_S \le \|\xi\|_Y \le (1 + 2\Lambda)\|\xi\|_S. \tag{7.41}$$

Note that $Y \sim X \oplus Q(S)$ and hence $Y \oplus P(S) \sim X \oplus S$. Since (u_n) is an unconditional basis for $P(S)$, we need only show that (e_n) is a quasi-greedy basis for Y.

Fix ξ with $\|\xi\|_Y = (2C\Lambda)^{-1}$ Note that

$$\sup |e_i^*(\xi)| \le 1,$$

and hence $\mathcal{G}_a\xi = 0$ for $a > 1$. So we may assume that $0 < a \le 1$. Let $r \ge 0$ be chosen so that $aM_r < 1$ but $aM_{r+1} > 1$. Then, by (7.41),

$$\begin{aligned}\|(E_1 + \cdots + E_r)(\xi - \mathcal{G}_a\xi)\|_Y &\le (1 + 2\Lambda)\sum_{i=1}^{r}\|E_i(\xi - \mathcal{G}_a\xi)\|_S \\ &\le (1 + 2\Lambda)aM_r < 1 + 2\Lambda.\end{aligned}$$

Thus,

$$\|(E_1 + \cdots + E_r)(\mathcal{G}_a\xi)\|_Y \le 1 + 2\Lambda + \|\xi\|_Y. \tag{7.42}$$

Note that

$$af(M_{r+2}) > aM_{r+1} > 1 \qquad \text{by } (ii),$$

and that

$$\delta_{r+2} < 1/M_{r+1} < a \qquad \text{by } (iii).$$

Suppose that $j \ge r + 3$. Then, by (7.41),

$$|\langle \xi, u_j^*\rangle| \le \Lambda\|E_j\xi\|_S \le \Lambda\|E_j\xi\|_Y \le 2C\Lambda\|\xi\|_Y = 1.$$

Hence

$$\|E_jP\xi\|_\infty \le \|u_j\|_\infty \le \delta_j \le \frac{1}{2}\delta_{r+2} < a/2.$$

It follows that $|E_jQ\xi| \geq \frac{1}{2}|E_j\mathcal{G}_a\xi|$ (coordinatewise). Hence

$$\|(\sum_{j\geq r+3} E_j)(\mathcal{G}_a\xi)\|_S \leq 2\|Q\xi\|_S \leq 2\|\xi\|_Y.$$

So

$$\begin{aligned} \|Q(\sum_{j\geq r+3} E_j)(\mathcal{G}_a\xi)\|_S \leq& \|Q\|\|(\sum_{j\geq r+3} E_j)(\mathcal{G}_a\xi)\|_S \\ &\leq 2(\Lambda+1)\|\xi\|_Y. \end{aligned} \tag{7.43}$$

Let

$$L_j := |\{k \in \sigma_j \colon |\xi_k| \geq a\}|.$$

Then, by (7.41) and the 1-unconditionality of S,

$$af(L_j) \leq \|E_j\xi\|_S \leq \|\xi\|_Y \leq 1.$$

Hence

$$af(L_j) \leq 1 < aM_{r+1} \leq af(M_{r+2}),$$

which implies that $L_j < M_{r+2}$. Thus,

$$|\langle \mathcal{G}_a\xi, u_j^*\rangle| \leq L_j\|u_j^*\|_\infty < M_{r+2}\delta_j.$$

Thus, if $j \geq r+3$, we have, by (iii) and (iv),

$$|\langle \mathcal{G}_a\xi, u_j^*\rangle| \leq M_{r+2}\delta_{r+3}2^{r+3-j} < 2^{r+3-j}.$$

Together with the triangle inequality, this gives

$$\|\sum_{j=r+3}^{\infty} \langle \mathcal{G}_a\xi, u_j^*\rangle b_j\|_X \leq 2. \tag{7.44}$$

Combining (7.43) and (7.44), we get

$$\|(\sum_{j\geq r+3} E_j)(\mathcal{G}_a\xi)\|_Y \leq 2(1+\Lambda)\|\xi\|_Y + 2. \tag{7.45}$$

We are left to estimate $\|E_j(\mathcal{G}_a\xi)\|_Y$ when $j = r+1, r+2$. But then by (7.41)

$$\begin{aligned} \|E_j(\mathcal{G}_a\xi)\|_Y &\leq (1+2\Lambda)\|E_j(\mathcal{G}_a\xi)\|_S \\ &\leq (1+2\Lambda)\|E_j\xi\|_S \leq (1+2\Lambda)\|\xi\|_Y. \end{aligned} \tag{7.46}$$

Combining (7.42), (7.45), and (7.46) gives

$$\|\mathcal{G}_a\xi\|_Y \leq 3 + 2\Lambda + (5+6\Lambda)\|\xi\|_Y = (6C\Lambda + 4C\Lambda^2 + 5 + 6\Lambda)\|\xi\|_Y.$$

Hence (e_n) is quasi-greedy with quasi-greedy constant $K \leq 6C\Lambda + 4C\Lambda^2 + 5 + 6\Lambda$. □

Corollary 7.3. *Suppose that X has a basis and contains a complemented subspace S with a symmetric basis, where S is not isomorphic to c_0. Then X has a quasi-greedy basis.*

Proof. The case $S = \ell_1$ is covered by Theorem 7.1 If $S \neq \ell_1$ then the basis of S is good. Hence $X \oplus S$ has a quasi-greedy basis. But $X \sim X \oplus S$. □

Theorem 7.4. *Suppose that X has a basis and contains a complemented subspace with a symmetric basis and finite cotype. Then X has an almost greedy basis.*

Proof. Combining Corollary 6.5 and Theorem 7.2 (or Theorem 7.1 for the case $S = \ell_1$) yields a basis that is simultaeously democratic and quasi-greedy. □

Remark 7.5. Non-commutative L_p spaces do not have an unconditional basis. However, under reasonable assumptions, it is proved in [9] that for $1 < p < \infty$ they have a basis (see also [10] for more examples associated with groups). Since non-commutative L_p spaces contain complemented copies of ℓ_p, we can apply Theorem 7.4 to obtain the existence of almost greedy bases in these spaces.

Remark 7.6. It is clear that if (b_n) and (b'_n) are *inequivalent* bases for X, then the corresponding almost greedy bases (e_n) and (e'_n) produced by the construction will be inequivalent. It is known that every Banach space with a basis has infinitely many inequivalent normalized conditional bases [16]. Hence, if X contains a complemented copy of ℓ_p for some $1 \leq p < \infty$, then X has infinitely many inequivalent almost greedy bases. This yields another proof of the existence of (infinitely many inequivalent) conditional quasi-greedy bases in ℓ_1 or ℓ_2 [4, 22].

8. Quasi-greedy bases in $\mathcal{L}_\infty$ spaces

This section makes heavy use of a theorem of Grothendieck and related results. So we begin by recalling these important facts.

Let $T : X \to Y$ be a continuous linear operator between Banach spaces X and Y. Then T is called *absolutely summing* if there exists a constant C such that for all sequence (x_n) in X, we have

$$(8.47) \qquad \sum_{n=1}^{\infty} \|T(x_n)\| \leq C \sup\{\sum_{n=1}^{\infty} |x^*(x_n)| \colon x^* \in X^*, \|x^*\| \leq 1\}.$$

(Note that the right-hand side of (8.47) equals $C \sup_{\varepsilon_n = \pm 1} \| \sum_{n=1}^{\infty} \varepsilon_n x_n \|$.) The least such constant C is denoted $\pi_1(T)$. A Banach space X is called a *GT space* [17] if every bounded linear operator from X to any Hilbert space H is absolutely summing. Grothendieck [7] proved that $L_1(\mu)$ spaces are GT spaces.

Recall that X is a $\mathcal{L}_\infty$ space if there exists $\lambda \geq 1$ and a directed net (F_α) of finite-dimensional subspaces of X, where each F_α is λ-isomorphic to an ℓ_∞^n space, such that $X = \overline{\cup_\alpha X_\alpha}$. This class includes every complemented subspace of a $C(K)$ space. If X is a $\mathcal{L}_\infty$ space then X^* is a GT space (see [17]). Bourgain [2] proved that the dual of the disc algebra is a GT space.

Let (e_n) be a basis for X. We say that (e_n) satisfies condition M_p $(1 \leq p \leq \infty)$ if there exists a constant C_p such that

$$\|(x^*(e_n))\|_p \leq C_p\|x^*\| \qquad (x^* \in X^*).$$

Note that M_p holds if and only if (e_n) is q-Hilbertian, i.e., it satisfies the upper q-estimate $(1/p + 1/q = 1)$

$$\|x\| \leq \frac{1}{C_p}\|(e_n^*(x))\|_q \qquad (x \in X).$$

The basis (e_n) is called *Hilbertian* if it satisfies M_2. We note that $C[0,1]$ has a Hilbertian basis [21].

Proposition 8.1. *Suppose that (e_n) is a semi-normalized thresholding-bounded Hilbertian basis for a Banach space X. If X^* is a GT space, then (e_n) is equivalent to the unit vector basis of c_0.*

Proof. We show that there exists a constant C such that

$$\sum_{n=1}^\infty |a_n| \leq C\|\sum_{n=1}^\infty a_n e_n^*\|,$$

which is equivalent to the result. We may assume without loss of generality that the Hilbertian constant C_2 equals one. Since X^* is a GT space, every bounded linear operator T from X^* to ℓ_2 is absolutely summing with $\pi_1(T) \leq B\|T\|$ for some absolute constant B. Since (e_n) is Hilbertian, the map $X^* \to \ell_2$ given by $x^* \mapsto (x^*(e_n))$ is bounded, with operator norm at most $C_2 = 1$, and hence absolutely summing. Thus,

$$\sum_{n=1}^\infty |x^*(e_n)| \leq B \sup_{\varepsilon_n=\pm 1} \|\sum_{n=1}^\infty \varepsilon_n x^*(e_n)e_n^*\| \quad (x^* \in X^*).$$

Fix $x^* = \sum_{n=1}^\infty a_n e_n^* \in [e_n^*]$. Choose signs (ε_n) such that

$$\|\sum_{n=1}^\infty a_n\varepsilon_n e_n^*\| \geq \frac{1}{B}\sum_{n=1}^\infty |a_n|.$$

Choose $x \in X$, with $\|x\| = 1$, such that

$$\sum_{n=1}^{\infty} \varepsilon_n a_n e_n^*(x) > \frac{1}{2B} \sum_{n \in A} |a_n|.$$

Let $\sigma = \{n \in \mathbb{N} \colon |e_n^*(x)| > (4B)^{-1}\}$. Clearly,

$$\sum_{n \in \sigma} \varepsilon_n a_n e_n^*(x) > \frac{1}{4B} \sum_{n=1}^{\infty} |a_n|. \tag{8.48}$$

Also,

$$\|\sum_{n \in \sigma} \varepsilon_n e_n^*(x) e_n\| \leq 2\theta(1)^2 \|\sum_{n \in \sigma} (\operatorname{sgn} e_n^*(x)) e_n\|$$

(by Proposition 4.2 since $\max |e_n^*(x)| \leq \theta(1)$)

$$= 2\theta(1)^2 (4B) (\frac{1}{4B} \|\sum_{n \in \sigma} (\operatorname{sgn} e_n^*(x)) e_n\|)$$

$$\leq 8\theta(1)^2 B (1 + \theta((4B\theta(1))^{-1}) \|\sum_{n \in \sigma} e_n^*(x) e_n\|$$

(by Proposition 4.4 since $(4B)^{-1} \leq |e_n^*(x)| \leq \theta(1)$ for $n \in \sigma$)

$$\leq 8\theta(1)^2 B (1 + \theta((4B\theta(1))^{-1}) \theta((4B\theta(1))^{-1}) \|x\|$$

(since $\sum_{n \in \sigma} e_n^*(x) e_n = \theta(1) \mathcal{G}_a(\theta(1)^{-1} x)$ for $a = (4B\theta(1))^{-1}$)

$$= 8\theta(1)^2 B (1 + \theta((4B\theta(1))^{-1}) \theta((4B\theta(1))^{-1})).$$

From (8.48), we get

$$\begin{aligned} \sum_{n=1}^{\infty} |a_n| &< 4B \sum_{n \in \sigma} \varepsilon_n a_n e_n^*(x) \\ &\leq 4B \|\sum_{n=1}^{\infty} a_n e_n^*\| \|\sum_{n \in \sigma} \varepsilon_n e_n^*(x) e_n\| \\ &\leq C \|\sum_{n=1}^{\infty} a_n e_n^*(x)\|, \end{aligned}$$

where $C = 32\theta(1)^2 B^2 (1 + \theta((4B\theta(1))^{-1}) \theta((4B\theta(1))^{-1})$. □

Lemma 8.2. *Suppose that (e_n) is a normalized thresholding-bounded basis for X. Let $\sigma \subset \mathbb{N}$ with $|\sigma| = N \geq 2$. Then, for every choice of* *signs* $(\varepsilon)_{n\in\sigma}$, *we have*

$$\|\sum_{n\in\sigma}\varepsilon_n e_n^*(x)e_n\| \le C(\log_2 N)\theta(1/N)\|x\| \qquad (x\in X),$$

where $C = 2 + 8\theta(1)^2(1+\theta(1/2))$.

Proof. Suppose that $\|x\| = 1/\theta(1)$, so that $x \in Q$ by (i) of Proposition 4.1. For $k \ge 0$, let

$$\tau_k = \{n \in \mathbb{N}\colon 2^{-k} \le |e_n^*(x)| < 2^{1-k}\}.$$

Then, for $k > [\log_2 N]$ and $n \in \tau_k$, we have $|e_n^*(x)| \le 2/N$. Hence

$$\|\sum_{k>[\log_2 N]}(\sum_{n\in\sigma\cap\tau_k}\varepsilon_n e_n^*(x)e_n)\| < \frac{2}{N}|\sigma| = 2\theta(1)\|x\|. \tag{8.49}$$

For $k \le [\log_2 N]$, we have

$$\|\sum_{n\in\sigma\cap\tau_k}\varepsilon_n e_n^*(x)e_n\| \le 2\theta(1)\|\sum_{n\in\sigma\cap\tau_k}2^{1-k}(\operatorname{sgn} e_n^*(x))e_n\|$$

(by Lemma 4.2)

$$\begin{aligned}
&\le 2\theta(1)^2\|\sum_{n\in\tau_k}2^{1-k}(\operatorname{sgn} e_n^*(x))e_n\| \\
&= 4\theta(1)^2\|\sum_{n\in\tau_k}2^{-k}(\operatorname{sgn} e_n^*(x))e_n\| \\
&\le 4\theta(1)^2(1+\theta(1/2))\|\sum_{n\in\tau_k}e_n^*(x)e_n\|
\end{aligned}$$

(by Lemma 4.4)

$$\begin{aligned}
&= 4\theta(1)^2(1+\theta(1/2))\|\mathcal{G}_{2^{-k}}x - \mathcal{G}_{2^{1-k}}x\| \\
&\le 8\theta(1)^2(1+\theta(1/2))\theta(2^{-k})\|x\|.
\end{aligned}$$

Hence

$$\sum_{k=1}^{[\log_2 N]}\|\sum_{n\in\tau_k\cap\sigma}\varepsilon_n e_n^*(x)\| \le 8\theta(1)^2(1+\theta(1/2))(\log_2 N)\theta(1/N)\|x\|. \tag{8.50}$$

Combining (8.49) and (8.50), we get

$$\|\sum_{n\in\sigma}\varepsilon_n e_n^*(x)e_n\| \le C(\log_2 N)\theta(1/N)\|x\|,$$

where $C = 2 + 8\theta(1)^2(1+\theta(1/2))$. ∎

Remark 8.3. By duality, we also have

$$\|\sum_{n\in\sigma}\varepsilon_n x^*(e_n)e_n^*\| \le C(\log_2 N)\theta(1/N)\|x\| \qquad (x^*\in X^*).$$

Lemma 8.4. *Suppose that* (e_n) *is a normalized thresholding-bounded basis for a Banach space* X *which satisfies* $\theta(a)\le Ca^{-\varepsilon}$ $(0<a\le 1)$*, where* C *and* $\varepsilon\in(0,1/2)$ *are positive constants. Suppose that* X^* *is a GT space and that* M_p *holds for some* $p>2$*. Then* M_r *holds whenever*

$$\frac{1}{r}<\frac{1}{p}+\frac{1}{2}-\varepsilon.$$

Proof. Let $1/s = 1/p+1/2$. Suppose that $\sigma\subset\mathbb{N}$, with $|\sigma|=N$, and that $(\eta_n)_{n\in\sigma}$ is any fixed choice of signs. Choose $x^*\in X^*$, with $\|x^*\|=1$, such that

$$x^*(\sum_{n\in\sigma}\eta_n e_n) = \|\sum_{n\in\sigma}\eta_n e_n\|.$$

Next consider $T\colon X^*\to\ell_2(\sigma)$ defined as follows:

$$Ty^* = (y^*(e_n)|x^*(e_n)|^{s-1})_{n\in\sigma} \qquad (y^*\in X^*).$$

Then, applying Hölder's inequality and using condition M_p, we get

$$\begin{aligned}\|Ty^*\| &= (\sum_{n\in\sigma}|x^*(e_n)|^{2s-2}|y^*(e_n)|^2)^{1/2}\\ &\le (\sum_{n\in\sigma}|x^*(e_n)|^s)^{1/2-1/p}(\sum_{n\in\sigma}|y^*(e_n)|^p)^{1/p}\\ &\le C_p(\sum_{n\in\sigma}|x^*(e_n)|^s)^{1/2-1/p}\|y^*\|.\end{aligned}$$

Hence $\|T\|\le C_p(\sum_{n\in\sigma}|x^*(e_n)|^s)^{1/2-1/p}$. Since X^* is a GT space, we have

$$\begin{aligned}\sum_{n\in\sigma}|x^*(e_n)|^s &= \sum_{n=1}^{N}|x^*(e_n)|\|Te_n^*\|\\ &\le B\|T\|\sup_{\varepsilon_n=\pm1}\|\sum_{n\in\sigma}\varepsilon_n x^*(e_n)e_n^*\|\\ &\le BC_p(\sum_{n\in\sigma}x^*(e_n)^s)^{1/2-1/p}\sup_{\varepsilon_n=\pm1}\|\sum_{n\in\sigma}\varepsilon_n x^*(e_n)e_n^*\|.\end{aligned}$$

Thus,

$$(\sum_{n\in\sigma}|x^*(e_n)|^s)^{1/s}\le BC_p\sup_{\varepsilon_n=\pm1}\|\sum_{n\in\sigma}\varepsilon_n x^*(e_n)e_n^*\|.$$

Since $|\sigma| = N$, Remark 8.3 gives

$$\sup_{\varepsilon_n=\pm 1} \|\sum_{n\in\sigma} \varepsilon_n x^*(e_n) e_n^*\| \le C'(\log_2 N)\theta(1/N)\|x^*\| = C'(\log_2 N)\theta(1/N),$$

where C' is independent of N. Hence

$$(\sum_{n\in\sigma} |x^*(e_n)|^s)^{1/s} \le BC'C_p(\log_2 N)\theta(1/N).$$

Thus,

$$\begin{aligned}\|\sum_{n\in\sigma} \eta_n e_n\| &= \sum_{n\in\sigma} \eta_n x^*(e_n)\\ &\le (\sum_{n\in\sigma} |x^*(e_n)|^s)^{1/s} N^{1-1/s}\\ &\le BC'C_p(\log_2 N)\theta(1/N) N^{1-1/s}.\end{aligned}$$

Now suppose that $y^* \in X^*$ with $\|y^*\| = 1$. For $a > 0$, let

$$\sigma(a) = \{n \colon |y^*(e_n)| \ge a\} \quad \text{and} \quad N(a) = |\sigma(a)|.$$

Then, for some choice of signs (η_n), we have

$$\begin{aligned}aN(a) &\le y^*(\sum_{n\in\sigma_a} \eta_n e_n)\\ &\le \|\sum_{n\in\sigma_a} \eta_n e_n\|\\ &\le BC'C_p(\log_2 N(a))\theta(1/N(a))N(a)^{1-1/s}\\ &\le BCC'C_p(\log_2 N(a))N(a)^{1-1/s+\varepsilon},\end{aligned}$$

using the hypothesis that $\theta(a) \le Ca^{-\varepsilon}$. Thus, for some constant C'', we have $N(a) \le C''a^{-t}$ provided t satisfies

$$\frac{1}{r} < \frac{1}{t} < \frac{1}{s} - \varepsilon.$$

This implies, by a standard calculation, that (e_n) satisfies M_r. $\square$

Theorem 8.5. *Let (e_n) be a semi-normalized thresholding-bounded basis for X satisfying $\theta(a) \le Ca^{-\varepsilon}$, where $0 < \varepsilon < 1/2$. If X^* is a GT space, then (e_n) is equivalent to the unit vector basis of c_0.*

Proof. For simplicity, we assume that the basis is normalized (the semi-normalized case is similar). Note that (e_n) satisfies M_∞. By applying Lemma 8.4 a total of $[(1-2\varepsilon)^{-1}] + 1$ times, starting with $p = \infty$, we finally obtain that (e_n) satisfies M_2. Now apply Proposition 8.1. $\square$

Corollary 8.6. *c_0 is the unique infinite-dimensional $\mathcal{L}_\infty$ space, up to isomorphism, with a quasi-greedy basis. Moreover, c_0 has a unique quasi-greedy basis up to equivalence.*

Remark 8.7. Szarek [19] proved that every Schauder basis of an infinite-dimensional $\mathcal{L}_\infty$ space contains a subsequence equivalent to the unit vector basis of c_0. Thus, c_0 is the only infinite-dimensional $\mathcal{L}_\infty$ space, up to isomorphism, with a super-democratic or a semi-greedy (by Proposition 3.3) basis.

Corollary 8.8. *The disc algebra (regarded as a real Banach space) does not have a quasi-greedy basis.*

Corollary 8.9. *Let X be an infinite-dimensional Banach space. Then X has a unique normalized (or unique semi-normalized) quasi-greedy basis up to equivalence if and only if X is isomorphic to c_0.*

Proof. The fact that c_0 has a unique quasi-greedy basis is the second assertion of Corollary 8.6. Suppose that X is a Banach space with a unique (up to equivalence) normalized (respectively, semi-normalized) quasi-greedy basis (e_n). Let (ε_n) be any choice of signs. Then $(\varepsilon_n e_n)$ is also quasi-greedy and normalized (respectively, semi-normalized). So, by uniqueness, $(\varepsilon_n e_n)$ is equivalent to (e_n), and so (e_n) is unconditional. In particular, X has a unique unconditional basis up to equivalence. It follows from the Lindenstrauss-Zippin theorem [13] that X is isomorphic to c_0, ℓ_1 or ℓ_2. But ℓ_1 [4] and ℓ_2 [22] have conditional quasi-greedy bases. Thus X is isomorphic to c_0. □

Acknowledgement. The authors thank Ted Odell for helpful discussions and several useful references.

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