Why Calculus Works

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1 Review of basic facts about $\mathbb{R}$

Here are some things to recall from earlier sections: The set of real numbers is denoted by $\mathbb{R}$. The set of rational numbers, denoted by $\mathbb{Q}$, is the set of all numbers that can be represented as a fraction $\frac{m}{n}$ where m, n are integers with $n \neq 0$. The set of irrational numbers, denoted $\mathbb{P}$, is $\mathbb{P} = \mathbb{R} - \mathbb{Q}$. There are irrational numbers, e.g., $\sqrt{2}$.

EXERCISE: Be careful about arithmetic with irrational numbers. Above all, don't divide by zero. Prove or disprove the following statements:

a) if $x \in \mathbb{Q}$ and $y \in \mathbb{P}$ then $x + y \in \mathbb{P}$
b) if $x \in \mathbb{Q}$ and $y \in \mathbb{P}$ then $xy \in \mathbb{P}$
c) if $x, y \in \mathbb{P}$ then $x + y \in \mathbb{P}$
d) if $x, y \in \mathbb{P}$ then $xy \in \mathbb{P}$.

Proposition 1.1 *A number is rational if and only if it has a decimal representation that is either terminating (= finite) or repeating.*

Warning: In Proposition 1.1, the word "repeating" means that a given block of the decimal representation repeats exactly. Unless you can actually see a string of digits in a number's decimal representation that repeats forever, you can't tell whether the decimal representation for that number is a repeating decimal. It is not enough to look at a number's first billion decimal digits because maybe the repetition begins in the billon and first digit. It took a very long time – until the 1800s – to decide whether famous numbers like π and e are rational. (They aren't.)

Rather than prove Proposition 1.1, we will give two examples. First consider the real number $x = 0.143143143143\cdots$. Then $1000x = 143.143143143\cdots$ so that $1000x - x = 143$ showing that $999x = 143$ and therefore $x = \frac{143}{999}$. Next, try to work out the decimal representation of $\frac{2}{11}$ by long division. We will do this in class.

a) We start asking how many times 11 goes into 20, and the answer is once, with a remainder of 9.

b) Next, we ask how many times does 11 go into 90 and the answer is eight times, with a remainder of 2.

c) Next we ask how many times 11 goes into 20, and we realize that we have asked that question before so we will get a repetition of the same answer.

d) Thus we see that $\frac{2}{11} = 0.14141414....$

Exercise: What rational number is $0.99999\cdots$?

Proposition 1.1 allows us to write down a number that is guaranteed to be irrational. Look at $x = 0.101001000100001000001\cdots$. The pattern of that number is clear, and it does not have a fixed block of digits that repeats over and over. Consequently, the number is not rational.

You learned to do addition, subtraction, and multiplication computations with terminating decimals in grade school. Computations with infinite decimals is trickier. For example, in base 10 notation, let $x = 0.8888888\cdots$ and $y = 0.55555555\cdots$ and you see that the carrying problems in the addition $x + y$ are complicated.

Proposition 1.2 *The set $\mathbb{Q}$ is countably infinite, i.e., there is a surjective and injective function f from the set of positive integers to the set $\mathbb{Q}$. Therefore it is possible to list $\mathbb{Q}$ as $\mathbb{Q} = \{q_n : n \in \mathbb{N}\}$.*

We also studied Cantor's diagonalization proof of the following:

Proposition 1.3 *The set of all real numbers is not countably infinite. Therefore it is not possible to list $\mathbb{R}$ as $\mathbb{R} = \{r_n : n \in \mathbb{N}\}$.*

From high school you know about the absolute value function defined as $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$. We use the absolute value to measure distance in $\mathbb{R}$. The distance between two real numbers x, y is $d(x, y) = |x - y|$. A crucial property of the absolute value function $|\cdot|$ is given in the next Lemma. It is called the triangle inequality.

Lemma 1.4 *For any real numbers x and y, $|x + y| \leq |x| + |y|$. Therefore, for any three real numbers a, b, c we have $|a - c| \leq |a - b| + |b - c|$.*

Proof: For any real number x, we have $x \leq |x|$. Now consider $|x + y|$. In case $x + y \geq 0$, then $|x+y| = x+y \leq |x|+|y|$ because $x \leq |x|$ and $y \leq |y|$. In case $x+y < 0$, then $|x+y| = -(x+y) = (-x)+(-y) \leq |-x|+|-y| = |x|+|y|$ because $-x \leq |-x| = |x|$ and $-y \leq |y|$. To prove the assertion about the numbers a, b and c, let $x = a - b$ and $y = b - c$. Then $|(a-b)+(b-c)| \leq |a-b|+|b-c|$, so that $|a - c| \leq |a - b| + |b - c|$ as claimed. □

The last part of the proof of Lemma 1.4 could be re-written as $|a - c| = |(a - b) + (b - c)| \leq |a-b|+|b-c|$ and now it illustrates the well-known mathematical trick called "adding zero". Some colleagues in other departments say that adding zero and multiplying by one are the only two things that mathematicians do well.

2 Special Properties of $\mathbb{R}$

In this section we introduce two properties of the set of real numbers (namely, the Least Upper Bound Axiom and Archimedes' property) that make $\mathbb{R}$ a very special set in which calculus is possible. The most important of the two is the Least Upper Bound Axiom. But first, some definitions.

Definition 2.1 *A subset $S \subseteq \mathbb{R}$ is bounded above if there is some number $b \in \mathbb{R}$ with $s \leq b$ for every $s \in S$, and then the number b is called an upper bound for S.*

Obviously, if b is an upper bound for S, then so is every number larger than b, so any set that is bounded above has many upper bounds.

EXERCISE: Given a useful negation of the statement "The set E is bounded above". (Do not reply "E is not bounded above" or "It is not true that E is bounded above." State exactly what must be true if E is not bounded.)

EXERCISE: Show that the empty set is bounded above. Then show that the empty set has no least upper bound. [Hint: Suppose 3 is not an upper bound for the empty set. Then....]

Definition 2.2 Least Upper Bound Axiom: *(LUB Axiom) If E is a nonempty subset of $\mathbb{R}$ that is bounded above, then E has a least upper bound, i.e., there is a number $L \in \mathbb{R}$ such that*

a) L is an upper bound for E, and

b) if $x < L$ then x is not an upper bound for E because some $e \in E$ has $x < e$.

If a) and b) hold, we write $L = \text{lub(E)}$ or $L = \sup(E)$.

Warning: There is another term that you know from earlier courses, namely $\max(E)$ where $E \subseteq \mathbb{R}$, by which one means the largest number in E. It is very dangerous to use that term in mathematics courses and we will do so rarely, because for many bounded sets $E \subseteq \mathbb{R}$ the term $\max(E)$ is undefined. For example, if E is the set $(0, 1)$, what is $\max(E)$? Or if $E = \{x \in \mathbb{Q} : x^2 < 2\}$, what is $\max(E)$?

Why is the LUB Axiom called an axiom? The reason is that it cannot be proved from the more usual algebraic and ordering properties of $\mathbb{R}$. How can one say that it cannot be proved? The reason is that the usual set $\mathbb{Q}$ of rational numbers satisfies the same algebraic laws as does the set $\mathbb{R}$ and, as we will show later, the LUB Axiom is not true for $\mathbb{Q}$.

If we were truly committed to showing that mathematics can be built up using only set theory, we would need to construct the real numbers and show that the set $\mathbb{R}$ satisfies the LUB axiom. This can be done, but we simply do not have time.

************Henceforth we will assume that $\mathbb{R}$ satisfies the LUB axiom.************

There is a second crucial property of $\mathbb{R}$ called "Archimedes' Property":

Definition 2.3 *(Archimedes' Property): Suppose $x \in \mathbb{R}$ has $x > 0$. Then there is a positive integer n with $\frac{1}{n} < x$. Equivalently, if $y > 0$ then there is some $n \in \mathbb{N}$ with $y < n$.*

Exercise Prove that the two versions of Archimedes' property logically equivalent, i.e., that each version implies the other. [Hint: Taking reciprocals reverses order.]

That the set $\mathbb{R}$ has Archimedes' property seems so obvious that one thinks there must be a way to prove it using the algebraic properties of $\mathbb{R}$. Unfortunately there is not, because there are mathematical things that behave like $\mathbb{R}$ from an algebraic point of view and yet do not have the Archimedean property. (One of the appendices to this chapter involves constructing a "non-Archimedean ordered field".) Remember that we are assuming that $\mathbb{R}$ has the LUB property. As our next result shows, we do not need to assume (because we can prove) that $\mathbb{R}$ has the Archimedean property.

Proposition 2.4 *The LUB Axiom implies Archimedes' Property.*

Proof: We consider the second version of Archimedes' Property. Suppose Archimedes' Property fails. Then there is some $x \in \mathbb{R}$ with the property that every $n \in \mathbb{N}$ has $n \le x$. Therefore $\mathbb{N}$ is a non-empty set that is bounded above, so that there is a real number $L = lub(\mathbb{N})$. Then $L - 0.5$ is not an upper bound for $\mathbb{N}$ so there is some $m \in \mathbb{N}$ with $L - 0.5 < m \le L$. But then $L < m + 1 \in \mathbb{N}$ showing that L is not an upper bound for $\mathbb{N}$. That contradiction completes the proof. □

Next is a consequence of Archimedes' property (and therefore also a consequence of the LUB Axiom).

Theorem 2.5 *Between any two real numbers there are a rational number and an irrational number.*

Proof: We may assume $x < y$ in $\mathbb{R}$. First consider the case where $0 < x < y$. Compute $\epsilon = y - x$. Then $\epsilon > 0$. Using Archimedes' axiom, find a positive integer n with $\frac{1}{n} < \epsilon$ and $\frac{1}{n} < x$. Now consider the sequence $\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \frac{4}{n}, \cdots$. There is some integer $k \ge 0$ with $\frac{k}{n} \le x < \frac{k+1}{n}$. [WHY?]. Because the distance from x to y is $\epsilon > \frac{1}{n}$ we know that $x < \frac{k+1}{n} < y$, showing that the rational number $\frac{k+1}{n}$ lies between x and y.

Next we show that if $0 < x < y$, then there is an irrational number between x and y. For each positive integer n, $\frac{\sqrt{2}}{n}$ is irrational [WHY?] and positive. Find N so that $\frac{\sqrt{2}}{N} < \epsilon$ and $\frac{\sqrt{2}}{N} < x$, and consider the sequence $\frac{\sqrt{2}}{N}, \frac{2\sqrt{2}}{N}, \frac{3\sqrt{2}}{N}, \cdots$. There is an integer k so that $\frac{k\sqrt{2}}{N} \le x < \frac{(k+1)\sqrt{2}}{N}$. Because the distance from x to y is greater than $\epsilon > \frac{\sqrt{2}}{N}$ we know that $\frac{(k+1)\sqrt{2}}{N} \in (x, y)$. But $\frac{(k+1)\sqrt{2}}{N}$ is irrational [WHY?], so the proof is complete if $0 < x < y$.

The second case where $x < y < 0$ is similar. Alternately, one could note that $0 < (-y) < (-x)$ and apply the first case to get a rational number q and an irrational number p between $-y$ and $-x$, and then the number $-q \in \mathbb{Q}$ and $-p \in \mathbb{P}$ are the numbers that we want.

The remaining cases to be considered are when $x \le 0 < y$ (in which case choose any $z \in (0, y)$ and apply case 1 to the interval (z, y)) and when $x < 0 \le y$ (in which case take any $w \in (x, 0)$ and apply case 2 to (x, w)).

No doubt there are more efficient ways to prove this theorem. □

There is a technical term that describes the conclusion of Theorem 2.5: a subset $D \subseteq \mathbb{R}$ is dense in $\mathbb{R}$ if for every open interval (x, y) we have $D \cap (x, y) \ne \emptyset$. Consequently, Theorem 2.5 asserts that both $\mathbb{Q}$ and $\mathbb{P}$ are dense subsets of $\mathbb{R}$.

Terminology There are interchangeable terms in the mathematical literature that you need to know. What we have called lub(E) for a set $E \subseteq \mathbb{R}$ is also called $\sup(E)$ which is read "supremum of E", and what we have called glb(F) for a set $F \subseteq \mathbb{R}$ is also called $\inf(F)$, which is read "infimum of F". All four of the terms $\text{lub}, \text{glb}, \sup$ and $\inf$ are in frequent use.

The next example highlights a crucial difference between the sets $\mathbb{R}$ and $\mathbb{Q}$ that was mentioned above and shows that the usual algebraic properties shared by $\mathbb{R}$ and $\mathbb{Q}$ are not enough to prove the LUB axiom.

Proposition 2.6 *The set $\mathbb{Q}$ does not satisfy the LUB axiom.*

Proof: We look at the set $S = \{x \in \mathbb{Q} : x^2 < 2\}$. The set S is certainly bounded above because every $x \in S$ has $x \le 10$. Suppose there is a rational number $r = \text{lub(S)}$. We will prove that $r^2 = 2$ and we know that not rational number can have that property.

After we learn that the function $f(x) = x^2$ is continuous, there will be an easy proof that $r^2 = 2$ but for now we must do extra work. We will show that both $r^2 < 2$ and $r^2 > 2$ are impossible, and we will be forced to conclude that $r^2 = 2$.

Claim 1: $r^2 < 2$ is impossible. Suppose $r^2 < 2$. Then let $\epsilon = 2 - r^2$, and we see that $\epsilon > 0$. From algebra we know that for any n, $(r + \frac{1}{n})^2 = r^2 + \frac{2r}{n} + \frac{1}{n^2}$. When we look at $\frac{2r}{n} + \frac{1}{n^2}$ we see that there is a sufficiently large n_0 having $\frac{2r}{n_0} + \frac{1}{n_0^2} < \epsilon$ and it follows that

$$(r + \frac{1}{n_0})^2 = r^2 + \frac{2r}{n_0} + \frac{1}{n_0^2} < r^2 + \epsilon = r^2 + (2 - r^2) = 2$$

so that $r + \frac{1}{n_0} \in S$ (because we are assuming that $r \in \mathbb{Q}$). But that is impossible because r is an upper bound for the set S. Therefore $r^2 < 2$ is impossible.

Claim 2: $r^2 > 2$ is impossible. Suppose $r^2 > 2$ and let $\delta = r^2 - 2$. Then $\delta > 0$. We know that $(r - \frac{1}{n})^2 = r^2 - \frac{2r}{n} + \frac{1}{n^2} < r^2$ so that $-\frac{2r}{n} + \frac{1}{n^2} < 0$ and we see that there is some n_1 so large that $|\frac{2r}{n_1} - \frac{1}{n_1^2}| < \delta$. But then $(r - \frac{1}{n_1^2})^2 = r^2 - \frac{2r}{n_1} - \frac{1}{n_1^2} > r^2 - \delta = r^2 - (r^2 - 2) = 2$ so that $(r - \frac{1}{n_1})^2 > 2$. Therefore, if $x \in S$, we have $x^2 < 2 < (r - \frac{1}{n_1^2})^2$ so that $x < (r - \frac{1}{n_1})$ showing that $r - \frac{1}{n_1}$ is an upper bound for S that is less than r. This is impossible because r is the least upper bound for the set S. Therefore $r^2 > 2$ is impossible. □

Here is a consequence of the LUB axiom that has important applications. It is called "Cantor's nested intervals theorem" or "Cantor's intersection theorem" and we will use it over and over in the part of mathematics called "real analysis" or "why Calculus works."

Proposition 2.7 *(Cantor's nested intervals theorem) Suppose that $I_n = [a_n, b_n]$ is a sequence of closed intervals with $I_1 \supseteq I_2 \supseteq \cdots$. Then $\bigcap\{I_n : n \ge 1\} \neq \emptyset$.*

Proof: Because $I_{n+1} \subseteq I_n$ we have $a_n \le a_{n+1}$ and $b_{n+1} \le b_n$. Let $E = \{a_n : n \ge 1\}$. Clearly E is non-empty. We claim that every point b_n is an upper bound for E, i.e. that $a_m \le b_n$ for every integer m. Fix m and n. First suppose $m \le n$. Then $a_m \le a_n \le b_n$. Next, suppose $n < m$. Then then $a_m \le b_m \le b_n$ and every b_n is an upper bond for E. Therefore there is a number $L = \text{lub(E)}$. Then $a_n \le L$ for every n, and because each b_n is an upper bound for E, we have $L \le b_n$ for every n. Therefore $L \in \bigcap\{I_n : n \ge 1\}$. □

Cantor's intersection theorem can give another proof that the set $\mathbb{R}$ is not countably infinite.

Corollary 2.8 *The set $\mathbb{R}$ is not countable.*

Proof: For contradiction, suppose $\mathbb{R}$ is countable. Then list the members of $\mathbb{R}$ without repetition as $\mathbb{R} = \{r_k : k \ge 1\}$. Choose an interval $I_1 = [a_1, b_1]$ with $a_1 < b_1$ that does not contain r_1. Choose an interval $I_2 = [a_2, b_2] \subseteq I_1$ with $a_2 < b_2$ that does not contain r_2. In general, if we already have $I_1, \cdots, I_n$ with $I_1 \supseteq I_2 \supseteq \cdots I_n$ with $I_n = [a_n, b_n]$, $a_n < b_n$ and $r_j \notin I_j$ for $1 \le j \le n$, then choose a closed interval $I_{n+1} = [a_{n+1}, b_{n+1}] \subseteq I_n$ in such a way that $a_{n+1} < b_{n+1}$ and $r_{n+1} \notin I_{n+1}$. Now apply the nested closed interval proposition to conclude that there is some real number $L \in \bigcap\{I_n : n \geq 1\}$. Because $L \in \mathbb{R} = \{r_j : j \geq 1\}$ there must be some j with $L = r_j$. But then $L \notin I_j$ so that $\bigcap\{I_n : n \geq 1\} \subseteq I_j$ guarantees that $L \notin \bigcap\{I_n : n \geq 1\}$ and that is impossible.□

Exercise: In the proof of Corollary 2.8, why did we need to say that $a_n < b_n$?

3 Definitions and examples for sequences

The set of positive integers is denoted by $\mathbb{N}$. A sequence in a set X is a function $f : \mathbb{N} \to X$. Our text focuses on the case where $X = \mathbb{R}$ but there can be sequences in any nonempty set. For historical reasons, we sometimes write "$x_1, x_2, \cdots$ is a sequence in the set X", or "$\langle x(n) \rangle$ is a sequence in X", by which we mean the function $f(n) = x_n$.

It is important to realize that a sequence $\langle x_n \rangle$ is not the same as the set $\{x_n : n \geq 1\}$. For example, the function $f(n) = (-1)^n$ is a sequence with infinitely many terms, while $\{f(n) : n \geq 1\}$ is a set with just two elements.

In the set $\mathbb{R}$ of real numbers, the ball with radius ϵ centered at the number p is

$$B(p, \epsilon) = (p - \epsilon, p + \epsilon).$$

This is just an open interval centered at p with radius ϵ; we call it a ball because many of the things we prove about $\mathbb{R}$ have analogs in the plane and in space, where the word "interval" cannot be used but the word "ball" can.

Definition 3.1 *A real number L is the limit of a sequence $\langle x_n \rangle$ if for each $\epsilon > 0$ there is an $N \in \mathbb{N}$ with the property that whenever $n \geq N, x_n \in B(L, \epsilon)$. If L is the limit of $\langle x_n \rangle$ then we say that $\langle x_n \rangle$ converges to L and write $x_n \to L$ or $\lim x_n = L$.*

That definition is ultra-important. We can restate it as: for each $\epsilon > 0$ there is a cut off point N such that $|L - x_n| < \epsilon$ whenever $n \geq N$. We can state this informally as "The sequence x_n eventually gets in, and stays in, every ϵ-ball centered at L."

It is crucial to be able to state what it means for a sequence x_n *not* to converge to the real number L: there is some $\epsilon > 0$ so that for every N, there is some $n \geq N$ with $x_n \notin (L - \epsilon, L + \epsilon)$. Here are two examples to consider: $x_n = (-1)^n$, and $y_n = n^2$. Neither converges to a real number.

Exercise: Consider the sequence $a_n = \sin(\frac{1}{n})$ where $\frac{1}{n}$ is measured in radians. Does this sequence converge to a real number? What about the sequence $b_n = \sin(n)$?

Lemma 3.2 *If $a_n \to L$ is a convergent sequence of real numbers then the set $\{a_n : n \geq 1\}$ is bounded.*

Proof: According to the definition of $a_n \to L$, there is some N such that $n \geq N$ guarantees that $a_n \in (L - 1, L + 1)$. Then $|a_n| \leq \max(|L - 1|, |L + 1|)$ for all $n \geq N$. Because the set $\{|L - 1|, |L + 1|, |a_1|, |a_2|, \cdots, |a_N|\}$ is finite, it has a maximum element M, and then we have that $\{a_n : n \geq 1\} \subseteq [-M, M]$ showing that $\{a_n : n \geq 1\}$ is a bounded set. □

EXERCISE In the previous proof, why didn't we say $|a_n| \leq |L+1|$ for all $n \geq N$?

Could it be that a convergent sequence $\langle x_n \rangle$ has two limits? In other words, is it true that if we could prove that $\langle x_n \rangle$ converges to both L and M, then $L = M$?

Proposition 3.3 *A convergent sequence in $\mathbb{R}$ cannot have two distinct limits.*

Proof: Suppose $L \neq M$ are both limits of the sequence $\langle x_n \rangle$. Then the number $\epsilon = |L - M|$ is positive. Because $x_n \to L$ there is a cut-off number N_L such that if $n \geq N_L$ then $|x_n - L| < \frac{\epsilon}{3}$ and a number N_M such that if $n \geq N_M$ then $|x_n - M| < \frac{\epsilon}{3}$. Consider any integer $n \geq \max(N_L, N_M)$. Then

$$\epsilon = |L - M| \leq |L - x_n| + |x_n - M| < \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon$$

and that is impossible. Therefore $L \neq M$ cannot happen. □

Exercise: Suppose a_n is a sequence of real numbers and that $a_n \to p \in \mathbb{R}$.

a) Show that if each $a_n > 0$, then $p \geq 0$.
b) Can we prove that if $a_n > 0$ for all n, then $p > 0$?

In other courses you will develop special tricks for finding the limits of certain sequences, and I don't mind if you use tricks like L'Hospital's rule to guess what a sequence's limit must be. However, for now, to actually prove that a sequence has a certain limit, we must use only algebra, the definition of limits, and the following result, often called the "Sandwich Theorem".

Theorem 3.4 *(Sandwich Theorem) Suppose a_n, b_n and c_n are sequences of real numbers with $a_n \leq b_n \leq c_n$ for all sufficiently large n. If $a_n \to L$ and $c_n \to L$ then $b_n \to L$*

Proof: There is some N_0 with $a_n \leq b_n \leq c_n$ for all $n \geq N_0$. Now suppose $\epsilon > 0$ is given. There are integers N_1, N_2 such that if $n \geq N_1$ then $a_n \in (L - \epsilon, L + \epsilon)$ and if $n \geq N_2$ then $c_n \in (L - \epsilon, L + \epsilon)$. Then if $n \geq \max(N_0, N_1, N_2)$ we have $b_n \in [a_n, c_n] \subseteq (L - \epsilon, L + \epsilon)$.□

Example 3.5 *As $n \to \infty$, the sequence $b_n = \frac{\sin(n)}{n} \to 0$.*

Proof: We don't have a good idea of what the sequence $\sin(n)$ does, except that $|\sin(n)| \leq 1$. Let $a_n = -\frac{1}{n}$ and $c_n = \frac{1}{n}$ for each n . Then $-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$ so that the sequence $b_n = \frac{\sin(n)}{n}$ is trapped between the sequence $a_n = -\frac{1}{n}$ and $c_n = \frac{1}{n}$. Therefore, the Sandwich Theorem shows that $\lim \frac{\sin(n)}{n} = 0$. □

EXERCISE Show that if $|x_n| \to 0$ then $x_n \to 0$ and give an example showing that $|y_n| \to 2$ does not guarantee that $y_n \to 2$.

Example 3.6 *Without L'Hospital's rule, show that* $\lim \frac{n^2+5n}{n^3} = 0$

Proof: For $n \geq 1$ algebra gives us

$$\frac{n^2 + 5n}{n^3} = \frac{n^2 + 5n}{n^3} * \frac{\frac{1}{n^3}}{\frac{1}{n^3}} = \frac{\frac{1}{n} + \frac{5}{n^2}}{1}.$$

Now $0 \leq \frac{1}{n} + \frac{5}{n^2} \leq \frac{1}{n} + \frac{5}{n} = \frac{6}{n}$. Therefore the quantity $\frac{1}{n} + \frac{5}{n^2}$ is trapped between the sequence that is constantly 0 and the sequence $\frac{6}{n}$ that converges to 0, so that the Sandwich Theorem guarantees that $\frac{n^2+5n}{n^3} = \frac{1}{n} + \frac{5}{n^2} \to 0.\Box$

A very useful tool for making estimates is the *binomial theorem* which allows us to expand $(a+b)^n$ into a sum of products of terms $a^{n-k}b^k$. You may have used Pascal's triangle to find the coefficients of the term $a^{n-k}b^k$ in that expansion.

Theorem 3.7 *For each positive integer n and each pair of real numbers a, b, we have*

$$(a+b)^n = \Sigma\Big\{\binom{n}{k} a^{n-k}b^k : 0 \leq k \leq n\Big\}$$

where $\binom{n}{k}$ is is the coefficient called "n choose k"[1] *In particular, $(1+b)^n = 1+nb+\frac{n(n-1)}{2}b^2+\cdots+b^n$.*

Example 3.8 $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$

Proof: We know that $n^{\frac{1}{n}} > 1$ for each positive integer $n > 1$. Let $a_n = n^{\frac{1}{n}} - 1$. Then $a_n > 0$. We must show that $a_n \to 0$.

Consider $n \geq 2$. Note that $1 + a_n = n^{\frac{1}{n}}$ so that $(1+a_n)^n = n$. By the binomial theorem we have

$$n = (1+a_n)^n = 1 + na_n + \frac{n(n-1)}{2}a_n^2 + \text{ other positive terms } \geq 1 + na_n + \frac{n(n-1)}{2}a_n^2$$

so that

$$n - 1 \geq na_n[1 + \frac{(n-1)a_n}{2}]$$

which gives us

$$\frac{(n-1)}{n} \geq a_n[1 + \frac{n-1}{2}a_n] \geq a_n[\frac{n-1}{2}a_n] = a_n^2[\frac{n-1}{2}].$$

Multiply by 2 and divide by $(n-1)$ (why is $(n-1) \neq 0$?) and we get

$$\frac{2(n-1)}{n(n-1)} \geq a_n^2 \geq 0$$

so $\frac{2}{n} \geq a_n^2 \geq 0$. The Sandwich Theorem applies to guarantee that $a_n^2 \to 0$. But then $a_n \to 0$ as required. $\Box$

EXERCISE: Show that if $a_n^2 \to 0$ then $a_n \to 0$ without using the square root function. Then give an example of a sequence b_n with $b_n^2 \to 4$ and yet b_n converges to neither of ± 2.

[1]The number "n choose k" is the number of k-element subsets of a set of size n. Clearly $\binom{n}{0} = 1 = \binom{n}{n}$ and the formula for $\binom{n}{k}$ is

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!},$$

so $\binom{n}{1} = n$ and $\binom{n}{2} = \frac{n(n-1)}{2}$.

Theorem 3.9 *: For $|r| < 1$, $\lim_{n\to\infty} r^n = 0$.*

Proof: If $r = 0$ there is nothing to prove. Consider the case where $0 < r < 1$. Find a rational number $q \in (r, 1)$. Then $0 \le r^n \le q^n$ for each $n \ge 1$ so that if we can show that $\lim_{n\to\infty} q^n = 0$, then the Sandwich theorem will give $\lim_{n\to\infty} r^n = 0$.

We may write $q = \frac{a}{b}$ where a, b are positive integers. Because $q \in (0, 1)$ we know that $a < b$ so that $b = a + k$ for some integer $k \ge 1$. The binomial theorem tells us that $b^n = (a + k)^n = a^n + na^{n-1}k+$ other positive terms, so that $b^n \ge a^n + na^{n-1}k$.

Taking reciprocals reverses inequalities (for example, $3 < 4$ and $\frac{1}{4} < \frac{1}{3}$) so that

$$\frac{1}{b^n} \le \frac{1}{a^n + na^{n-1}k}$$

and now when we multiply by the positive number a^n we get

$$\frac{a^n}{b^n} \le \frac{a^n}{a^n + na^{n-1}k} = \frac{a^n}{a^{n-1}(a + nk)} = \frac{a}{a + nk}.$$

At this point we have $0 \le r^n \le \frac{a^n}{b^n} \le \frac{a}{a+nk}$. Because $\lim_{n\to\infty} \frac{a}{a+nk} = 0$, the Sandwich Theorem guarantees that $0 \le \lim_{n\to\infty} (\frac{a^n}{b^n}) \le \lim_{n\to\infty} (\frac{a}{a+nk}) = 0$ as required for the case where $r > 0$.

Finally, if $-1 < r < 0$, we know that $0 < |r| < 1$ so that the first part of the proof shows that $|r|^n \to 0$ and now an earlier exercise gives $r^n \to 0$. □

What to learn from the theorem about r^n

(1) Sometimes rational numbers are easier to deal with than arbitrary real numbers.

(2) The binomial theorem strikes again.

Exercise: Prove that $\lim \sqrt[n]{n+1} = 1$. Now find $\lim \sqrt[n]{a^n + b^n}$ assuming that $a > b > 0$. [Hint: The Sandwich theorem could help.]

Exercise Assume that $e = \lim_{n\to\infty}(1 + \frac{1}{n})^n$ and determine the value of $\lim_{n\to\infty}(1 + \frac{1}{n^2})^n$. (Make sure you can do this via L'Hospital's rule, and without L'Hospital's rule.)

Exercise: What is wrong with the following argument (except that the conclusion is incorrect)? "We know that $lim_{n\to\infty}\left(1 + \frac{1}{n}\right) = 1$ and that $lim_{n\to\infty}1^n = 1$ and therefore $e = \lim(1 + \frac{1}{n})^n = lim_{n\to\infty}1^n = 1$."

4 Arithmetic and limits

Our next major result deals with the arithmetic of convergent sequences.

Theorem 4.1 *Suppose $p_n \to s$ and $q_n \to t$ where s and t are real numbers. Then*

(i) $p_n + q_n \to s + t$
(ii) $p_n q_n \to st$
(iii) provided $t \neq 0$, $\frac{p_n}{q_n} \to \frac{s}{t}$.

Proof: A key to the proof is the triangle inequality which says that $|x + y| \leq |x| + |y|$ for every $x, y \in \mathbb{R}$. To prove (a), suppose $\epsilon > 0$. We must show that for some N we have $p_n + q_n \in (s + t - \epsilon . s + t + \epsilon)$ for every $n \geq N$. Translated into absolute value notation, what we want is $|(s + t) - (p_n + q_n)| < \epsilon$ for all sufficiently large n. We know that $p_n \to s$ so that there is some N_1 such that $|s - p_n| < \frac{\epsilon}{2}$ for each $n \geq N_1$ and similarly there is some N_2 such that $|t - q_n| < \frac{\epsilon}{2}$ for each $n \geq N_2$. Consider any $n \geq \max(N_1, N_2)$. Then we have

$$(*) \quad |(s + t) - (p_n + q_n)| = |(s - p_n) + (t - q_n)| \leq |s - p_n| + |t - q_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

as required.

To prove (b) we use a trickier argument. This time we want to show that $|st - p_n q_n|$ can be made small, by relating it to the two quantities we know we can make small, namely $|s - p_n|$ and $|t - q_n|$. The key idea is

$$(**) \quad |st - p_n q_n| = |st - sq_n + sq_n - p_n q_n| \leq |st - sq_n| + |sq_n - p_n q_n| = |s||t - q_n| + |s - p_n||q_n|.$$

Because $|t - q_n|$ can be made as small as we like, the quantity $|t - q_n|$ times the constant $|s|$ can be made as small as we want (details in a minute). Analogously, $|s - p_n|$ can be made as small as we like, but the product $|s - p_n| * |q_n|$ is much trickier because we are multiplying the smaller and smaller term $|s - p_n|$ by the <u>variable</u> quantity $|q_n|$. Luckily Lemma 3.2 comes to our aid, because it gives us a constant $M > 0$ having $|q_n| \leq M$ for all values of n. Combined with equation (**), that allows us to say that

$$(***) \quad |st - p_n q_n| \leq |s||t - q_n| + |s - p_n||q_n| \leq |s| * |t - q_n| + |s - p_n| * M.$$

To finish the proof of (b), we want to make each of the terms $|s| * |t - q_n|$ and $|s - p_n| * M$ less than $\frac{\epsilon}{2}$ for all sufficiently large n. That is easily done, as follows: Provided $s \neq 0$, there is some N_3 such that $n \geq N_3$ implies $|t - q_n| < \frac{\epsilon}{2|s|}$ because $q_n \to t$, and there is some N_4 such that $|s - p_n| < \frac{\epsilon}{2M}$ whenever $n \geq N_4$. Then if $n \geq \max(N_3, N_4)$ we have both $|t - q_n| < \frac{\epsilon}{2|s|}$ and $|s - p_n| < \frac{\epsilon}{2M}$ so that

$$(****) \quad |st - p_n q_n| \leq |s| * |t - q_n| + |s - p_n| * M < |s| * \frac{\epsilon}{2|s|} + M\frac{\epsilon}{2M} = \epsilon,$$

as required.

[Question: Where would the above argument break down if $s = 0$ and what would you do to fix the problem?]

To prove c), we start by showing that if $\lim q_n = t$ with $t \neq 0$, then $\lim \frac{1}{q_n} = \frac{1}{t}$. This will involve showing that $|\frac{1}{t} - \frac{1}{q_n}|$ can be made arbitrarily small. Note that $|\frac{1}{t} - \frac{1}{q_n}| = |\frac{q_n - t}{tq_n}|$ and that $|t - q_n|$ can be made as small as we want, because $\lim q_n = t$. However, the denominator tq_n is variable and might get very close to 0 in which case the entire fraction $|\frac{q_n - t}{tq_n}|$ could get very large even though its numerator is small. Here is how we solve that problem. Because $\lim q_n = t \neq 0$ we know that there is some N such that for every $n \geq N$, we have $|q_n| > \frac{|t|}{2} > 0$. Then for $n \geq N$, $\frac{1}{|q_n|} < \frac{2}{|t|}$ so that $\frac{1}{|tq_n|} < \frac{2}{|t^2|}$. Denote the constant $\frac{2}{|t^2|}$ by K. At this stage, for $n \geq N$, we have

$$|\frac{1}{t} - \frac{1}{q_n}| = \frac{|q_n - t|}{|tq_n|} \leq |q_n - t| * K.$$

Now, given any $\epsilon > 0$, find M so large that whenever $n \geq M$, we have $|q_n - t| < \frac{\epsilon}{K}$. Then, if $n \geq \max(N, M)$ we have

$$|\frac{1}{t} - \frac{1}{q_n}| = \frac{|q_n - t|}{|tq_n|} \leq |q_n - t| * K \leq \frac{\epsilon}{K} * K = \epsilon.$$

To complete the proof of (c), note that $\frac{p_n}{q_n} = p_n * \frac{1}{q_n}$ so that part (b) assures us that $\lim \frac{p_n}{q_n} = \frac{s}{t}$. □

It is ultra important when applying Theorem 4.1 that both L and M must be real numbers. For example, you know that $a_n = \frac{1}{n} \to 0$ and that $b_n = n \to \infty$ (something we will discuss in a moment). But you cannot conclude that $1 = \lim a_n b_n = (\lim a_n)(\lim b_n) = 0 * \infty$.

Exercise: Suppose $a_n \to p$ and $b_n \to p$. Prove that $|a_n - b_n| \to 0$. [How is the distance from a_n to b_n related to the distances from a_n to p and from b_n to p?]

5 The extended real numbers and infinite limits

The symbols ∞ and $-\infty$ are not real numbers. They are symbols that lie at opposite ends of the real number line, and the set $[-\infty, \infty]$ is sometimes called the "extended real number line". It is very dangerous to do arithmetic with these new symbols. For example $\infty - \infty$ and $\frac{\infty}{\infty}$ are simply undefined. L'Hospital's rule gives ways for dealing with them in some cases.

For some sequences a_n that do not converge to any real number, it makes sense to write $\lim a_n = \infty$. Here is the technical definition of that idea.

Definition 5.1 *Let a_n be a sequence of real numbers. We write $\lim a_n = \infty$ if for each M there is an N such that $a_n \geq M$ whenever $n \geq N$. We write $\lim a_n = -\infty$ if for each M there is an N such that $a_n \leq M$ for each $n \geq N$.*

In the definition of $\lim a_n = L$ for a real number L we used the intervals $(L-\epsilon, L+\epsilon)$ to describe what the phrase "close to L" meant. In the above definition, the the half lines (M, ∞) are used to describe being "close to $+\infty$". We can describe $\lim a_n = \infty$ informally by saying that the terms of the sequence a_n eventually get larger and stay larger than any fixed number M.

EXERCISE: Prove from the definition that $\lim \frac{n^2}{n+1} = \infty$.

EXERCISE Prove from the definition that if $a_n > 0$ and $\lim a_n = 0$, then $\lim \frac{1}{a_n} = \infty$

Exercise: Suppose $a_n \leq b_n$ for all n and that $a_n \to \infty$. Use the definitions to show that $b_n \to \infty$. [This is a kind of sandwich theorem for infinite limits.]

6 Monotone sequences and subsequences

Given a sequence a_n, we obtain a subsequence by choosing positive integers $n_1 < n_2 < \cdots$ and then looking at $a_{n_1}, a_{n_2}, a_{n_3}, \cdots$. Clearly $n_1 \geq 1, n_2 \geq 2$ and in general $n_k \geq k$.

A crucial issue about subsequences is that one cannot repeat terms of a_n as in $a_1, a_3, a_3, a_7, \cdots$ and one cannot change the order in which terms appear, as in $a_5, a_2, a_7, \cdots$. If we think of a sequence as a function $f : \mathbb{N} \to \mathbb{R}$, then a subsequence is the function f restricted to an infinite subset of $\mathbb{N}$.

Lemma 6.1 *Suppose $a(n) \to p$. Then every subsequence $a(n_k)$ also converges to p*

EXERCISE Prove that if $a(n) \to p$, then every subsequence $a(n_k)$ also converges to p. It is useful to recall that for a subsequence $a(n_k)$ of $a(n)$, we always have $n_k \geq k$.

It is often clumsy to type a statement like $a_{n_{k+1}} \geq a_{n_k}$ and it is easier to type $a(n_{k+1}) \geq a(n_k)$. For that reason, we sometimes write sequences and subsequences as functions evaluated at various positive integers (which is technically more accurate even if it is less traditional).

A sequence a_n is **increasing** if $a_n \leq a_{n+1}$ for every $n \in \mathbb{N}$. The sequence is **strictly increasing** if $a_n < a_{n+1}$ for every n. The terms "decreasing" and "strictly decreasing" are defined analogously and a sequence that is either increasing or decreasing is said to be "monotonic" or "monotone".

Here is a theorem with a particularly elegant proof.

Theorem 6.2 *Any sequence of real numbers has a monotone subsequence.*

Proof: Let $f(n) = a_n$ be a sequence. We say that a positive integer n is a "peak index" of the sequence if $a_m \leq a_n$ whenever $m \geq n$. Let P be the set of all peak indices of the sequence. The set P is either finite or infinite and we consider those two cases separately.

In case P is finite, let n_1 be a positive integer larger than each integer in P. Then n_1 is not a peak index of the original sequence, so there is a number $n_2 > n_1$ with $a(n_1) < a(n_2)$. The number $n_2 \notin P$ so that n_2 is not a peak index of $a(n)$ and therefore some n_3 has $a(n_2) < a(n_3)$. Given $n_1 < \cdots < n_k$ and $a(n_k)$ with $a(n_1) < \cdots < a(n_k)$, we know that $n_k \notin P$ and therefore some integer $n_{k+1} > n_k$ has $a(n_k) < a(n_{k+1})$. Then $\langle a(n_k) \rangle$ is our monotone subsequence.

In case P is infinite, list P in increasing order (without repetitions) as $P = \{n_1, n_2, n_3, \cdots\}$. For each $k, a(n_k)$ is a peak index of the original sequence, so that because $n_{k+1} > n_k$ we know that $a(n_{k+1}) \leq a(n_k)$. Therefore $\langle a(n_k) \rangle$ is the required monotone subsequence. $\square$

Most people would try (unsuccessfully) to prove Theorem 6.2 in an inductive or recursive fashion. In such a case, one might say "Let $n_1 = 1$ and if you have already chosen finitely many terms $a(n_1), a(n_2), \cdots a(n_k)$ in a monotone way, e.g., with $a(n_1) \leq a(n_2) \leq \cdots \leq a(n_k)$, choose an integer $n_{k+1} > n_k$ with $a(n_k) \leq a(n_{k+1})$." But how do you know there is such an integer n_{k+1}? If there is, good for you. If there isn't, then some of your earlier choices were wrong, and perhaps your decision to search for an increasing subsequence was wrong because, maybe, the subsequence for which you should be aiming is decreasing. The proof given for Theorem 6.2 is one that contains a flash of inspiration that gets around the direction-switching problem that might otherwise come up.

Proposition 6.3 *Any bounded monotone sequence converges.*

Proof: Consider the case where $a_1 \leq a_2 \leq \cdots$ is an increasing sequence. Then the set $E = \{a_n : n \geq 1\}$ is a bounded nonempty set of real numbers, so it has a least upper bound L. We claim $\lim a_n = L$. Let $\epsilon > 0$. The $L - \epsilon$ is not an upper bound for the set E so there is some N with $L - \epsilon < a_N \leq L$. For any $n \geq N$ we have $a_N \leq a_n \leq L$ so that $|L - a_n| < \epsilon$, as required. $\square$

EXERCISE: Suppose a_n is a bounded decreasing sequence. Prove that a_n converges.

Exercise: Suppose a_n is a sequence of integers with $a_n \in \{0,1,2,3,4,5,6,7,8,9\}$ for each n. Let $Sum_1 = \frac{a_1}{10^1}$, $Sum_2 = \frac{a_1}{10^1} + \frac{a_2}{10^2}$ and

$$Sum_n = \frac{a_1}{10^1} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n}.$$

Show that Sum_n is a monotone sequence. Also show that $Sum_n \leq 1$ for each n. Explain why the sequence $Sum_1, Sum_2, \cdots$ has a limit in $\mathbb{R}$. As you will see in a few pages, this limit is denoted by $\Sigma_1^\infty \frac{a_n}{10^n}$. Explain how this is related to Cantor's proof that the set $\mathbb{R}$ is not countable.

Our next theorem is named after two mathematicians (Bolzano and Weierstrass) who discovered it in the 1800s.

Theorem 6.4 *(Bolzano-Weierstrass Theorem) Any bounded sequence of real numbers has a convergent subsequence.*

Proof: Suppose $a(n)$ is a sequence of real numbers that is bounded. Using Theorem 6.2 we find a monotone subsequence $a(n_k)$. But then $a(n_k)$ is both bounded and monotone, so that Proposition 6.3 guarantees that $a(n_k)$ converges.$\Box$

Here is a second proof of the Bolzano-Weierstrass theorem that is based on bisections. Many theorems are proved by the bisection method.

Second Proof of Bolzano-Weierstrass Theorem: Because the sequence $a(n)$ is bounded, there is a closed interval I_1 that contains every $a(n)$. Let $n_1 = 1$ (so that $a(n_1) \in I_1$). Divide I_1 into two equal-length closed subintervals. At least one subinterval contains infinitely many terms of $a(n)$ and we call that part I_2. Let n_2 be the first integer with $a(n_2) \in I_2$ and having $n_2 > n_1$. Note that the width of I_2 is one half of the length of I_1

For induction hypothesis, suppose we have closed intervals $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_k$ and integers $n_1, \cdots, n_k$ where

a) the diameter of I_k is $\frac{1}{2^{k-1}}$ times the diameter of I_1
b) I_k contains infinitely many terms of the sequence $a(n)$,
c) $n_1 < n_2 < \cdots < n_k$ where $a(n_j) \in I_j$ for $1 \leq j \leq k$.

Now divide I_k into two equal-length closed intervals. At least one of them must contain infinitely many terms of $a(n)$ and we denote that subinterval by I_{k+1}. Note that the diameter of I_{k+1} is one half of the diameter of I_k, so that the diameter of I_{k+1} is $\frac{1}{2^k}$ times the diameter of I_1. Because the set I_{k+1} contains infinitely many terms of $a(n)$ there is a first integer n_{k+1} with $a(n_{k+1}) \in I_{k+1}$ and $n_{k+1} > n_k$.

This recursive construction gives us closed bounded intervals $I_1 \supseteq I_2 \supseteq \cdots$ where the diameter of I_k is $\frac{1}{2^{k-1}}$ times the diameter of I_1. Therefore $\bigcap\{I_k : k \geq 1\} \neq \emptyset$. Let $L \in \bigcap\{I_k : k \geq 1\} \neq \emptyset$. We claim that $a(n_k) \to L$. To prove that assertion, suppose $\epsilon > 0$ and choose K so large that $\frac{1}{2^{K-1}} < \epsilon$. Because the diameter of I_K is $\frac{1}{2^{K-1}} < \epsilon$ and $L \in I_K$ we see that $I_K \subseteq (L-\epsilon, L+\epsilon)$. But for every $k \geq K$ we have $a(n_k) \in I_k \subseteq I_K$, showing that $a_k \in (L-\epsilon, L+\epsilon)$ for every $k \geq K$. $\Box$

7 First application of sequences: Continuous functions

Once we understand sequences we can sat exactly what we meant in Calculus by a continuous function.

Definition 1: A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at a point p if for every sequence x_n with $\lim x_n = p$ we have $\lim f(x_n) = f(p)$.

Intuitively, Definition 1 says that for any sequence x_n that approximates the number p, then the image sequence $f(x_n)$ must approximate $f(p)$. How could that fail to happen? Here are two examples.

Example: Let's look at a function that is not continuous because it has a jump in its graph. Define $f(x) = x+1$ if $x \leq 2$ and $f(x) = x^2$ if $x > 2$. Imagine a sequence x_n that approaches the number 2 from above. Then $f(x_n) = (x_n)^2$ which approaches 4 from above, and yet $f(2) = 2+1 = 3$, showing that $\lim f(x_n) \neq f(2)$. A discontinuity of this type is called a jump discontinuity.

Example: Now we look at a stranger type of discontinuity. For $x \neq 0$ define $g(x) = \sin(\frac{1}{x})$, and define $g(0) =$ a value that we will try to determine in a moment. In class I will draw a portion of the graph. When x is close to, but not equal to, zero, a very small change in x causes $\frac{1}{x}$ to move through a very long segment of the x-axis. For example, sliding x from $\frac{1}{100}$ to $\frac{1}{1000}$ causes $\frac{1}{x}$ to move through the segment from 100 to $1,000$, and as $\frac{1}{x}$ moves from 100 to 1,000, $\sin(\frac{1}{x})$ moves through hundreds of cycles of length 2π, with the value of $\sin(\frac{1}{x})$ swinging from -1 to 0 and then to 1 and then to 0 and then back to -1. What should be the value of $g(x)$ at $x = 0$? First consider a sequence of points on the x-axis that cause $g(x)$ to have value $+1$: we use $a_1 = \frac{2}{\pi}$, $b_2 = \frac{2}{5\pi}$, $a_3 = \frac{2}{9\pi}, \cdots$. For each $n \geq 1$ we have $g(a_n) = 1$ so that because $a_n \to 0$ we must have $g(0) = \lim_{n\to\infty} g(a_n) = 1$ if we expect $g(x)$ to be continuous at $x = 0$. But now look at a sequence points on the x axis that cause $g(x) = 0$: we use $b_n = \frac{1}{n\pi}$ and because $b_n \to 0$ we see that if g is to be continuous, we must have $g(0) = \lim_{n\to\infty} g(b_n) = 0$. At this point, we know that there is no way to define $g(0)$ in such a way that the function $g(x)$ will be continuous at $x = 0$. This kind of discontinuous behavior is sometimes called an oscillatory discontinuity.

Example: If $x \neq 0$ we define $h(x) = x\sin(\frac{1}{x})$ and define $h(0) = 0$. Because the values of the sine function are all trapped between -1 and $+1$, we see that the graph of $h(x)$ is trapped between the straight lines $y = -x$ and $y = x$. Now the Sandwich Theorem shows us that if x_n is any sequence with $x_n \to 0$, then $\lim_{n\to\infty} h(x) = 0$. Therefore h is continuous at $x = 0$. The function h is what some call damped oscillation. Other examples of this type are $k(x) = x^2\sin(\frac{1}{x})$ and $k(0) = 0$ whose graph is trapped between the parabolic graphs $y = x^2$ and $y = -x^2$.

Your Calculus course might have included a definition of continuity that is different from ours. Here is the definition of continuity of $f(x)$ at a point p that your Calculus course might have contained.

Definition 2: A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at p provided for each $\epsilon > 0$ there is some $\delta > 0$ with the property that if $|x - p| < \delta$ then $|f(x) - f(p)| < \epsilon$.

Luckily, as our next proposition shows, Definitions 1 and 2 are logically equivalent, and sometimes one is better suited to an individual problem than the other.

Proposition 7.1 *Suppose $f : \mathbb{R} \to \mathbb{R}$ and $p \in \mathbb{R}$. Then f is continuous at p in the sense of Definition 1 if and only if f is continuous at p in the sense of Definition 2.*

Proof: First we show that Definition 1 implies Definition 2. Suppose that f is continuous at p in the sense of Definition 1, and suppose f does *not* satisfy Definition 2. (We will ge a contradiction.) Because f does not satisfy Definition 2, there is some positive number ϵ so that no corresponding δ can be found. Consequently, no matter what $\delta > 0$ someone proposes, there is an x value less than δ units from p for which $f(x)$ is not within ϵ units of $f(p)$. In particular, then, for each number $\frac{1}{n}$ there is some $x_n \in (p - \frac{1}{n}, p + \frac{1}{n})$ for which $f(x_n) \notin (p - \epsilon, p + \epsilon)$. Then x_n is a sequence that converges to p and yet $f(x_n)$ stays far away from $f(p)$, so the image sequence $f(x_n)$ does not converge to $f(p)$, contradicting Definition 1. Therefore, if f satisfies Definition 1 then f satisfies Definition 2.

To show that Definition 2 implies Definition 1, suppose f satisfies Definition 2, and suppose $x_n \to p$. To show that f satisfies Definition 1 we must show that $f(x_n) \to f(p)$. To show that $f(x_n) \to f(p)$, suppose $\epsilon > 0$ is given. Use Definition 2 to find the positive number δ corresponding to the given ϵ with the property that if $|x - p| < \delta$, then $|f(x) - f(p)| < \epsilon$. Because $x_n \to p$, there is some cutoff N so that if $n \geq N$, then $x_n \in (p - \delta, p + \delta)$, i.e., $|p - x_n| < \delta$. Using the fact that δ corresponds to ϵ in the sense of Definition 2, we know that for each $n \geq N, |f(x_n) - f(p)| < \epsilon$. But that is exactly what we need to do to show that $f(x_n) \to f(p)$, so we see that f satisfies Definition 1. Therefore, if f satisfies Definition 2, then f satisfies Definition 1. $\square$

Definition To say that a function $f : \mathbb{R} \to \mathbb{R}$ is <u>continuous on $\mathbb{R}$</u> means that f is continuous at every point of $\mathbb{R}$.

You may recall the array of theorems that continuous functions satisfy. Using sequences, most are easily proved. Here are some examples.

Proposition 7.2 *Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and suppose the interval $[a, b]$ is given. Then there are numbers $\hat{M}$ and M such that $\{f(x) : a \leq x \leq b\} \subseteq [\hat{M}, M]$, i.e., f is a bounded function for $x \in [a, b]$.*

Proof: We will show that there is some number M with $f(x) \leq M$ for each $x \in [a, b]$. A similar argument shows that there will be another number $\hat{M}$ with $\hat{M} \leq f(x)$ for all $x \in [a, b]$. Then we will have $f(x) \in [\hat{M}, M]$ for all $x \in [a, b]$

For contradiction, suppose there is no such number M. Then for each positive integer n, there is some $x(n) \in [a, b]$ with $n < f(x(n)))$. Because $x(n)$ is a bounded sequence, the Balzano-Weierstrass theorem gives us a convergent subsequence $x(n_k)$ which has a limit $p \in [a, b]$. Then $f(p) = \lim f(x(n_k))$. But the fact that $f(x(n_k)) > n_k \geq k$ means that $f(p) = lim f(x(n_k))$ mut be greater than each integer k, and that is impossible. $\square$

Theorem 7.3 *(Maximum Value Theorem) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and $[a, b]$ is given. Then there is some $c \in [a, b]$ with the property that $f(x) \leq f(c)$ for all $x \in [a, b]$.*

Proof: From Proposition 7.2 we know that the set $S = \{f(x) : x \in [a, b]\}$ is nonempty and bounded, so the LUB axiom gives us the number $L = \text{lub(S)}$. What we do not know (yet) is that $L \in S$, i.e., that L is one of the output values of the function f.

For each $n \geq 1$, then number $L - \frac{1}{n}$ is not an upper bound for S so that there is some $x(n) \in [a, b]$ with $L - \frac{1}{n} < f(x(n)) \leq L$. The sequence $< x(n) >$ is a bounded sequence of real numbers so that, by the Balzano-Weierstrass theorem, there is a subsequence $< x(n_k) >$ that converges to some number p. Because $a \le x(n_k) \le b$ for all k, we know that $p = \lim_{k\to\infty} x(n_k) \in [a, b]$. Because f is continuous, we have $f(p) = f(\lim_{k\to\infty} x(n_k)) = \lim_{k\to\infty} f(x(n_k)) = L$. Consequently, p is the point of $[a, b]$ that we wanted. □

Corollary 7.4 *(Minimum Value Theorem) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and $[a, b]$ is given. Then there is some $d \in [a, b]$ with the property that $f(d) \le f(x)$ for all $x \in [a, b]$.*

Proof: To prove this Minimum Value Theorem, one could slightly modify the proof of Theorem 7.3 by looking at glb(S) rather than lub(S), but there is another way. Let $g(x) = -f(x)$ for all $x \in \mathbb{R}$. Then g has a largest output value for $x \in [a, b]$ and the largest output value of g corresponds to the smallest output value of f for $x \in [a, b]$. □

Exercise Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and that $x_n \in \mathbb{R}$ has $\lim_{n\to\infty} x_n = L$. Also suppose that $f(x_n) < 0$ for all n.

a) Prove that $f(L) \le 0$.

b) Give an example showing that we cannot claim that $f(L) < 0$.

Theorem 7.5 *(Intermediate Value Theorem) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and $[a, b]$ is given. Suppose $f(a) < f(b)$. If $f(a) < V < f(b)$ then for some $x \in (a, b)$ we have $f(x) = V$.*

Proof: Let $T = \{x \in [a, b] : f(x) < V\}$. Then $a \in T$ so that $T \ne \emptyset$. Because $T \subseteq [a, b]$ the LUB axiom gives a number $L =$ lub(T) and $L \in [a, b]$. For each $n \ge 1$ there are points $x(n) \in T$ with $L - \frac{1}{n} < x(n) \le L$. Then $lim_{n\to\infty} x(n) = L$. Because f is continuous, $f(L) = f(\lim_{n\to\infty} x(n)) = \lim_{n\to\infty} f(x(n))$ so that $f(L) \le V$ because each $x(n) \in T$ so that $f(x(n)) < V$.

To show that $f(L) = V$ we will show that $f(L) \ge V$. When combined with $f(L) \le V$, this will give $f(L) = V$. First note that L cannot equal b because we already know that $f(L) \le V < f(b)$. Therefore $L < b$. Let $z(n) \in (L, b]$ be any sequence converging to L. e.g., $z(n) = L + \frac{b-L}{n}$. Because $L =$ lub(T) and $L < z(n)$ we know that $z(n) \notin T$ so that $f(z(n)) \ge V$ for each n. Because f is continuous we know that $f(L) = f(\lim_{n\to\infty} z(n)) = \lim_{n\to\infty} f(z(n)) \ge V$. At this point we have both $f(L) \le V$ and $f(L) \ge V$, and we conclude that $f(L) = V$ as required. □

Exercise Suppose $g : [0, 1] \to [0, 1]$ is a continuous function. Show that for some $x \in [0, 1]$ we have $g(x) = x$. This result is an example of a "fixed point theorem" and such theorems are central ideas in applied mathematics. [Hint: For contradiction, suppose that $g(x)$ is never equal to x and consider $f(x) = x - g(x)$. Show $f(0) < 0 < f(1)$.]

8 Second Application of sequences: the topology of $\mathbb{R}$

What is "topology" in the title of this section? For our course the following answer is good enough. Using sequences, we will define what we mean by "sequentially closed sets" or, more simply, "closed sets." We will study the properties of closed sets and show how they are linked to continuous functions. Then we will use the closed sets to define what we will call "open sets." Now we can say that topology is whatever can be studied using only these open sets and closed sets.

Definition 8.1 *A subset $E \subseteq \mathbb{R}$ is a* <u>*closed set*</u> *(also known as "sequentially closed") if it is true that whenever $e(n) \in E$ and $e(n) \to p$, then $p \in E$.*

Example 8.2 It is easy to check that if each term of the sequence a_n belongs to the interval $[c, d]$ and if $a_n \to L$, then $L \in [c, d]$ so that the segment $[c, d]$ is what we called a closed set. But there are closed sets that are not of the form $[c, d]$. For example, the set $[0, 1] \cup [3, 4]$ is also a closed set and is not an interval of any kind. Also, $[1, \infty)$ is a closed set and there are closed set that are much more complicated.

Exercise Show that the set $\mathbb{Z}$ of all integers (positive, negative, and zero) is a closed set. [Hint: What are the convergent sequences of integers?] Show that the set $\mathbb{Q}$ of rational numbers is not closed, and neither is the set $\mathbb{P} = \mathbb{R} - \mathbb{Q}$.

Theorem 8.3 *If $E_1, E_2, \cdots, E_n$ is a finite sequence of closed sets in $\mathbb{R}$, then $E_1 \cup E_2 \cup \cdots \cup E_n$ is a closed set, and if $\mathcal{F}$ is any collection of closed subsets of $\mathbb{R}$, no matter how large, then $\bigcap \mathcal{F}$ is a closed set.*

Proof: Suppose $E = E_1 \cup \cdots \cup E_n$ and suppose $a(m)$ is a sequence of points in E that converges to a point $p \in \mathbb{R}$. For $i = 1, 2, \cdots, n$, let $K_i = \{n \geq 1 : a(n) \in E_i\}$. Then $\mathbb{N} = K_1 \cup \cdots \cup K_n$. Because the infinite set $\mathbb{N}$ is broken into a finite number of subsets K_i, there is some i for which K_i is an infinite set, say $K_i = \{n_1, n_2, n_3 \cdots\}$ listed in increasing order. Then $a(n_k)$ is a subsequence of $a(n)$ with $a(n_k) \in E_i$ for every n_k and we know that $a(n_k) \to p$. But E_i is a closed set, so that $p \in E_i \subseteq E$. Hence E is closed.

To prove the second assertion, suppose $x(n)$ is a sequence of points in the set $F = \bigcap \mathcal{F}$ with $x(n) \to p$. For each $F \in \mathcal{F}$ we know that each $x(n) \in F$ and that F is a closed set. Therefore, $p \in F$ for each $F \in \mathcal{F}$, which means that $p \in \bigcap \mathcal{F}$ as required.$\square$

Example 8.4 *For each $x \in \mathbb{R}$, let $F_x = [x, \infty)$ and let $\mathcal{F} = \{F_x : x \in \mathbb{R}\}$. According to Theorem 8.3, the set $\bigcap \mathcal{F}$ must be a closed set. But $\bigcap \mathcal{F} = \emptyset$, so $\emptyset$ must be a closed set.*

EXERCISE: Use the <u>definition</u> of closed set to prove that both $\emptyset$ and $\mathbb{R}$ are closed subsets of $\mathbb{R}$.

EXERCISE: In this exercise we construct one of the most interesting closed sets of real numbers. Start with $C_1 = [0, 1]$ and remove the middle third interval $(\frac{1}{3}, \frac{2}{3})$ to form the closed set $C_2 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ which has two halves. From each half, remove the middle third open interval to form the closed set $C_3 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$ which consists of four smaller intervals. From each of those four, remove the middle third open interval, giving a closed set C_4 that is the union of eight closed intervals. Continue this process of deleting middle thirds, obtaining a sequence C_n of closed sets with $C_{n+1} \subseteq C_n$ for each n. Let $C = \bigcap \{C_n : n \geq 1\}$. This set is called the <u>Cantor set</u> and is one of the most surprising sets of real numbers.

a) Explain why C is a closed set.

b) Explain why $C \neq \emptyset$ by finding infinitely many points of C.

c) Show that C is uncountable. [Hint: Every real number has a ternary (= base 3) expansion. As with decimal expansions, some numbers have both a finite ternary expansion and an infinite expansion. Show that any real number whose infinite ternary expansion does not contain the digit 1 must belong to each C_n. How many infinite ternary expansions exist that do not contain the digit 1?]

We already have two equivalent definitions of "continuous function" an not we add a third.

Proposition 8.5 *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function. The following are equivalent:*

a) f is continuous at each $p \in \mathbb{R}$ (defined using sequences);

b) For each $p \in \mathbb{R}$ and each $\epsilon > 0$, there is a $\delta > 0$ with the property that if $|x - p| < \delta$, then $|f(x) - f(p)| < \epsilon$;

c) For each closed subset T of the codomain $\mathbb{R}$, the set $f^{-1}[T]$ is a closed set in the domain $\mathbb{R}$.

Proof: We already know that a) $\iff$ b), so we will show that a) $\iff$ c).

Suppose a) holds, and suppose T is a closed subset of the codomain $\mathbb{R}$. Write $S = f^{-1}[T]$. To show that S is closed. suppose $x_n \in S$ and $x_n \to p$. We must show that $p \in S$. Because $x_n \in S$ we know that $f(x_n) \in T$ and because a) holds, we know that $f(x_n) \to f(p)$. Because T is given as a closed set, we know that $f(p) \in T$ and therefore $p \in f^{-1}[T] = S$, as required.

Now suppose that c) holds. To prove a), consider any sequence x_n or real numbers that converges to a number p. For contradiction, suppose $f(x_n)$ does not converge to $f(p)$. Then there is some $\epsilon > 0$ such that for each proposed cut-off number N, some larger n has $f(x_n) \notin (f(p) - \epsilon, f(p) + \epsilon)$. Inductively choose integers $n_1 < n_2 < \cdots$ with the property that $f(x(n_k)) \notin (f(p) - \epsilon, f(p) + \epsilon)$, so that if we let $T = (-\infty, f(p) - \epsilon] \cup [f(p) + \epsilon, \infty)$ then T is a closed set and has $f(x(n_k)) \in T$ for each k. Then $x(n_k)) \in f^{-1}[T]$ for each k and $f^{-1}[T]$ is a closed subset of the domain $\mathbb{R}$ in the light of c). But because $x(n_k))$ is a subsequence of x_n, we know that $x(n_k)) \to p$ so that p must be a point of the closed set $f^{-1}[T]$ and therefore $f(p) \in T$. But that is not true because $f(p) \in (f(p) - \epsilon, f(p) + \epsilon)$, so we have obtained a contradiction. Therefore $f(x_n) \to f(p)$ so we have proved a). □

Definition 8.6 *A subset G of $\mathbb{R}$ is an <u>open set</u> provided $\mathbb{R} - G$ is a closed set.*

It is easy to see that any open interval (a, b) is an open set by looking at $\mathbb{R} - (a, b)$, but there are open sets that are not open intervals, e.g., $(1, 2) \cup (3, 4)$.

Warning: What is the negation of "A is a closed set"? Don't fall into the trap of saying that the negation is "A is an open set" because $\mathbb{R}$ and $\mathbb{R} - \mathbb{R} = \emptyset$ are both open and closed, and the sets $\mathbb{Q}$ and $\mathbb{R} - \mathbb{Q}$ are neither open nor closed in $\mathbb{R}$.

Theorem 8.7 *If $G_1, G_2, \cdots, G_n$ is a finite sequence of open sets, then $\bigcap\{G_i : 1 \le i \le n\}$ is an open set, and if $\mathcal{G}$ is any collection of open sets, finite or infinite, then $\bigcup \mathcal{G}$ is an open set.*

Proof: We prove the second assertion. To show that $\bigcup \mathcal{G}$ is open, we must show that $\mathbb{R} - \bigcup \mathcal{G}$ is a closed set. By de Morgan's laws, $\mathbb{R} - \bigcup \mathcal{G} = \bigcap \{\mathbb{R} - G : G \in \mathcal{G}\}$ which is an intersection of closed sets and therefore (see Theorem 8.3) is a closed set. $\square$

EXERCISE: Show that if $G_1, G_2, \cdots, G_n$ is a finite sequence of open sets, then $\bigcap\{G_i : 1 \leq i \leq n\}$ is an open set. [This could be an induction proof.]

EXERCISE: Let $F_n = [\frac{1}{n}, 1]$. We know that each F_n is a closed set in $\mathbb{R}$. Show that $\bigcup\{F_n : n \geq 2\}$ is not a closed set. Find a collection of open sets in $\mathbb{R}$ whose intersection is not open.

Remember that in $\mathbb{R}$, the open interval with center p and radius ϵ is the set $(p - \epsilon, p + \epsilon)$ which we denoted by $B(p, \epsilon)$ and sometimes call an open ball.

Theorem 8.8 *The following statements about a subset $G \subseteq \mathbb{R}$ are equivalent:*

(i) G is an open subset of $\mathbb{R}$ (i.e., $\mathbb{R} - G$ is closed);
(ii) For all $p \in G$ there is an $\epsilon_p > 0$ with $B(p, \epsilon_p) \subseteq G$;
(iii) G is a union of an infinite or finite number of open balls.

Proof: To show $i) \Rightarrow ii)$ suppose G is an open subset of $\mathbb{R}$. Then $\mathbb{R} - G$ is closed. Suppose that assertion (ii) is false. Then there is a $p \in G$ such that for each $n \geq 1$, the ball $B(b, \frac{1}{n}) \not\subseteq G$. This allows us to find points $a(n) \in B(p, \frac{1}{n})$ with $a(n) \notin G$. Then $a(n) \in \mathbb{R} - G$ and $a(n) \to p$ so that, because $\mathbb{R} - G$ is a closed set, $p \in \mathbb{R} - G$. But that is impossible because $p \in G$. Hence (i) implies (ii).

To show that (ii) implies (iii), suppose the set G satisfies (ii). Then $G = \bigcup\{B(p, \epsilon_p) : p \in G\}$ which is exactly what is required to prove (iii).

To show that (iii) implies (i), we may apply Theorem 8.7 because each open ball is an open set in $\mathbb{R}$ as noted above. $\square$

Exercise: Suppose G is a non-empty, bounded, open set in $\mathbb{R}$. For two real numbers a, b, let $Int(a, b) = [a, b]$ if $a \leq b$ and let $Int(a, b) = [b, a]$ if $b \leq a$. We define a relation $\sim$ for points of the set G as follows: we write $a \sim b$ provided $Int(a, b) \subseteq G$.

a) Show that $\sim$ is an equivalence relation on the set G.

b) For each $a \in G$ let $cls(a)$ be the equivalence class to which a belongs. Show that if $x, y, z \in G$ have $x < y < z$ and $x, z \in cls(a)$, then $y \in cls(a)$.

c) Show that for each $a \in G$, there are real numbers u, v with $cls(a) = (u, v)$

d) Conclude that each nonempty bounded open subset of $\mathbb{R}$ is the union of a pairwise-disjoint collection $\mathcal{J}$ of open intervals (i.e., if J_1, J_2 are elements of $\mathcal{J}$, either $J_1 = J_2$ or else $J_1 \cap J_2 = \emptyset$).

e) With $\mathcal{J}$ as in (d), show that $card(\mathcal{J}) \leq card(\mathbb{N})$. [Hint: each $J \in \mathcal{J}$ contains a rational number.]

f) What if the open set G is not bounded?

Now that we know what "open set" means, we can add another equivalent statement to our characterization of continuous functions, and we have:

Proposition 8.9 *Suppose $f : \mathbb{R} \to \mathbb{R}$ is given. The following assertions are equivalent:*

a) f is continuous on $\mathbb{R}$ (meaning that f is continuous at each $p \in \mathbb{R}$ as defined using sequences);

b) For each $p \in \mathbb{R}$, given any $\epsilon > 0$ there is a $\delta > 0$ with the property that if $|p - x| < \delta$, then $|f(p) - f(x)| < \epsilon$;

c)) for each open set $G \subseteq \mathbb{R}$, the set $f^{-1}[G]$ is an open subset of $\mathbb{R}$;

d) for each closed set $C \subseteq \mathbb{R}, f^{-1}[C]$ is a closed subset of $\mathbb{R}$

Proof: We already know that a) and b) and d) are equivalent. Assertions c) and d) are equivalent because for any set $T \subseteq \mathbb{R}$, $f^{-1}[\mathbb{R} - T] = \mathbb{R} - f^{-1}[T]$ together with the fact that the complement of any open set is closed, and vice-versa. □

9 More on sequences: Cauchy sequences

In the third application of sequences (in the next section on infinite series) we will need one more idea. If we have a sequence a_n and a number L, we know how to determine whether or not a_n converges to L. But what if L is not given? What if all you have are the terms of the sequence a_n? Without having the number L in hand, it is often harder to determine whether a given sequence a_n converges to something. Of course, if the sequence is monotone and bounded, we know that it converges to some real number. But what if the sequence is not monotone? Giving a way to check convergence that involves only the terms of the sequence is the goal of this section.

Lemma 9.1 *Suppose $a_n \to L$. Then for each $\epsilon > 0$ there is an integer N with the property that whenever $m, n \geq N$, we have $|a_m - a_n| < \epsilon$*

Proof: Let $\epsilon > 0$. Because $a_n \to L$, there is an integer N such that if $n \geq N$, then $|a_n - L| < \frac{\epsilon}{2}$. But then if $m, n \geq N$ we have

$$|a_m - a_n| = |a_m - L + L - a_n| \leq |a_m - L| + |L - a_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Definition 9.2 *A sequence a_n is a Cauchy sequence if for each $\epsilon > 0$ there is a cut-off integer N such that if $m, n \geq N$, then $|a_m - a_n| < \epsilon$.*

Exercise: Show that the following alternate definition is equivalent to the one given above: A sequence a_n is Cauchy provided that for each $\epsilon > 0$ there is some N with the property that for every $m \geq N$, $|a_M - a_N| < \epsilon$.

It follows from Lemma 9.1 that any convergent sequence is a Cauchy sequence. That the converse is true in the set $\mathbb{R}$ (but not in $\mathbb{Q}$) is another crucial difference between $\mathbb{R}$ and $\mathbb{Q}$.

Theorem 9.3 *A sequence $a(n)$ of real numbers converges to a real number if and only if $a(n)$ is a Cauchy sequence.*

Proof: Lemma 9.1 is half of this theorem. For the converse, suppose $a(n)$ is a Cauchy sequence of real numbers. Our proof has two steps. First we show that $a(n)$ is bounded and therefore (see Bolzano-Wierestrass) has a subsequence $a(n_k)$ that converges to some real number L, and second we show that the original sequence $a(n)$ also converges to L.

To show that $a(n)$ is bounded, let $\epsilon = 1$. Because $a(n)$ is Cauchy, there is some N_1 with the property that whenever $m, n \geq N_1$ we have $|a_m - a_n| < 1$. Then $a_n \in [a(N_1) - 1, a(N_1) + 1]$ whenever $n \geq N_1$ so that eventually the terms of a_n are bounded. Compute the number M which is the maximum of the finitely many numbers $|a(N_1) - 1|, |a(N_1) + 1|, |a(1)|, |a(2)|, \cdots, |a(N_1)|$. Then $|a_n| \leq M$ for every n, so the sequence is bounded. Therefore the Balzano-Weierstrass theorem shows that there is a subsequence $a(n_k)$ that converges to some number L.

To complete the proof, we will show that the original sequence also converges to L. To show that $a(n) \to L$, consider any $\epsilon > 0$. Find N_2 so large that if $m, n \geq N_2$ then $|a(m) - a(n)| < \frac{\epsilon}{3}$. Find N_3 so large that if $k \geq N_3$ then $|L - a(n_k)| < \frac{\epsilon}{3}$. Compute $K = \max(N_2, N_3)$ and suppose $n \geq K$. Then $n_k \geq K \geq N_2$ so that $|a(n) - a(n_k)| < \frac{\epsilon}{3}$ and $n_k \geq N_3$ so that $|a(n_k) - L| < \frac{\epsilon}{3}$. This gives

$$|a_n - L| = |a(n) - a(n_k) + a(n_k) - L| \leq |a(n) - a(n_k)| + |a(n_k) - L| < \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon$$

as required. □

Example 9.4 *To see that Theorem 9.3 fails in $\mathbb{Q}$, let $a(n)$ be the first n digits of the decimal representation of $\sqrt{2}$. Clearly $a(n) \to \sqrt{2}$ so the sequence $a(n)$ is Cauchy. However, there does not exist any rational number to which $a(n)$ converges.*

EXERCISE: Suppose $a_n > 0$ and $\lim a_n = L \in \mathbb{R}$. Define a new sequence $b_n = (a_1 * a_2 * \cdots * a_n)^{\frac{1}{n}}$. Prove that $\lim b_n = L$.

EXERCISE: Suppose a_n is a sequence of positive real numbers and that $\lim \frac{a_{n+1}}{a_n} = L < 1$. Prove that $\lim a_n = 0$. [Hint: Choose a number $r \in (L, 1)$. Then for some N we have $0 < \frac{a_{n+1}}{a_n} < r$ for every $n \geq N$. Then $a_{N+1} < r a_N$, and $a_{N+2} < r a_{N+1} < r(r a_N) = r^2 a_N$. What about a_{N+k}?]

EXERCISE: Suppose $a_n > 0$ and $\lim \frac{a_{n+1}}{a_n} = 0$. Prove that $\lim \sqrt[n]{a_n} = 0$. [Hint: Fix any $r \in (0, 1)$. Then there is an integer $N = N(r)$ such that $0 \leq \frac{a_{n+1}}{a_n} < r$ for every $n \geq N$. Show that $a(N + k) < a(N)r^k$ for each $k \geq 1$, and then consider $\sqrt[N+k]{a(N+k)} < \sqrt[N+k]{a(N)r^k}$. Somehow, this will lead you to see that $\lim_{k\to\infty} \sqrt[N+k]{a(N+k)} \leq 1 * r$ and consequently $\lim_{n\to\infty} \sqrt[n]{a_n} \leq r$, and this is one step away from $\lim_{n\to\infty} \sqrt[n]{a_n} = 0$.]

10 Third application of sequences: Infinite series of real numbers

An infinite series is a sum of infinitely many numbers, and for any $k \geq 1$ we write $\Sigma_k^\infty a_i = a_k + a_{k+1} + a_{k+2} + \cdots$. What could it mean to add up infinitely many numbers? Ordinary arithmetic tells us that we can add up any *finite* list of real numbers, and that we get the same answer independent of the order in which we add members of the list. Even in ancient times, people wondered whether infinite lists of numbers could be added up, and because such additions seemed to involve strange contradictions, infinite additions were banned from mathematics and were treated as undefined objects. Then two thousand years passed and in the early 1800s, we discovered how to make sense of certain infinite additions, using sequences of what we will call "partial sums."

Definition 10.1 *If a_n is a sequence of real numbers, then the n^{th} partial sum is $PS_n = a_1 + \cdots + a_n$. If the sequence $PS_1, PS_2, PS_3, \cdots$ converges to a real number L, then we define $\Sigma_1^\infty a_n = L$ and we say that the series $\Sigma_1^\infty a_n$ converges, or sums, to L.*

There are explicit formulas for the sum of certain types of series. A geometric series is a series of the form $\Sigma_{j=k}^\infty ar^j$. The key feature of a geometric series is that each term of the sum is r times its predecessor, and then r is called the common ratio of the series.

Proposition 10.2 *(Geometric Series) If $|r| < 1$ then $\Sigma_0^\infty r^n = \frac{1}{1-r}$*

Proof: For a number r, we have the partial sum $PS_n = 1 + r + r^2 + \cdots + r^n = \frac{1-r^{n+1}}{1-r}$, as can be verified by cross-multiplying or by using induction. Consequently we have a formula for the n^{th} partial sum of the series $\Sigma_0^\infty r^n$. Provided $|r| < 1$ we know that $r^{n+1} \to 0$ and therefore

$$\Sigma_0^\infty r^k = \lim_{n\to\infty}(1 + r + r^2 + \cdots + r^n) = \lim_{n\to\infty} \frac{1 - r^{n+1}}{1 - r} = \frac{1}{1 - r}.$$

Exercise Find the values of $\Sigma_0^\infty(\frac{2}{3})^n$ and $\Sigma_3^\infty(0.5)^n$.

Exercise: Use mathematical induction to show that for each $n \geq 1$, we have $1+r+\cdots+r^n = \frac{1-r^{n+1}}{1-r}$.

Other infinite series, e.g., $1 + 2 + 3 + 4 + \cdots$, obviously cannot be summed up. The first step toward understanding infinite series is to figure out which series converge.

Proposition 10.3 *Suppose $a_n \geq 0$ for each n. Then $\Sigma_1^\infty a_n$ converges if and only if the sequence PS_n of partial sums is bounded.*

Proof: If the series converges, then we know that the sequence PS_n of partial sums is convergent and is therefore bounded. Conversely, suppose there is a number M such that $PS_n \leq M$ for every n. Because $a_i \geq 0$ for each i, we know that $PS_1 \leq PS_2 \leq \cdots$ is an increasing sequence. But any bounded increasing sequence converges to its least upper bound. □

The Cauchy criterion for convergence of a sequence can be applied to a sequence of partial sums, and this gives:

Proposition 10.4 *(Cauchy Criterion for Series) The infinite series $\Sigma_1^\infty a_i$ converges if and only if for each $\epsilon > 0$ there is an N such that whenever $N \leq m \leq n$, $|\Sigma_m^n a_i| < \epsilon$.*

Proof: Consider the partial sum sequence $PS_n = \Sigma_1^n a_i$. This sequence converges if and only if it is Cauchy, i.e., if and only if for each $\epsilon > 0$ there is an N such that if $N \leq m \leq n, |\Sigma_m^n a_i| = |S_n - S_{m-1}| < \epsilon$. □

Example 10.5 *(The Harmonic Series) The series* $\Sigma_1^\infty \frac{1}{n}$ *does not converge.*

Proof: Look at the sum of the third and fourth terms, then the fifth through eighth terms, then the ninth through 16^{th} terms, and so on. We have

$\frac{1}{3} + \frac{1}{4} > 2 * \frac{1}{4} = \frac{1}{2}$
$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 4 * \frac{1}{8} = \frac{1}{2}$
$\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16} > 8 * \frac{1}{16} = \frac{1}{2}$

In general the sum of terms numbered above 2^n and numbered less that or equal to 2^{n+1} consists of 2^n terms each larger than $\frac{1}{2^{n+1}}$ so that the sum has value greater than $\frac{1}{2}$. Consequently the series $\Sigma_1^\infty \frac{1}{n}$ does not satisfy the Cauchy condition, and therefore cannot converge. □

Definition 10.6 *A series* $\Sigma_1^\infty a_n$ *is* <u>*absolutely convergent*</u> *if* $\Sigma_1^\infty |a_n|$ *converges.*

Proposition 10.7 *An absolutely convergent series is convergent.*

Proof: If $\Sigma_1^\infty |a_n|$ converges, then it satisfies the Cauchy criterion, so that given any $\epsilon > 0$ there is an N such that $\Sigma_m^n |a_i| < \epsilon$ whenever $N \le m < n$. But then $|\Sigma_m^n a_i| \le \Sigma_m^n |a_i| < \epsilon$ so that the series $\Sigma_1^\infty a_i$ satisfies the Cauchy criterion and therefore converges. □

Warning As we will see a little later, the alternating series $\Sigma_1^\infty \frac{(-1)^n}{n}$ is convergent, but Example 10.5 shows that it is not absolutely convergent.

EXERCISE: You know that for any real numbers x and y, $|x + y| \le |x| + |y|$. Use induction to prove that if $x_1, x_2, \cdots, x_n$ are real numbers, then $|x_1 + x_2 + \cdot + x_n| \le |x_1| + |x_2| + \cdots + |x_n|$.

The study of series becomes more complicated when positive and negative terms are mixed. A very nice series of this type is an <u>alternating series</u>, i.e. one in which the signs of terms alternate between positive and negative. Probably the most famous alternating series is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ which we write as $\Sigma\{(-1)^{n+1}\frac{1}{n} : n \ge 1\}$. The fundamental theorem about alternating series is the following.

Proposition 10.8 *Suppose* $0 \le a_{n+1} \le a_n$ *for each* n*. Then* $\Sigma\{(-1)^n a_n : n \ge 1\}$ *converges if and only if* $\lim a_n = 0$

Proof: Our series is $-a_1 + a_2 - a_3 + a_4 - a_5 + \cdots$. We study the partial sums of the alternating series. Because each $a_n \ge 0$ we have $PS(1) = -a_1 \le -a_1 + a_2 = PS(2)$. We claim that $PS(3)$ lies between $PS(1)$ and $PS(2)$. Note that $PS(3) = PS(2) - a_3 \le PS(2)$ and that $PS(3) = PS(1) + (a_2 - a_3)$ so that because $a_3 \le a_2$ we have $(a_2 - a_3) \ge 0$ so $PS(1) \le PS(3)$. Combining these two inequalities, we have $PS(1) \le PS(3) \le PS(2)$. Next we claim that $PS(4)$ lies between $PS(3)$ and $PS(2)$ because $PS(4) = PS(3) + a_4$ which is $\ge PS(3)$. Next, $PS(4) = PS(2) - a_3 + a_4$ which is $\le PS(2)$ because $-a_3 + a_4 \le 0$. Consequently we have $[PS(3), PS(4)] \subseteq [PS(1), PS(2)]$. Similar arguments show that we have $[PS(5), PS(6)] \subseteq [PS(3), PS(4)]$.

If we write $I_1 = [PS(1), PS(2)]$, $I_2 = [PS(3), PS(4)]$, $I_3 = [PS(5), PS(6)]$, $I_4 = [PS(7), PS(8)]$, etc. and we obtain a decreasing sequence of closed, bounded intervals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$. Cantor's Intersection Theorem guarantees that the intersection $\bigcap\{I_n : n \ge 1\} \ne \emptyset$ and because the length of I_n is a_{2n} and $a_{2n} \to 0$, we know that the intersection $\bigcap\{I_n : n \ge 1\}$ is a set with exactly one point, say L, and that $PS(n) \to L$. But that is exactly the definition of the statement $L = \Sigma\{(-1)^n a_n : n \geq 1\}$. $\square$

Example The alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges, but does not converge absolutely.

11 Appendix: Greatest Lower Bounds

A set $S \subseteq \mathbb{R}$ is bounded below if there is a number a having $a \leq a$ for every $s \in S$, and then a is called a lower boind for S.

i **Greatest Lower Bound Axiom**: if E is a nonempty subset of $\mathbb{R}$ that is bounded below, then E has a greatest lower bound (i.e., there is a number m with $m \leq e$ for every $e \in E$ and if $m < y$ then y is not a lower bound for E because some $e \in E$ has $e < y$).

The statements of the LUB axiom and of the GLB axiom would look neater if we did not need to specify that $E \neq \emptyset$, but that is an integral part of the axiom because as proved in an exercise above, the empty set is bounded above but has no LUB.

A set $E \subseteq \mathbb{R}$ is bounded if it both bounded above and bounded below. In other words, a set E is bounded if and only if there exist numbers $a < b$ with $E \subseteq [a, b]$.

We do not need to assume that $\mathbb{R}$ satisfies the GLB Axiom because

Proposition 11.1 *The LUB Axiom implies the GLB axiom, and conversely.*

Proof: Using the LUB axiom, we must show that if $F \subseteq \mathbb{R}$ is non-empty and bounded below, then F has a greatest lower bound. Let E be the set of all lower bounds for F. Then $E \neq \emptyset$ because F has lower bounds, and E is bounded above because $F \neq \emptyset$ and any element of F is an upper bound for E. The LUB axiom now gives us a number L that is the least upper bound of E.

We claim that L is a lower bound for F. Consider any $f \in F$. Then f is an upper bound for E, so that L, being the least of all upper bounds, must have $L \leq f$. Therefore L is a lower bound for F.

Next, we claim that L is the greatest lower bound for F. For suppose $L < y$. If $y \leq f$ for every $f \in F$, then y is a lower bound for F and so $y \in E$. But then $y \leq L$ because L is an upper bound for E, and we are forced to conclude that $L < y \leq L$ which is impossible. Therefore L is the greatest lower bound for F, as required.

The proof of the converse is analogous and is left as an exercise. $\square$

EXERCISE: Show that the GLB axiom implies the LUB axiom.

Exercise: Suppose S is a nonempty set of real numbers and suppose S is bounded below. Explain why the set $T = \{-x : x \in S\}$ is bounded above. How is $lub(T)$ related to $glb(S)$? It might help to draw several examples using different sets S.

12 Appendix: Generalizing the notion of continuity

So far, we have defined continuity only for functions from $\mathbb{R}$ to $\mathbb{R}$, but there is a more general notion. Suppose S is a closed subset of $\mathbb{R}$ and $f : S \to \mathbb{R}$ is a function. We say that f is continuous on S if for every sequence $x_n \in S$ having $x_n \to p$ then $f(x_n) \to f(p)$. Note that the point p must be in S because S is closed; this is important because otherwise $f(p)$ would not be defined.

Exercise Now suppose A and B are closed sets and that $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ are both continuous. Suppose also that $f(x) = g(x)$ for every $x \in A \cap B$. Prove that the two-part function $h : A \cup B \to \mathbb{R}$ defined by $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ is $x \in B$ will be continuous. (Why do we need to know that $f(x) = g(x)$ for every $x \in A \cap B$?)

Exercise Use induction to prove that if $A_1, A_2, \cdots, A_n$ is a finite sequence of closed sets and if $f_i : A_i \to \mathbb{R}$ is continuous for each i, and if $f_i(x) = f_j(x)$ for every $x \in A_i \cap A_j$, then there is a continuous function $h : (A_1 \cup A_2 \cup \cdots \cup A_n) \to \mathbb{R}$ with the property that $h(x) = f_i(x)$ for every $x \in A_i$.

Exercise: Show that the following cannot happen: $[0, 1] \subseteq A \cup B$ where A and B are disjoint closed sets and each contains at least one point of $[0, 1]$. [Hint: Let $C = A \cap [0, 1]$ and $D = B \cap [0, 1]$. Then C and D are closed sets and $C \cup D = [0, 1]$, and there is a continuous function $h : [0, 1] \to \mathbb{R}$ having $h(x) = -1$ is $x \in C$ and $h(x) = +1$ if $x \in D$. Then explain why this violates the Intermediate Value Theorem. Finally, explain why you needed to know that each of A and B contains at least one point of $[0, 1]$.]

Exercise: Suppose $f : [0, 1] \to \mathbb{R}$ is continuous. Show that there is a continuous $F : \mathbb{R} \to \mathbb{R}$ that has $F(x) = f(x)$ for all $x \in [0, 1]$. [Hint: You must define the new function F for $x \leq 0$ and for $x \geq 1$.]

Exercise: We know that the set $\mathbb{Z}$ of all integers is a closed subset of $\mathbb{R}$.

a) Show that every function $f : \mathbb{Z} \to \mathbb{R}$ is continuous.

b) Show that if $f : |IZ \to \mathbb{R}$ is any function, then there is a continuous function $F : \mathbb{R} \to \mathbb{R}$ having $F(n) = f(n)$ for each $n \in \mathbb{Z}$. [Hint: One such F will consist of straight lines of various slopes.]

Exercise: Suppose A is a closed subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$ is continuous. Show that there is a continuous $F : \mathbb{R} \to \mathbb{R}$ with $f(x) = f(x)$ for each $x \in A$. [Hint: The open set $G = \mathbb{R} - A$ is a union of open intervals, no two of which overlap. Where are the endpoints of those open intervals?]

13 Appendix: a non-Archimedean ordered field

Earlier in this chapter we said that the usual properties of $\mathbb{R}$ and $\mathbb{Q}$ are not enough to prove Archimedes' Principle because there are mathematical things that behave like $\mathbb{R}$ from an algebraic point of view and yet do not have the Archimedean property. In this Appendix, we present a sequence of exercises that lead to the construction of such an object. The exercises below tie together to make a project in which you build a mathematical object of a certain kind. This is something that mathematicians do all the time.

A field is a set F with two operations usually called $+$ and $*$ that behave just like the familiar addition and multiplication in the sets $\mathbb{Q}$ and $\mathbb{R}$ of rational and real numbers. In particular, there is an element called the <u>additive identity</u> that is usually denoted by 0 and that satisfies $\forall a \in F,\ a+0=a$ and another element called the <u>multiplicative identity</u> that is usually denoted by 1 and that satisfies $\forall a \in F,\ a*1=a$. In addition, for each $a \in F$ there is some $x \in F$ that satisfies $a+x=0$ and if $b \neq 0$ then there is some $x \in F$ with $b*x=1$. The element x is called the "additive inverse of a" and the element y is called the "multiplicative inverse of b". The operations $+$ and $*$ are commutative and satisfy the distributive law $\forall a,b,c \in F, a*(b+c)=a*b+a*c$. Both $\mathbb{Q}$ and $\mathbb{R}$ with the usual meanings of $+$ and $*$ are fields, and so is the clock arithmetic $\mathbb{Z}_5$ (but not $\mathbb{Z}_6$).

In the rest of this project, we consider only fields with the property that no finite sum of copies of the multiplicative identity 1 *can equal the additive identity* 0. *(Consequently,* $\mathbb{Z}_5$ *is excluded.) Any such field must be infinite.* An <u>ordered field</u> is a field equipped with a linear or total order $<$ that interacts well with the operations $+$ and $*$. For example, if $a<b$ and $0<c$, then $a*c<b*c$ and if $a<b$ and $c<d$ then $a+c<b+d$. For each $n \in \mathbb{N}$ the sum of n-many copies of the multiplicative identity 1 in the field F will be denoted by n. To say that an ordered field is <u>Archimedean</u> means that for every $a \in F$ there is some $n \in F$ with $a<n$ (or, equivalently, if $a>0$ then there is some n such that $\frac{1}{n}<a$, where $\frac{1}{n}$ means the multiplicative inverse of n. The familiar fields $\mathbb{Q}$ and $\mathbb{R}$ are Archimedean ordered fields.

****The goal of this project is to construct an ordered field that is *not* Archimedean.****

Let P be the set of polynomials with rational coefficients, including the zero polynomial $\bar{0}$, and let $P^* = P - \{\bar{0}\}$. Let $S = P \times P^* = \{(a,b) : a \in P, b \in P^*\}$ and define a relation on S by the rule that $(a,b) \sim (c,d)$ if there is some integer N such that $\frac{a(x)}{b(x)} = \frac{c(x)}{d(x)}$ for all $x>N$. For example $(1,x) \sim ((x-2),(x(x-2))$. Remember that a and b are polynomials with $b \neq \bar{0}$, the zero polynomial, so that for large enough values of x, $b(x) \neq 0$.

Exercise 1: $\sim$ is an equivalence relation on F.

Let $cls(a,b)$ denote the equivalence class to which (a,b) belongs. Define that $cls(a,b) \prec cls(c,d)$ if there is some N such that for each $x>N$ we have $\frac{a(x)}{b(x)} < \frac{c(x)}{d(x)}$. [Remember that a,b,c,d are polynomials so that $a(x),b(x),c(x),d(x)$ are real numbers.]

Exercise 2: Show that if $cls(a,b) \prec cls(c,d)$ and $cls(c,d) \prec cls(e,f)$, then $cls(a,b) \prec cls(e,f)$.

Exercise 3: Show that for $(a,b),(c,d) \in S$ exactly one of the following holds: $cls(a,b)=cls(c,d)$ or $cls(a,b) \prec cls(c,d)$ or $cls(c,d) \prec cls(a,b)$.

Let $F = \{cls(a,b) : (a,b) \in S\}$. Define two operations $\oplus$ and $\odot$ on F as follows: for $(cls(a,b), cls(c,d)) \in F$,

$$cls(a,b) \oplus cls(c,d) = cls(ad+bc,bd) \text{ and } cls(a,b) \odot cls(c,d) = cls(ac,bd).$$

Exercise 4: Show that in the previous definition $cls(ad+bc,bd)$ and $cls(ac,bd)$ are both in F.

Exercise 5: Show that both operations $\oplus$ and $\odot$ are well-defined.

Exercise 6: For any rational number r, let $\bar{r}$ be the constant polynomial whose value is r. Show that $cls(\bar{0}, \bar{1})$ is the additive identity for $\oplus$ and that $cls(\bar{1}, \bar{1})$ is the multiplicative identity for $\odot$ in F.

Exercise 7: Given any $cls(a, b) \in F$, find an additive inverse for $cls(a, b)$, and given any $cls(c, d) \neq cls(\bar{0}, \bar{1})$, find a multiplicative inverse for $cls(c, d)$. Where did you use the fact that $cls(c, d) \neq cls(\bar{0}, \bar{1})$?

Exercise 8: Show that if $cls(a, b) \prec cls(c, d)$ and $cls(e, f) \prec cls(g, h)$ then $cls(a, b) \oplus cls(e, f) \prec cls(c, d) \oplus cls(g, h)$.

Exercise 9: Show that if $cls(a, b) \prec cls(c, d)$ and $cls(0, 1) \prec cls(e, f)$ then $cls(a, b) \odot cls(e, f) \prec cls(c, d) \odot cls(e, f)$.

There are a few other things to check (such as that both operations $\oplus$ and $\odot$ are commutative and satisfy the required distributive law). They are easy to check and we will skip them. Therefore we now know that F with the operations $\oplus$ and $\odot$, and with the ordering $\prec$ is an ordered field.

Exercise 10: Show that $(F, \oplus, \odot, \prec)$ is not Archimedean. [Hint: First show that for each $n \in \mathbb{N}$, the sum of n copies of the multiplicative identity $cls(\bar{1}, \bar{1}) \in F$ is $cls(\bar{n}, \bar{1})$ where $\bar{n}$ and $\bar{1}$ are the constant polynomials with values n and 1 respectively. Then find some $cls(a, b) \in F$ with the property that for each $n \in \mathbb{N}$, $cls(\bar{n}, \bar{1}) \prec cls(a, b)$.]

Exercise 11: Prove that the set F is countable.

Exercise 12: Where did we use the fact that in $cls(a, b)$, both a and b are polynomials and not other kinds of functions like $\sin(x)$?

Exercise 13. Later in life you will prove that the countable linearly ordered set $(F, \prec)$ is just a copy of the usually ordered set of rational numbers, meaning that there is a function $g : (\mathbb{Q}, <) \rightarrow (F, \prec)$ that is injective, surjective, and has the property that if $x < y$ in $\mathbb{Q}$ then $g(x) \prec g(y)$ in $(F, \prec)$. It follows that there are at least two ordered fields $(\mathbb{Q}, +, *, <)$ and $(F, \oplus, \odot, \prec)$ that can be constructed in the countable set Q and that are very different from each other because the first is Archimedean and the second is not. Other than $(F, \oplus, \odot, \prec)$, are there any other countable ordered fields that are non-archimedean?