BUILDING HIGHLY CONDITIONAL ALMOST GREEDY AND QUASI-GREEDY BASES IN BANACH SPACES

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Abstract. It is known that for a conditional quasi-greedy basis $\mathcal{B}$ in a Banach space $\mathbb{X}$, the associated sequence $(k_m[\mathcal{B}])_{m=1}^\infty$ of its conditionality constants verifies the estimate $k_m[\mathcal{B}] = \mathcal{O}(\log m)$ and that if the reverse inequality $\log m = \mathcal{O}(k_m[\mathcal{B}])$ holds then $\mathbb{X}$ is non-superreflexive. Indeed, it is known that a quasi-greedy basis in a superreflexive quasi-Banach space fulfils the estimate $k_m[\mathcal{B}] = \mathcal{O}(\log m)^{1-\epsilon}$ for some $\epsilon > 0$. However, in the existing literature one finds very few instances of spaces possessing quasi-greedy basis with conditionality constants "as large as possible." Our goal in this article is to fill this gap. To that end we enhance and exploit a technique developed by Dilworth et al. in [16] and craft a wealth of new examples of both non-superreflexive classical Banach spaces having quasi-greedy bases $\mathcal{B}$ with $k_m[\mathcal{B}] = \mathcal{O}(\log m)$ and superreflexive classical Banach spaces having for every $\epsilon > 0$ quasi-greedy bases $\mathcal{B}$ with $k_m[\mathcal{B}] = \mathcal{O}(\log m)^{1-\epsilon}$. Moreover, in most cases those bases will be almost greedy.

1. Introduction

Let $\mathcal{B} = (\mathbf{x}_j)_{j=1}^\infty$ be a (Schauder) basis for a Banach space $\mathbb{X}$ and let $\mathcal{B}^* = (\mathbf{x}_j^*)_{j=1}^\infty$ be its sequence of coordinate functionals. By default all our bases (and basic sequences) will be assumed to be *semi-normalized*, i.e., $0 < \inf_{j\in\mathbb{N}} \|\mathbf{x}_j\| \le \sup_{j\in\mathbb{N}} \|\mathbf{x}_j\| < \infty$. We will denote by $S_m[\mathcal{B}, \mathbb{X}]$

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the m-th partial sum projection with respect to $\mathcal{B}$, i.e.,

$$S_m[\mathcal{B},\mathbb{X}](f)=\sum_{j=1}^{m}\mathbf{x}_j^*(f)\,\mathbf{x}_j,\quad f\in\mathbb{X}.$$

Given a subset A of $\mathbb{N}$, the *coordinate projection* on A is (when well defined) the linear operator $S_A[\mathcal{B},\mathbb{X}]\colon\mathbb{X}\to\mathbb{X}$ given by

$$f\mapsto\sum_{j\in A}\mathbf{x}_j^*(f)\,\mathbf{x}_j.$$

A basis $\mathcal{B}=(\mathbf{x}_j)_{j=1}^\infty$ is *bimototone* if $\|S_A[\mathcal{B},\mathbb{X}]\|\le 1$ for every interval A of integers in $\mathbb{N}$. Every semi-normalized basis in a Banach space $\mathbb{X}$ is normalized (i.e., $\|x_j\|=1$ for all j) and bimonotone under a suitable renorming of $\mathbb{X}$.

A basis $\mathcal{B}$ is *unconditional* if and only if $\sup_{A \text{ finite}}\|S_A[\mathcal{B},\mathbb{X}]\|<\infty$. Thus, in some sense, the *conditionality* of $\mathcal{B}$ can be measured in terms of the growth of the sequence

$$k_m[\mathcal{B},\mathbb{X}]:=\sup_{|A|\le m}\|S_A[\mathcal{B},\mathbb{X}]\|,\quad m\in\mathbb{N}.$$

Recall that a basis $\mathcal{B}=(\mathbf{x}_j)_{j=1}^\infty$ is said to be *quasi-greedy* if it is semi-normalized and there is a constant C such that

$$\|f-S_F[\mathcal{B},\mathbb{X}](f)\|\le C\|f\| \tag{1.1}$$

whenever $f\in\mathbb{X}$ and $F\subseteq\mathbb{N}$ are such that $|\mathbf{x}_j^*(f)|\le|\mathbf{x}_k^*(f)|$ for all $j\in\mathbb{N}\setminus F$ and all $k\in F$. The least constant C such that (1.1) holds is known as the quasi-greedy constant of the basis (see [4, Remark 4.2]).

The next theorem summarizes the connection that exists between superreflexivity and the conditionality constants of quasi-greedy bases. Recall that a space $\mathbb{X}$ is said to be *superreflexive* if every Banach space finitely representable in $\mathbb{X}$ is reflexive.

Theorem 1.1 (see [6, 7, 16, 24])**.** *Let $\mathbb{X}$ be a Banach space.*

(a) *If $\mathcal{B}$ is a quasi-greedy basis for $\mathbb{X}$ then $k_m[\mathcal{B},\mathbb{X}]\lesssim\log m$ for $m\ge 2$.*
(b) *If $\mathbb{X}$ is non-superreflexive then there is a quasi-greedy basis $\mathcal{B}$ for a Banach space $\mathbb{Y}$ finitely representable in $\mathbb{X}$ with $k_m[\mathcal{B},\mathbb{Y}]\gtrsim\log m$ for $m\ge 2$.*
(c) *If $\mathbb{X}$ is superreflexive and $\mathcal{B}$ is quasi-greedy then there is $0<a<1$ such that $k_m[\mathcal{B}]\lesssim(\log m)^a$ for $m\ge 2$.*
(d) *For every $0<a<1$ there is a quasi-greedy basis $\mathcal{B}$ for a Banach space $\mathbb{Y}$ (namely, $\mathbb{Y}=\ell_2$) finitely representable in $\mathbb{X}$ with $k_m[\mathcal{B},\mathbb{Y}]\gtrsim(\log m)^a$ for $m\ge 2$.*

Theorem 1.1 characterizes both the superreflexivity and the lack of superreflexivity of a Banach space $\mathbb{X}$ in terms of the growth of the conditionality constants of quasi-greedy bases. It could be argued that the quasi-greedy bases whose existence is guaranteed in parts (b) and (d) lie outside the space $\mathbb{X}$, and that, although this approach is consistent when dealing with "super" properties, in truth it does not tackle the question of the existence of a quasi-greedy basis with large conditionality constants in the space $\mathbb{X}$ itself! Hence this discussion naturally leads to the following two questions relative to a given Banach space:

Question A. Pick a non-superreflexive Banach space $\mathbb{X}$ with a quasi-greedy basis. Is there a quasi-greedy basis $\mathcal{B}$ for $\mathbb{X}$ with $k_m[\mathcal{B},\mathbb{X}] \approx \log m$ for $m \geq 2$?

Question B. Pick a non-superreflexive Banach space $\mathbb{X}$ with a quasi-greedy basis. Given $0 < a < 1$, is there a quasi-greedy basis $\mathcal{B}$ for $\mathbb{X}$ with $k_m[\mathcal{B},\mathbb{X}] \gtrsim (\log m)^a$ for $m \geq 2$?

Questions A and B can be regarded as a development of the query initiated by Konyagin and Telmyakov in 1999 [30] of finding conditional quasi-greedy bases in general Banach spaces, and which has evolved towards the more specific quest of finding quasi-greedy bases "as conditional as possible." The reader will find a detailed account of this process in the papers [7, 16, 19, 23, 28, 40].

Let us outline the state of the art of those two questions. Garrigós and Wojtaszczyk proved in [24] that Question B has a positive answer for $\mathbb{X} = \ell_p$, $1 < p < \infty$. As for Question A, it is known that Lindenstrauss' basic sequence in ℓ_1, the Haar system in $\mathrm{BV}(\mathbb{R}^d)$ for $d \geq 2$, and the unit-vector system in the Konyagin-Telmyakov space $KT(\infty, p)$ for $1 < p < \infty$, are all quasi-greedy basic sequences with conditionality constants as large as possible (see [12,23]). Moreover, in [24] it is proved that the answer to Question A is positive for $\ell_1 \oplus \ell_2 \oplus c_0$, and in [6] that the same holds true for mixed-norm spaces of the form $(\bigoplus_{n=1}^\infty \ell_1^n)_q$ $(1 < q < \infty)$, providing this way the first-known examples of reflexive Banach spaces having quasi-greedy bases with conditionality constants as large as possible. More recently, the authors constructed in [7] the first-known examples of Banach spaces of nontrivial type and nontrivial cotype for which the answer to Question A is positive. These spaces are $\mathcal{W}^0_{p,q} \oplus \mathcal{W}^0_{p,q} \oplus \ell_2$, $1 < p, q < \infty$, where $\mathcal{W}^0_{p,q}$ and $\mathcal{W}_{p,q}$ are the interpolation spaces

$$\mathcal{W}^0_{p,q} = (v^0_1, c_0)_{\theta,q}, \quad \text{and} \quad \mathcal{W}_{p,q} = (v_1, \ell_\infty)_{\theta,q}, \quad p = 1/(1-\theta), \quad (1.2)$$

defined from the space of sequences of bounded variation

$$v_1 = \{(a_j)_{j=1}^\infty \colon |a_1| + \sum_{j=2}^\infty |a_j - a_{j-1}| < \infty\},$$

and the subspace v_1^0 of v_1 resulting from the intersection of v_1 with c_0. Here and throughout this paper, $(\mathbb{X}_0, \mathbb{X}_1)_{\theta,q}$ denotes the Banach space obtained by applying the real interpolation method to the Banach couple $(\mathbb{X}_0, \mathbb{X}_1)$ with indices θ and q. Let us recall that, in light of [16, Theorem 8.5], the only quasi-greedy basis for a Banach space whose dual has the Grothendieck Theorem Property is the unit vector basis of c_0. Consequently, the answer to Question A is negative for c_0. Any other $\mathcal{L}_\infty$-space, despite having a basis (see [27, Theorem 5.1]), does not have a quasi-greedy basis.

In this article we develop the necessary machinery that permits to extend the scant list of known Banach spaces for which the answer either to Question A or to Question B is positive. Moreover, in a wide class of Banach spaces, the examples of bases we provide are not only quasi-greedy but are *almost greedy*. Recall that a basis $\mathcal{B} = (\mathbf{x}_j)_{j=1}^\infty$ for a Banach space $\mathbb{X}$ is *almost greedy* if there is a constant C such that

$$\|f - S_F[\mathcal{B}, \mathbb{X}](f)\| \le C \|f - S_A[\mathcal{B}, \mathbb{X}](f)\|$$

whenever $f \in \mathbb{X}$, $|A| \le |F| < \infty$, and $|\mathbf{x}_j^*(f)| \le |\mathbf{x}_k^*(f)|$ for any $j \in \mathbb{N} \setminus F$ and $k \in F$. Almost greedy bases were characterized in [17] as those bases that are simultaneously quasi-greedy and democratic. In fact, in this characterization democracy can be replaced with super-democracy. A basis $\mathcal{B}$ is *super-democratic* if there is a sequence $(\lambda_m)_{m=1}^\infty$ such that

$$\left\| \sum_{j\in A} \varepsilon_j \mathbf{x}_j \right\| \approx \lambda_{|A|}$$

for any $A \subseteq \mathbb{N}$ finite and any $(\varepsilon_j)_{j\in A}$ sequence of signs, in which case $(\lambda_m)_{m=1}^\infty$ is equivalent to its fundamental function, defined by

$$\varphi_m[\mathcal{B}, \mathbb{X}] = \sup_{|A|\le m} \left\| \sum_{j\in A} \mathbf{x}_j \right\|, \quad m \in \mathbb{N}.$$

A more demanding concept than super-democracy is that of bi-democracy. A basis $\mathcal{B}$ is said to be *bi-democratic* if

$$\varphi_m[\mathcal{B}, \mathbb{X}]\, \varphi_m[\mathcal{B}^*, \mathbb{Y}] \lesssim m \text{ for } m \in \mathbb{N},$$

where $\mathbb{Y}$ is the closed linear span of the basic sequence $\mathcal{B}^*$ in $\mathbb{X}^*$.

Our study includes, among other spaces, the finite direct sums

$$D_{p,q} := \begin{cases} \ell_p \oplus \ell_q \text{ if } 1 \le p, q < \infty, \\ \ell_p \oplus c_0 \text{ if } 1 \le p < \infty \text{ and } q = 0, \end{cases}$$

the matrix spaces

$$Z_{p,q} := \begin{cases} \ell_q(\ell_p) \text{ if } 1 \le p, q < \infty, \\ c_0(\ell_p) \text{ if } 1 \le p < \infty \text{ and } q = 0, \\ \ell_q(c_0) \text{ if } p = 0 \text{ and } 1 \le q < \infty, \end{cases}$$

and the mixed-norm spaces of the family

$$B_{p,q} := (\oplus_{n=1}^\infty \ell_p^n)_q,\ p \in [1, \infty],\ q \in \{0\} \cup [1, \infty),\ q \ne 0 \text{ when } p = \infty.$$

We use $(\oplus_{n=1}^\infty \mathbb{X}_n)_q$ to denote the direct sum of the Banach spaces $\mathbb{X}_n$ in the ℓ_q-sense (in the c_0-sense if $q = 0$).

In this article we shall prove that the answer to Question A is positive for all non-superreflexive spaces in the aforementioned list. We also show that in all superreflexive spaces in the above list the answer to Question B is positive. These results will appear in Section 4. Previously, in Sections 2 and 3, we introduce the tools that we will use to achieve our goal.

Throughout this article we follow standard Banach space terminology and notation as can be found, e.g., in [8]. In what follows we would like to single out the notation and terminology that is more commonly employed. We deal with real or complex Banach spaces, and $\mathbb{F}$ will denote the underlying scalar field. A weight will be a sequence of positive scalars. Given families of positive real numbers $(\alpha_i)_{i\in I}$ and $(\beta_i)_{i\in I}$, the symbol $\alpha_i \lesssim \beta_i$ for $i \in I$ means that $\sup_{i\in I} \alpha_i/\beta_i < \infty$. If $\alpha_i \lesssim \beta_i$ and $\beta_i \lesssim \alpha_i$ for $i \in I$ we say $(\alpha_i)_{i\in I}$ are $(\beta_i)_{i\in I}$ are equivalent, and we write $\alpha_i \approx \beta_i$ for $i \in I$. Applied to Banach spaces, the symbol $\mathbb{X} \approx \mathbb{Y}$ means that the spaces $\mathbb{X}$ and $\mathbb{Y}$ are isomorphic, while the symbol $\mathbb{X} \lesssim_c \mathbb{Y}$ means that $\mathbb{X}$ is isomorphic to a complemented subspace of $\mathbb{Y}$. The norm of a linear operator $T\colon \mathbb{X} \to \mathbb{Y}$ will be denoted either by $\|T\|_{\mathbb{X}\to\mathbb{Y}}$ or, when the Banach spaces $\mathbb{X}$ and $\mathbb{Y}$ are clear from context, simply by $\|T\|$. Given families of Banach spaces $(\mathbb{X}_i)_{i\in I}$ and $(\mathbb{Y}_i)_{i\in I}$, the symbol $\mathbb{X}_i \lesssim_c \mathbb{Y}_i$ for $i \in I$ means that the spaces $\mathbb{X}_i$ are uniformly isomorphic to complemented subspaces of $\mathbb{Y}_i$, i.e., there are linear operators $L_i\colon \mathbb{X}_i \to \mathbb{Y}_i$, $T_i\colon \mathbb{Y}_i \to \mathbb{X}_i$ such that $T_i \circ L_i = \mathrm{Id}_{\mathbb{X}_i}$ and $\sup_i \|T_i\|\|L_i\| < \infty$. Similarly the symbol $\mathbb{X}_i \approx \mathbb{Y}_i$ for $i \in I$ means that Banach-Mazur distance from $\mathbb{X}_i$ to $\mathbb{Y}_i$ is uniformly bounded. We write $\mathbb{X} \oplus \mathbb{Y}$ for the Cartesian product of the Banach spaces $\mathbb{X}$ and $\mathbb{Y}$

endowed with the norm

$$\|(x,y)\|_{\mathbb{X}\oplus\mathbb{Y}} = \|x\| + \|y\|, \quad x \in \mathbb{X},\ y \in \mathbb{Y}.$$

As it is customary, we put $\delta_{j,k} = 1$ if $j = k$ and $\delta_{j,k} = 0$ otherwise. Given $j \in \mathbb{N}$, the j-th unit vector is defined by $\mathbf{e}_j = (\delta_{j,k})_{k=1}^\infty$ and the *unit-vector system* will be sequence $\mathcal{E} := (\mathbf{e}_j)_{j=1}^\infty$. We denote by $\mathbf{e}_j^*$ the j-th coordinate functional defined on $\mathbb{F}^\mathbb{N}$ by $(a_k)_{k=1}^\infty \mapsto a_j$, and by $S_A \colon \mathbb{F}^\mathbb{N} \to \mathbb{F}^\mathbb{N}$ the coordinate projection on a set $A \subseteq \mathbb{N}$. Given $m \in \mathbb{N}$, S_m will be be coordinate projection on the set $\{1, \dots, m\}$.

The linear span of a family $(x_i)_{i\in I}$ in a Banach space will be denoted by $\langle x_i \colon i \in I\rangle$, and its closed linear span by $[x_i \colon i \in I]$. Other more specific notation will be introduced on the spot when needed.

2. Preliminary results

Most of the ideas behind the results we include in this preliminary section have appeared more or less explicitly in the literature before. Nonetheless, for the sake of clarity and completeness, we shall include the statements of the results we need in the form that best suits our purposes and the sketches of their proofs.

Definition 2.1. Given a basis $\mathcal{B}$ for a Banach space $\mathbb{X}$, we define the sequence $(L_m[\mathcal{B},\mathbb{X}])_{m=1}^\infty$ by

$$L_m[\mathcal{B},\mathbb{X}] = \sup\left\{ \frac{\|S_A[\mathcal{B},\mathbb{X}](f)\|}{\|f\|} \colon \max(\operatorname{supp}(f)) \le m,\ A \subseteq \mathbb{N} \right\}.$$

When the space $\mathbb{X}$ is clear from context we will drop it and simply write $L_m[\mathcal{B}]$. Likewise, we will drop $\mathbb{X}$ from the notation of all the other concepts that we introduced in Section 1 involving a basis $\mathcal{B}$ for a Banach space.

Notice that $\mathcal{B}$ is unconditional if and only if $\sup_m L_m[\mathcal{B}] < \infty$. Hence, the growth of the sequence $(L_m[\mathcal{B}])_{m=1}^\infty$ provides also a measure of the conditionality of the basis. Since $L_m[\mathcal{B}] \le k_m[\mathcal{B}]$ for all $m \in \mathbb{N}$, any result establishing that the size of the members of the sequence $(L_m[\mathcal{B}])_{m=1}^\infty$ is large, is a stronger statement than the corresponding one enunciated in terms of $(k_m[\mathcal{B}])_{m=1}^\infty$.

The papers [7, 24] draw attention to the fact that, in some cases, the sequence $(L_m[\mathcal{B}])_{m=1}^\infty$ is more fit than the "usual" sequence of conditionality constants $(k_m[\mathcal{B}])_{m=1}^\infty$ for transferring conditionality properties from a given basis to a basis constructed from it. This is the reason why we establish all the instrumental results of this section in terms of the conditionality constants $(L_m[\mathcal{B}])_{m=1}^\infty$. Notice that, in contrast to $(k_m[\mathcal{B}])_{m=1}^\infty$, the sequence $(L_m[\mathcal{B}])_{m=1}^\infty$ is not necessarily doubling.

This leads us to use a doubling function in our statements. Recall that a function $\delta\colon [0,\infty) \to [0,\infty)$ is said to be *doubling* if for some non-negative constant C one has $\delta(2t) \le C\delta(t)$ for all $t \ge 0$.

Given sequences $\mathcal{B}_0 = (\mathbf{x}_j)_{j=1}^\infty$ and $\mathcal{B}_1 = (\mathbf{y}_j)_{j=1}^\infty$ in Banach spaces $\mathbb{X}$ and $\mathbb{Y}$, respectively, their *direct sum* $\mathcal{B}_0 \oplus \mathcal{B}_1$ will be the sequence in $\mathbb{X} \times \mathbb{Y}$ given by

$$\mathcal{B}_0 \oplus \mathcal{B}_1 = ((\mathbf{x}_1, 0), (0, \mathbf{y}_1), (\mathbf{x}_2, 0), (0, \mathbf{y}_2), \dots, ((\mathbf{x}_j, 0), (0, \mathbf{y}_j), \dots).$$

Lemma 2.2 (cf. [23]). *Suppose that $\mathcal{B}_0 = (\mathbf{x}_j)_{j=1}^\infty$ and $\mathcal{B}_1 = (\mathbf{y}_j)_{j=1}^\infty$ are bases for some Banach spaces $\mathbb{X}$ and $\mathbb{Y}$, respectively. Assume that $L_m[\mathcal{B}_0] \gtrsim \delta(m)$ for $m \in \mathbb{N}$ for some non-decreasing doubling function $\delta\colon [0,\infty) \to [0,\infty)$. Then $\mathcal{B}_0 \oplus \mathcal{B}_1$ is a basis for $\mathbb{X} \oplus \mathbb{Y}$ with $L_m[\mathcal{B}_0 \oplus \mathcal{B}_1] \gtrsim \delta(m)$ for $m \in \mathbb{N}$. Moreover:*

(a) *If $\mathcal{B}_0$ and $\mathcal{B}_1$ are quasi-greedy so is $\mathcal{B}_0 \oplus \mathcal{B}_1$.*
(b) *If $\mathcal{B}_0$ and $\mathcal{B}_1$ are super-democratic, both with fundamental function equivalent to $(\lambda_m)_{m=1}^\infty$, then $\mathcal{B}_0 \oplus \mathcal{B}_1$ is super-democratic with fundamental function equivalent to $(\lambda_m)_{m=1}^\infty$*

Proof. It is similar to the proof of [23, Proposition 6.1], so we omit it. □

Our next lemma follows an idea from [40] for constructing quasi-greedy bases. To state it properly, it will be convenient to introduce some additional notation. Given $N \in \mathbb{N}$ and a Banach space $\mathbb{X} \subseteq \mathbb{F}^\mathbb{N}$ for which the unit-vector system is a basis, $\mathbb{X}^{(N)}$ will be the N-dimensional space $[\mathbf{e}_j \colon 1 \le j \le N]$ regarded as a subspace of $\mathbb{X}$. As it is customary, we will write

$$\ell_p^N := \ell_p^{(N)},\ 1 \le p < \infty, \quad \text{and} \quad \ell_\infty^N := c_0^{(N)}.$$

More generally, given a basis $\mathcal{B} = (\mathbf{x}_j)_{j=1}^\infty$ for a Banach space $\mathbb{X}$ and $N \in \mathbb{N}$ we will consider the closed linear span of the truncated finite sequence $(\mathbf{x}_j)_{j=1}^N$, i.e.,

$$\mathbb{X}^{(N)}[\mathcal{B}] = [\mathbf{x}_j \colon 1 \le j \le N].$$

Let $\mathcal{B} = (\mathbf{x}_j)_{j=1}^\infty$ be a basis in a Banach space $\mathbb{X}$. Given a sequence of positive integers $(N_n)_{n=1}^\infty$ we define a sequence $(\mathbf{z}_k)_{k=1}^\infty$ that we denote $\bigoplus_{n=1}^\infty (\mathbf{x}_j)_{j=1}^{N_n}$ in the space $\mathbb{X}^\mathbb{N}$ by

$$\mathbf{z}_k = (\underbrace{0, \dots, 0}_{r-1 \text{ times}}, \mathbf{x}_j, 0, \dots, 0, \dots),$$

where, for a given $k \in \mathbb{N}$, the integers r and j are univocally determined by the relations $k = j + \sum_{n=1}^{r-1} N_n$ and $1 \le j \le N_n$.

Lemma 2.3. *Suppose that $\mathcal{B} = (\mathbf{x}_j)_{j=1}^{\infty}$ is a basis for a Banach space $\mathbb{X}$ with $L_m[\mathcal{B}] \gtrsim \delta(m)$ for $m \in \mathbb{N}$, for some non-decreasing doubling function $\delta\colon [0,\infty) \to [0,\infty)$. Let $(N_n)_{n=1}^{\infty}$ be a sequence of positive integers verifying*

$$M_r := \sum_{n=1}^{r} N_n \lesssim N_{r+1} \text{ for } r \in \mathbb{N}.$$

Then

(a) *The sequence $\mathcal{B}_0 = \bigoplus_{n=1}^{\infty} (\mathbf{x}_j)_{j=1}^{N_n}$ is a basis for the Banach space $(\bigoplus_{n=1}^{\infty} \mathbb{X}^{(N_n)}[\mathcal{B}])_p$, $p \in \{0\} \cup [1,\infty)$, with $L_m[\mathcal{B}_0] \gtrsim \delta(m)$ for $m \in \mathbb{N}$.*
(b) *If $\mathcal{B}$ is quasi-greedy so is $\mathcal{B}_0$.*
(c) *If $\mathcal{B}$ is super-democratic with fundamental function equivalent to $(m^{1/p})_{m=1}^{\infty}$ for some $1 \le p < \infty$, so is $\mathcal{B}_0$.*

Proof. It is clear that $\mathcal{B}_0$ is a quasi-greedy basis with the same quasi-greedy constant as $\mathcal{B}$, hence we need only take care of obtaining an estimate for $(L_m[\mathcal{B}_0])_{m=1}^{\infty}$. Let C_1 be such that $C_1 L_m[\mathcal{B}] \ge \delta(m)$ for all $m \in \mathbb{N}$. Let $C_2 > 1$ be such that $M_r \le C_2 N_r$ for all $r \in \mathbb{N}$. Since δ is doubling there is a constant C_3 such that $C_3 \delta(m) \ge \delta(2C_2 m)$ for all $m \in \mathbb{N}$. Given $m \ge M_1$, pick $r \in \mathbb{N}$ such that $M_r \le m < M_{r+1}$. We have

$$\begin{aligned} C_1 C_3 L_m[\mathcal{B}_0] &\ge C_1 C_3 \max\{L_{N_r}[\mathcal{B}], L_{m-M_r}[\mathcal{B}]\} \\ &\ge C_3 \max\{\delta(N_r), \delta(m-M_r)\} \\ &\ge \max\{\delta(2C_2 N_r), \delta(2C_2(m-M_r))\} \\ &\ge \max\{\delta(2M_r), \delta(2m-2M_r))\} \\ &= \delta(\max\{2M_r, 2m-2M_r\}) \\ &\ge \delta(m), \end{aligned}$$

as desired. □

The spaces $\mathcal{W}_{p,q}^0$ and $\mathcal{W}_{p,q}$ ($1 < p < \infty$, $1 \le q < \infty$) defined in (1.2) were introduced and studied by Pisier and Xu [36]. It is verified that $\mathcal{W}_{p,q}^0 \approx \mathcal{W}_{p,q}$. Moreover, when $q > 1$ these spaces have nontrivial type and nontrivial cotype and they are pseudo-reflexive.

Our next proposition is a new addition to the study of Pisier-Xu spaces, which will be used below. Recall that given $1 \le q < \infty$ and a scalar sequence $\mathbf{w} = (w_n)_{n=1}^{\infty}$, the Lorentz sequence space $d_q(\mathbf{w})$ consists of all sequences f in c_0 whose non-increasing rearrangement

$(a_n^*)_{n=1}^\infty$ verifies

$$\|f\|_{d_q(\mathbf{w})} = \left(\sum_{n=1}^\infty (a_n^*)^q w_n \right)^{1/q} < \infty.$$

In the case when $\mathbf{w} = (n^{q/p-1})_{n=1}^\infty$ for some $1 \le p < \infty$ we have that $\ell_{p,q} := d_q(\mathbf{w})$ is the classical sequence Lorentz space of indices p and q.

Proposition 2.4. *Let $1 < p < \infty$ and $1 \le q < \infty$. Then*

$$\ell_{p,q} \lesssim_c \mathcal{W}_{p,q}^0.$$

In fact, $(\mathbf{e}_{2j-1})_{j=1}^\infty$ is a complemented basic sequence isometrically equivalent to the unit vector system in $\ell_{p,q}$.

Proof. Put $p = 1/(1-\theta)$. Consider the linear maps $L, T \colon \mathbb{F}^\mathbb{N} \to \mathbb{F}^\mathbb{N}$ defined by

$$L((a_j)_{j=1}^\infty) = (a_1, 0, a_2, 0, \dots, a_j, 0, a_{j+1}, 0, \dots), \tag{2.1}$$

$$T((a_j)_{j=1}^\infty) = (a_1 - a_2, a_3 - a_4, \dots, a_{2j-1} - a_{2j}, \dots). \tag{2.2}$$

We have $\|L \colon \ell_1 \to v_1^0\| \le 1$, $\|L \colon c_0 \to c_0\| \le 1$, $\|T \colon v_1^0 \to \ell_1\| \le 1$, and $\|T \colon c_0 \to c_0\| \le 2$. Taking into account that

$$(\ell_1, c_0)_{\theta,q} = (\ell_1, \ell_\infty)_{\theta,q} = \ell_{p,q}$$

(see, e.g., [13, Theorem 1.9]), interpolation gives $\|L \colon \ell_{p,q} \to \mathcal{W}_{p,q}^0\| \le 1$ and $\|T \colon \mathcal{W}_{p,q}^0 \to \ell_{p,q}\| \le 2^\theta$. Since $T(L(f)) = f$ for every $f \in \mathbb{F}^\mathbb{N}$ we are done. □

Corollary 2.5. *Let $1 < p < \infty$ and $1 \le q < \infty$. Then $\ell_q \lesssim_c \mathcal{W}_{p,q}^0$.*

Proof. In light of Proposition 2.4, it suffices to see that $\ell_q \lesssim_c \ell_{p,q}$. By [33, Proposition 4] we have $\ell_q \lesssim_c \ell_{p,q}$ if $q \le p$ and $\ell_{q'} \lesssim_c \ell_{p',q'}$ otherwise. We conclude the proof by dualizing (see [10, Theorem 1]). □

3. The Dilworth-Kalton-Kutzarova method, revisited

Recall that a basis is said to be *subsymmetric* if it is unconditional and equivalent to all of its subsequences. If $(\mathbf{x}_j)_{j=1}^\infty$ is a subsymmetric basis in a Banach space $(\mathbb{S}, \|\cdot\|_\mathbb{S})$ then there is a constant C such that

$$C^{-1} \left\| \sum_{j=1}^\infty a_j \mathbf{x}_j \right\|_\mathbb{S} \le \left\| \sum_{j=1}^\infty \varepsilon_j a_j \mathbf{x}_{\phi(j)} \right\|_\mathbb{S} \le C \left\| \sum_{j=1}^\infty a_j \mathbf{x}_j \right\|_\mathbb{S}$$

for all $\sum_{j=1}^\infty a_j \mathbf{x}_j \in \mathbb{S}$, all sequence of signs $(\varepsilon_j)_{j=1}^\infty$, and all increasing maps ϕ, in which case the basis is said to be C-subsymmetric. Every subsymmetric basis is quasi-greedy and super-democratic, hence almost greedy.

A *subsymmetric sequence space* will be a Banach space $\mathbb{S} \subseteq \mathbb{F}^{\mathbb{N}}$ for which the unit-vector system is a 1-subsymmetric basis. Note that every Banach space equipped with a subsymmetric basis is isomorphic to a subsymmetric sequence space (see, e.g., [11]).

Given $f \in \mathbb{F}^{\mathbb{N}}$ and $A \subseteq \mathbb{N}$ finite, we put

$$\operatorname{Av}(f, A) = \frac{1}{|A|} \left(\sum_{j \in A} a_j \right).$$

The *averaging projection* with respect to a sequence $\sigma = (\sigma_n)_{n=1}^{\infty}$ of disjoint finite subsets of $\mathbb{N}$ is the map $P_\sigma \colon \mathbb{F}^{\mathbb{N}} \to \mathbb{F}^{\mathbb{N}}$ defined by

$$(a_j)_{j=1}^{\infty} \mapsto (b_k)_{k=1}^{\infty}, \quad b_k = \operatorname{Av}(f, \sigma_n) \text{ if } k \in \sigma_n.$$

We denote by $Q_\sigma = \operatorname{Id}_{\mathbb{F}^{\mathbb{N}}} - P_\sigma$ its complementary projection. A sequence $(\sigma_n)_{n=1}^{\infty}$ of subsets of $\mathbb{N}$ is said to be *ordered* when

$$\max(\sigma_n) < \min(\sigma_{n+1}), \quad n \in \mathbb{N}.$$

Note that any ordered sequence consists of disjoint finite subsets. Let us recall the following result (see [32, Propostion 3.a.4]).

Theorem 3.1. *Let $(\mathbb{S}, \|\cdot\|_{\mathbb{S}})$ be a subsymmetric sequence space and σ an ordered sequence of subsets of $\mathbb{N}$. Then P_σ is bounded from $\mathbb{S}$ into $\mathbb{S}$ with $\|P_\sigma\|_{\mathbb{S}\to\mathbb{S}} \le 2$.*

An *ordered partition* of $\mathbb{N}$ will be an ordered sequence $(\sigma_n)_{n=1}^{\infty}$ of subsets of $\mathbb{N}$ with $\mathbb{N} = \cup_n \sigma_n$. Notice that ordered partitions consist of integer intervals. In fact, if $\sigma = (\sigma_n)_{n=1}^{\infty}$ and we let

$$M_r = \sum_{n=1}^{r} |\sigma_n|, \quad r \in \mathbb{N} \cup \{0\},$$

we have $\sigma_n = [1 + M_{n-1}, M_n]$. Theorem 3.1 immediately gives

$$\|P_\sigma(f)\|_{\mathbb{S}} \le 2\|f\|_{\mathbb{S}}, \quad \|Q_\sigma(f)\|_{\mathbb{S}} \le 3\|f\|_{\mathbb{S}}, \quad f \in c_{00} \tag{3.1}$$

for any ordered partition σ and any subsymmetric sequence space $\mathbb{S}$. Another consequence of Theorem 3.1 is that subsymmetric bases are bi-democratic. To be precise for any subsymmetric sequence space $(\mathbb{S}, \|\cdot\|_{\mathbb{S}})$ we have (see [32, Proposition 3.a.6])

$$m \le \left\| \sum_{j \in 1}^{m} \mathbf{e}_j \right\|_{\mathbb{S}} \left\| \sum_{j=1}^{m} \mathbf{e}_j^* \right\|_{\mathbb{S}^*} \le 2m, \quad m \in \mathbb{N}. \tag{3.2}$$

Remark 3.2. Notice that if $\mathcal{B}$ is a subsymmetric basis for a Banach space $\mathbb{S}$ then $\mathbb{S} \oplus \mathbb{S} \approx \mathbb{S}$. To see this it suffices to consider the subsequences $\mathcal{B}_o$ and $\mathcal{B}_e$ consisting, respectively, of the odd and the even terms of $\mathcal{B}$.

Then, on the one hand, we have that $\mathcal{B}_o \oplus \mathcal{B}_e$ is equivalent to $\mathcal{B}$ and, on the other hand, it is equivalent to $\mathcal{B} \oplus \mathcal{B}$.

Given a subsymmetric sequence space $(\mathbb{S}, \|\cdot\|_{\mathbb{S}})$ and an ordered partition $\sigma = (\sigma_n)_{n=1}^\infty$ of $\mathbb{N}$ we will put

$$\Lambda_m = \left\| \sum_{j=1}^m \mathbf{e}_j \right\|_{\mathbb{S}}, \quad \Lambda_m^* = \frac{m}{\Lambda_m}, \quad m \in \mathbb{N}, \text{ and} \tag{3.3}$$

$$\mathbf{v}_n = \frac{1}{\Lambda_{|\sigma_n|}} \sum_{j \in \sigma_n} \mathbf{e}_j, \quad \mathbf{v}_n^* = \frac{1}{\Lambda_{|\sigma_n|}^*} \sum_{j \in \sigma_n} \mathbf{e}_j^*, \quad n \in \mathbb{N}. \tag{3.4}$$

We have that $(\mathbf{v}_n)_{n=1}^\infty$ is a normalized basic sequence in $\mathbb{S}$ and

$$P_\sigma(f) = \sum_{n=1}^\infty \mathbf{v}_n^*(f)\, \mathbf{v}_n, \quad f \in \mathbb{F}^{\mathbb{N}}.$$

Consequently, by inequality (3.1),

$$\|f\|_{\mathbb{S}} \le \|Q_\sigma(f)\|_{\mathbb{S}} + \left\| \sum_{n=1}^\infty \mathbf{v}_n^*(f)\, \mathbf{v}_n \right\|_{\mathbb{S}} \le 5\|f\|_{\mathbb{S}}, \quad f \in \mathbb{S}. \tag{3.5}$$

That is, the middle term in (3.5) defines an equivalent norm for $\mathbb{S}$. Replacing the basic sequence $(\mathbf{v}_n)_{n=1}^\infty$ with an arbitrary basis provides a method, invented in [16], for constructing Banach spaces with special types of bases. Let us give a precise description of this method. Suppose $\mathcal{B} = (\mathbf{x}_n)_{n=1}^\infty$ is a semi-normalized basis for a Banach space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$. Consider the linear mapping $H\colon c_{00} \to c_{00} \times \mathbb{X}$ given by

$$f \mapsto \left(Q_\sigma(f), \sum_{n=1}^\infty \mathbf{v}_n^*(f)\, \mathbf{x}_n \right), \tag{3.6}$$

and define a gauge on c_{00}, by

$$\|f\|_{\mathcal{B},\mathbb{S},\sigma} = \|H(f)\|_{\mathbb{S}\oplus\mathbb{X}} = \|Q_\sigma(f)\|_{\mathbb{S}} + \left\| \sum_{n=1}^\infty \mathbf{v}_n^*(f)\, \mathbf{x}_n \right\|_{\mathbb{X}}.$$

The authors of [16] considered the sequence Banach space obtained by the completion of $(c_{00}, \|\cdot\|_{\mathcal{B},\mathbb{S},\sigma})$. In order to justify the validity of this procedure, called for short the *DKK-method* from now on, we need a couple of lemmas.

Lemma 3.3. *Let $\mathcal{B}$ be a (semi-normalized) basis of a Banach space $\mathbb{X}$. Let $\mathbb{S}$ be a subsymmetric sequence space and σ be an ordered partition of $\mathbb{N}$. Then there are constants C_1 and C_2 such that*

$$C_1^{-1}\|f\|_{\mathbb{S}} \le \|f\|_{\mathcal{B},\mathbb{S},\sigma} \le C_2\|f\|_{\mathbb{S}},$$

for all sequences f supported on σ_k for some $k \in \mathbb{N}$. Moreover, if $\mathcal{B}$ is normalized, then $C_1 = 1$ and $C_2 = 5$.

Proof. Without loss of generality assume that $\mathcal{B} = (\mathbf{x}_n)_{n=1}^\infty$ is normalized. Let $\mathbf{v}_n$ and $\mathbf{v}_n^*$ be for $n \in \mathbb{N}$ as in (3.4). Let $k \in \mathbb{N}$ and $f \in c_{00}$ be such that $\operatorname{supp} f \subseteq \sigma_k$. Since $\mathbf{v}_n^*(f) = 0$ for $n \neq k$ we have

$$\|f\|_{\mathcal{B},\mathbb{S},\sigma} = \|Q_\sigma(f)\|_{\mathbb{S}} + |\mathbf{v}_k^*(f)|\,\|\mathbf{x}_k\|_{\mathbb{X}} = \|Q_\sigma(f)\|_{\mathbb{S}} + \left\| \sum_{n=1}^\infty \mathbf{v}_n^*(f)\,\mathbf{v}_n \right\|_{\mathbb{S}}.$$

We conclude the proof by appealing to (3.5). □

Lemma 3.4. *Let $\mathbb{S}$ be a subsymmetric sequence space, and $\sigma = (\sigma_n)_{n=1}^\infty$ be an ordered partition of $\mathbb{N}$. Let $A \subseteq \mathbb{N}$ finite. With $(\mathbf{v}_n^*)$ as is (3.4) and $B = \cup_{n\in A}\sigma_n$ we have*

$$\mathbf{v}_n^*(S_B(f)) = \begin{cases} \mathbf{v}_n^*(f) & \text{if } n \in A \\ 0 & \text{otherwise,} \end{cases}$$

and

$$S_B(Q_\sigma(f)) = Q_\sigma(S_B(f))$$

for all $f \in c_{00}$.

Proof. The first equality is clear from the definition and implies that $S_B(P_\sigma(f)) = P_\sigma(S_B(f))$. Thus the second identity holds. □

From Lemma 3.3 and Lemma 3.4 we deduce the existence of a constant C such that

$$|a_k| \le C\|f\|_{\mathcal{B},\mathbb{S},\sigma}$$

for all $k \in \mathbb{N}$ and all $f = (a_j)_{j=1}^\infty \in c_{00}$. We infer that $(c_{00}, \|\cdot\|_{\mathcal{B},\mathbb{S},\sigma})$ is a normed space whose completion can be carried out inside c_0. Thus we can safely give the following definition.

Definition 3.5. Let $\mathcal{B}$ be a basis of a Banach space $\mathbb{X}$. Let $\mathbb{S}$ be a subsymmetric sequence space and σ be an ordered partition of $\mathbb{N}$.

- We define the space $\mathbb{Y}[\mathcal{B},\mathbb{S},\sigma]$ as the completion of $(c_{00}, \|\cdot\|_{\mathcal{B},\mathbb{S},\sigma})$ carried out inside c_0.
- $\mathbb{Y}^{(N)}[\mathcal{B},\mathbb{S},\sigma]$ will be the N-dimensional space $[\mathbf{e}_j \colon 1 \le j \le N]$ regarded as a subspace of $\mathbb{Y}[\mathcal{B},\mathbb{S},\sigma]$.

The next theorem summarizes some early properties of the DKK-method. For the sake of expositional ease we include a sketch of their proof.

Theorem 3.6 (cf. [16])**.** *Let $\mathcal{B} = (\mathbf{x}_j)_{j=1}^\infty$ be a basis for a Banach space $\mathbb{X}$, let $\mathbb{S}$ be a subsymmetric sequence space, and $\sigma = (\sigma_n)_{n=1}^\infty$ be an ordered partition of $\mathbb{N}$. We have:*

(a) *The unit-vector system* $\mathcal{E}=(\mathbf{e}_n)_{n=1}^\infty$ *is a semi-normalized basis for* $\mathbb{Y}[\mathcal{B},\mathbb{S},\sigma]$.
(b) $\mathbb{Y}[\mathcal{B},\mathbb{S},\sigma]\approx Q_\sigma(\mathbb{S})\oplus\mathbb{X}$. *In fact, the mapping* H *in* (3.6) *extends to an isomorphism from* $\mathbb{Y}[\mathcal{B},\mathbb{S},\sigma]$ *onto* $Q_\sigma(\mathbb{S})\oplus\mathbb{X}$.
(c) *If* $M_r=\sum_{n=1}^r|\sigma_n|$ *for* $k\in\mathbb{N}$ *then*

$$\mathbb{Y}^{(M_r)}[\mathcal{B},\mathbb{S},\sigma]\approx Q_\sigma(\mathbb{S}^{(M_r)})\oplus\mathbb{X}^{(r)}[\mathcal{B}].$$

(d) *If* $\mathcal{B}'$ *is a basis for a Banach space* $\mathbb{X}'$ *then (up to an equivalent norm)* $\mathbb{Y}[\mathcal{B},\mathbb{S},\sigma])=\mathbb{Y}[\mathcal{B}',\mathbb{S},\sigma]$ *if and only if the bases* $\mathcal{B}$ *and* $\mathcal{B}'$ *are equivalent.*

Proof. Set $Q=Q_\sigma$ and let H be as in (3.6). By Lemma 3.4 we have

$$H(S_{M_r}(f))=(S_{M_r}(Q(f)),S_r[\mathcal{B}](f)),\quad r\in\mathbb{N},\ f\in c_{00}.$$

Hence, the partial-sum projections $(S_{M_r})_{r=1}^\infty$ are uniformly bounded. Combining with Lemma 3.3 gives (a).

The mapping H is continuous by definition and so it extends to a continuous map $\tilde{H}$ from $\mathbb{Y}[\mathcal{B},\mathbb{S},\sigma]$ into $Q(\mathbb{S})\oplus\mathbb{X}$. Let us prove that $\tilde{H}$ is an onto isomorphism. Put $V=c_{00}\cap Q(\mathbb{S})$, $X=\langle\mathbf{x}_k\colon k\in\mathbb{N}\rangle$, and let L be the linear map from X into c_{00} defined by $L(\mathbf{x}_n)=\mathbf{v}_n$ for all $n\in\mathbb{N}$. It is straightforward to check that the map

$$G\colon V\times X\to c_{00},\quad (g,h)\mapsto g+L(h)$$

is continuous and verifies $G(H(f))=f$ for all $f\in c_{00}$, and also $H(G(g,h))=(g,h)$ for all $g\in V$ and $h\in X$. Hence, in order to obtain (b) it only remains to see that V is a dense subspace of $Q(\mathbb{S})$. Let $f\in Q(\mathbb{S})$ and $\epsilon>0$. Pick $g\in U$ such $\|f-g\|_{\mathbb{S}}\le\varepsilon/\|Q\|$. Since

$$\|f-Q(g)\|_{\mathbb{S}}=\|Q(f)-Q(g)\|_{\mathbb{S}}\le\|Q\|\,\|f-g\|_{\mathbb{S}}\le\epsilon$$

and $Q(g)\in V$ we are done.

We obtain (c) by restricting the isomorphism $\tilde{H}$ to $\mathbb{Y}^{(M_r)}[\mathcal{B},\mathbb{S},\sigma]$.

Assume that $\mathcal{B}'=(\mathbf{x}_n')_{n=1}^\infty$ is a basis for $\mathbb{X}'$. Let L' and G' be the operators corresponding, respectively, to L and G when replacing $\mathcal{B}$ with $\mathcal{B}'$. We have that $\mathbb{Y}[\mathcal{B},\mathbb{S},\sigma]=\mathbb{Y}[\mathcal{B}',\mathbb{S},\sigma]$ if and only if $H\circ G'$ extends to an isomorphism from $Q(\mathbb{S})\oplus\mathbb{X}$ onto $Q(\mathbb{S})\oplus\mathbb{X}'$. It is straightforward to check that

$$H\left(G'\left(g,\sum_{n=1}^\infty a_n\,\mathbf{x}_n\right)\right)=\left(g,\sum_{n=1}^\infty a_n\,\mathbf{x}_n'\right),\quad g\in V,\ (a_n)_{n=1}^\infty\in c_{00}.$$

Hence, $\mathbb{Y}[\mathcal{B},\mathbb{S},\sigma]=\mathbb{Y}[\mathcal{B}',\mathbb{S},\sigma]$ if and only if the mapping

$$\sum_{n=1}^\infty a_n\,\mathbf{x}_n\mapsto\sum_{n=1}^\infty a_n,\mathbf{x}_n',\quad (a_n)_{n=1}^\infty\in c_{00}$$

extends to an isomorphism from $\mathbb{X}$ onto $\mathbb{X}'$. □

Part (d) of Theorem 3.6 alerts us that in the case when the basis $\mathcal{B}$ is conditional so is the unit-vector system of $\mathbb{Y}[\mathcal{B},\mathbb{S},\sigma]$. Indeed, it is possible to obtain a relation between the conditionality constants of both bases.

Lemma 3.7. *Let $\mathcal{B}$ be a basis for a Banach space $\mathbb{X}$, let $\mathbb{S}$ be a subsymmetric sequence space, and $\sigma=(\sigma_n)_{n=1}^\infty$ be an ordered partition of $\mathbb{N}$. If $M_r=\sum_{n=1}^r |\sigma_n|$ for all $r\in\mathbb{N}$, then*

$$L_{M_r}[\mathcal{E},\mathbb{Y}[\mathcal{B},\mathbb{S},\sigma]]\ge L_r[\mathcal{B},\mathbb{X}].$$

Proof. Let Λ_n, $\mathbf{v}_n$, and $\mathbf{v}_n^*$ be defined for all $n\in\mathbb{N}$ as in (3.3) and (3.4). Put $m_0=0$. Given a non-null r-tuple $(a_n)_{n=1}^r$ we define a sequence $f=(b_j)_{j=1}^\infty$ by

$$b_j=\begin{cases} a_n/\Lambda_{|\sigma_n|} & \text{if } j\in\sigma_n \text{ for some } 1\le n\le r,\\ 0 & \text{if } j>M_r.\end{cases}$$

We have $\mathbf{v}_n^*(f)=a_n$ for all $n\in\mathbb{N}$, $P_\sigma(f)=f$ and, then, $Q_\sigma(f)=0$. Consequently, for any $A\subseteq\mathbb{N}$, putting $B=\cup_{n\in A}\sigma_n$, we have

$$L_{M_r}[\mathcal{E},\mathbb{Y}[\mathcal{B},\mathbb{S},\sigma]]\ge\frac{\|S_B(f)\|_{\mathcal{B},\mathbb{S},\sigma}}{\|f\|_{\mathcal{B},\mathbb{S},\sigma}}=\frac{\|S_A(\sum_{n=1}^r a_n\mathbf{x}_n)\|_{\mathbb{X}}}{\|\sum_{n=1}^r a_n\mathbf{x}_n\|_{\mathbb{X}}}.$$

We finish the proof by taking the supremum on $(a_n)_{n=1}^r$. □

Proposition 3.8. *Let $\mathcal{B}$ be a basis for a Banach space $\mathbb{X}$, let $\mathbb{S}$ be a subsymmetric sequence space and $\sigma=(\sigma_n)_{n=1}^\infty$ be an ordered partition of $\mathbb{N}$. Assume that $L_m[\mathcal{B}]\gtrsim\delta(m)$ for $m\in\mathbb{N}$ for some non-decreasing doubling function $\delta\colon[0,\infty)\to[0,\infty)$ and that*

$$\log\left(\sum_{n=1}^r|\sigma_n|\right)\lesssim r \text{ for } r\in\mathbb{N}. \tag{3.7}$$

Then $L_m[\mathcal{E},\mathbb{Y}[\mathcal{B},\mathbb{S},\sigma]]\gtrsim\delta(\log m)$ for $m\in\mathbb{N}$.

Proof. If $M_r=\sum_{n=1}^r|\sigma_n|$, there is $C_1>0$ such that $\log M_r\le C_1(r-1)$ for all $r\ge 2$. Let $C_2\in(0,\infty)$ be such that $\delta(C_1x)\le C_2\delta(x)$ for all $x\ge 0$. Let $C_3\in(0,\infty)$ be such that $\delta(m)\le C_3L_m[\mathcal{B}]$ for all $m\in\mathbb{N}$. Given $m\in\mathbb{N}$ with $m\ge M_1$, pick $r\ge 1$ such that $M_r\le m<M_{r+1}$. Invoking Lemma 3.7 we get

$$\begin{aligned} C_2C_3L_m[\mathcal{E},\mathbb{Y}[\mathcal{B},\mathbb{S},\sigma]] &\ge C_2C_3L_{M_r}[\mathcal{E},\mathbb{Y}[\mathcal{B},\mathbb{S},\sigma]]\\ &\ge C_2C_3L_r[\mathcal{B},\mathbb{X}]\\ &\ge C_2\delta(r)\end{aligned}$$

$$\begin{aligned} &\geq \delta(C_1 r) \\ &\geq \delta(\log M_{r+1}) \\ &\geq \delta(\log m). \end{aligned}$$

□

3.1. **Democracy-like properties from the DKK-method.** The super-democracy of the unit-vector system in sequence spaces obtained by the DKK-method will be inferred by means of embeddings involving Lorentz sequence spaces.

Given a weight $\mathbf{w} = (w_n)_{n=1}^\infty$, the weak Lorentz sequence space $d_1^\infty(\mathbf{w})$ consists of all sequences $f = (a_n)_{n=1}^\infty \in c_0$ whose non-increasing rearrangement $(a_n^*)_{n=1}^\infty$ verifies

$$\|f\|_{d_1^\infty(\mathbf{w})} = \sup_m a_m^* \sum_{n=1}^m w_n = \sup_{a>0} a \sum_{n=1}^{|\{j \colon |a_j|>a\}|} w_n < \infty.$$

It is well-known that an embedding of the form

$$d_1(\mathbf{w}) \subseteq \mathbb{Y} \subseteq d_1^\infty(\mathbf{w})$$

implies that the unit-vector basis of the sequence space $\mathbb{Y}$ is super-democratic with fundamental function equivalent to $(\sum_{n=1}^m w_n)_{m=1}^\infty$. Conversely, any almost greedy basis fulfils such embeddings (see [1, Theorem 3.1]).

In the case when $(\mathbb{S}, \|\cdot\|_\mathbb{S})$ is a subsymmetric sequence space, the sequence $(\Lambda_n)_{n=1}^\infty$ defined as in (3.4) is the fundamental function of the unit-vector system and if $\mathbf{w} = (\Lambda_n - \Lambda_{n-1})_{n=1}^\infty$ then

$$\|f\|_{d_1^\infty(\mathbf{w})} \leq \|f\|_\mathbb{S} \leq \|f\|_{d_1(\mathbf{w})} \tag{3.8}$$

(cf. [5, Theorem 6.1]). Note that while $(\Lambda_n)_{n=1}^\infty$ is non-decreasing, the sequence $(\Lambda_n/n)_{n=1}^\infty$ is non-increasing (see [17, comments below Theorem 3.1]). Of course, this monotonicity can be expressed as

$$\frac{\Lambda_m}{m} \leq \frac{\Lambda_n}{n}, \quad m \geq n, \tag{3.9}$$

but also in the form

$$\Lambda_n - \Lambda_{n-1} \leq \frac{\Lambda_n}{n}, \quad n \in \mathbb{N}. \tag{3.10}$$

Theorem 3.9 (cf. [16, Proposition 6.1])**.** *Let $\mathcal{B}$ be a basis for a Banach space $\mathbb{X}$, let $(\mathbb{S}, \|\cdot\|_\mathbb{S})$ be a subsymmetric sequence space, and $\sigma = (\sigma_n)_{n=1}^\infty$ be an ordered partition of $\mathbb{N}$. Assume that*

$$M_r := \sum_{n=1}^r |\sigma_n| \lesssim |\sigma_{r+1}| \text{ for } r \in \mathbb{N}. \tag{3.11}$$

Let $\mathbf{w} = (\Lambda_n - \Lambda_{n-1})_{n=1}^\infty$ *be as in* (3.4) *and* $\mathbf{w}' = (\Lambda_n/n)_{n=1}^\infty$*. Then*

$$d_1(\mathbf{w}') \subseteq \mathbb{Y}[\mathcal{B}, \mathbb{S}, \sigma] \subseteq d_1^\infty(\mathbf{w})$$

(with continuous embeddings).

Proof. Let $C_\sigma = \sup_r M_r/|\sigma_r|$. Assume that $\mathcal{B} = (\mathbf{x}_n)_{n=1}^\infty$ is a bi-monotone and normalized basis for $(\mathbb{X}, \|\cdot\|_\mathbb{X})$.

Note that if $j \in \sigma_n$ then, by (3.9),

$$\frac{\Lambda_j}{j} \ge \frac{\Lambda_{M_n}}{M_n} \ge \frac{\Lambda_{|\sigma_n|}}{C_\sigma |\sigma_n|}.$$

If $f = (a_j)_{j=1}^\infty \in c_{00}$,

$$\begin{aligned}\left\| \sum_{n=1}^\infty \mathbf{v}_n^*(f) \mathbf{x}_n \right\|_\mathbb{X} &\le \sum_{n=1}^\infty |\mathbf{v}_n^*(f)| \\ &\le \sum_{n=1}^\infty \frac{\Lambda_{|\sigma_n|}}{|\sigma_n|} \sum_{j\in\sigma_n} |a_j| \\ &\le C_\sigma \sum_{j=1}^\infty \frac{\Lambda_j}{j} |a_j|.\end{aligned}$$

Hence, appealing to the rearrangement inequality and to (3.9),

$$\left\| \sum_{n=1}^\infty \mathbf{v}_n^*(f) \mathbf{x}_n \right\|_\mathbb{X} \le C_\sigma \|f\|_{d_1(\mathbf{w}')}.$$

By (3.1), (3.8) and (3.10), we also have

$$\|Q_\sigma(f)\|_\mathbb{S} \le 3\|f\|_\mathbb{S} \le 3\|f\|_{d_1(\mathbf{w})} \le 3\|f\|_{d_1(\mathbf{w}')}.$$

Combining, we get

$$\|f\|_{\mathcal{B},\mathbb{S},\sigma} \le (3 + C_\sigma)\|f\|_{d_1(\mathbf{w}')}.$$

Let us now look at the lower estimate. Suppose $a > 0$ and let $B := \{k \in \mathbb{N} \colon |a_k| > a\}$. Put $t = C_\sigma/(1 + C_\sigma)$ and define

$$A = \{n \in \mathbb{N} \colon |\mathbf{v}_n^*(f)| < ta\Lambda_{|\sigma_n|}\} = \{n \in \mathbb{N} \colon |\operatorname{Av}(f, \sigma_n)| < ta\}.$$

If $k \in B \cap \sigma_n$ and $n \in A$, we have

$$|\mathbf{e}_k^*(Q_\sigma(f))| = |a_k - \operatorname{Av}(f, \sigma_n)| \ge |a_k| - |\operatorname{Av}(f, \sigma_n)| > (1-t)a = \frac{a}{1 + C_\sigma}.$$

Consequently, if $B_0 := B \cap (\cup_{n\in A}\sigma_n)$, we have

$$\|Q_\sigma(f)\|_\mathbb{S} \ge \frac{a\Lambda_{|B_0|}}{1 + C_\sigma}.$$

If m is the largest integer in $\mathbb{N}\setminus A$ we have $B_1 := B\setminus(\cup_{n\in A}\sigma_n)\subseteq \cup_{n=1}^m \sigma_n$. Therefore $|B_1|\le C_\sigma|\sigma_m|$ and then there is an integer N with $\max\{|B_1|,|\sigma_m|\}\le N\le C_\sigma|\sigma_m|$. By (3.9),

$$\Lambda_{|B_1|}\le \Lambda_N \le \frac{N}{|\sigma_m|}\Lambda_{|\sigma_m|}\le \frac{C_\sigma}{ta}|\mathbf{v}_m^*(f)| = \frac{1+C_\sigma}{a}|\mathbf{v}_m^*(f)|.$$

Combining,

$$\|f\|_{\mathcal{B},\mathbb{S},\sigma}\ge \|Q_\sigma(f)\|_{\mathbb{S}}+|\mathbf{v}_m^*(f)|\ge \frac{a}{1+C_\sigma}(\Lambda_{|B_0|}+\Lambda_{|B_1|})\ge \frac{a}{1+C_\sigma}\Lambda_{|B|}.$$

Hence

$$\|f\|_{d_1^\infty(\mathbf{w})}\le (1+C_\sigma)\|f\|_{\mathcal{B},\mathbb{S},\sigma}.$$

□

Following [17], we say that a weight $(\lambda_m)_{m=1}^\infty$ has the *lower regularity property* (LRP for short) if there a positive integer b such

$$2\lambda_m\le \lambda_{bm},\quad m\in\mathbb{N}.$$

We will also need the so-called *upper regularity property* (URP for short). We say that $(\lambda_m)_{m=1}^\infty$ has the URP if there is an integer $b\ge 3$ such that

$$\lambda_{bm}\le \frac{b}{2}\lambda_m,\quad m\in\mathbb{N}.$$

Corollary 3.10. *Assume that the hypotheses of Theorem 3.9 hold and that $(\Lambda_n)_{n=1}^\infty$ has the LRP. Then*

$$d_1(\mathbf{w})\subseteq \mathbb{Y}[\mathcal{B},\mathbb{S},\sigma]\subseteq d_1^\infty(\mathbf{w}).$$

Proof. From the LRP and (3.9) (see, e.g., [1, Lemma 2.12]) we infer that $(\Lambda_n)_{n=1}^\infty$ satisfies the Dini condition

$$\sum_{n=1}^m \frac{\Lambda_n}{n}\le C_d\Lambda_m \text{ for all } m\in\mathbb{N}$$

for some constant C_d. Consequently, $d_1(\mathbf{w})\subseteq d_1(\mathbf{w}')$. Combining with Theorem 3.9 yields

$$d_1(\mathbf{w})\subseteq \mathbb{Y}[\mathcal{B},\mathbb{S},\sigma]\subseteq d_1^\infty(\mathbf{w}).$$

Quantitatively, if $C_\sigma=\sup_r M_r/|\sigma_r|$,

$$\frac{1}{1+C_\sigma}\|f\|_{d_1^\infty(\mathbf{w})}\le \|f\|_{\mathcal{B},\mathbb{S},\sigma}\le (3+C_\sigma)C_d\|f\|_{d_1(\mathbf{w})}.$$

for all $f\in c_{00}$. □

3.2. **Quasi-greediness in bases from the DKK-method.** The embeddings provided by Lemma 3.10 are considered by some authors as a property which ensures in a certain sense the optimality of the compression algorithms with respect to the basis (see [21]). The authors of [1] point out that almost greediness is a stronger condition. In this Section we show that an extra property on the symmetric sequence space, namely URP, fills the gap between those two properties.

Lemma 3.11. *Let* $(\mathbb{S}, \|\cdot\|_{\mathbb{S}})$ *be a subsymmetric sequence space and* $\sigma = (\sigma_n)_{n=1}^\infty$ *be an ordered partition of* $\mathbb{N}$. *Let* Λ_n^* *and* $\mathbf{v}_n^*$ *for* $n \in \mathbb{N}$ *be as in* (3.3) *and* (3.4). *Then*

$$|\mathbf{v}_n^*(S_A(f))| \le 2\frac{\Lambda^*_{|A|}}{\Lambda^*_{|\sigma_n|}} \|S_{\sigma_n}(f)\|_{\mathbb{S}}$$

whenever $n \in \mathbb{N}$, $A \subseteq \sigma_n$ *and* $f \in c_{00}$.

Proof. Assume without loss of generality that $\operatorname{supp}(f) \subseteq \sigma_n$. Let $v^* = \sum_{j\in A} \mathbf{e}_j^*$. Taking into account (3.9) and (3.2) we obtain

$$|\mathbf{v}_n^*(S_A(f))| = \frac{\Lambda_{|\sigma_n|}}{|\sigma_n|} |v^*(f)| \le \frac{\Lambda_{|\sigma_n|}}{|\sigma_n|} \|v^*\|_{\mathbb{S}^*} \|f\|_{\mathbb{S}} \le 2\frac{\Lambda_{|\sigma_n|}}{|\sigma_n|} \frac{|A|}{\Lambda_{|A|}} \|f\|_{\mathbb{S}},$$

as desired. □

For further reference let us write down the following easy result.

Lemma 3.12. *Let* $\sigma = (\sigma_n)_{n=1}^\infty$ *be an ordered partition of* $\mathbb{N}$. *Let* Λ_n *and* $\mathbf{v}_n^*$ *for* $n \in \mathbb{N}$ *be as in* (3.4) *with* $\mathbb{S} = \ell_1$. *Then*

$$|\mathbf{v}_n^*(S_A(f))| \le \frac{\Lambda_{|A|}}{\Lambda_{|\sigma_n|}} |\mathbf{v}_n^*(f)| + \sum_{j\in\sigma_n} |a_j - \operatorname{Av}(f, \sigma_n)|$$

whenever $n \in \mathbb{N}$, $A \subseteq \sigma_n$, *and* $f = (a_j)_{j=1}^\infty \in c_{00}$.

Proof. We have $\Lambda_m = m$ for all m. Hence

$$\begin{aligned} |\mathbf{v}_n^*(S_A(f))| &= \left| \sum_{j\in A} a_j \right| \\ &\le |A| \, |\operatorname{Av}(f, \sigma_n)| + \left| \sum_{j\in A} a_j - \operatorname{Av}(f, \sigma_n) \right| \\ &\le |A| \, |\operatorname{Av}(f, \sigma_n)| + \sum_{j\in\sigma_n} |a_j - \operatorname{Av}(f, \sigma_n)| \\ &= \frac{\Lambda_{|A|}}{\Lambda_{|\sigma_n|}} |\mathbf{v}_n^*(f)| + \sum_{j\in\sigma_n} |a_j - \operatorname{Av}(f, \sigma_n)|. \end{aligned}$$

□

Lemma 3.13. *Let $(\lambda_n)_{n=1}^\infty$ be a non-decreasing sequence of positive scalars such that $(\lambda_n/n)_{n=1}^\infty$ is non-increasing. Then*

$$\frac{k}{k+n}\lambda_n + \frac{n}{k+n}\lambda_k \le 2\lambda_k$$

for all k, $n \in \mathbb{N}$.

Proof. In the case when $k \le n$ we have

$$\frac{k}{k+n}\lambda_n + \frac{n}{k+n}\lambda_k \le \frac{n}{k+n}\lambda_k + \frac{n}{k+n}\lambda_k = \frac{2n}{n+k}\lambda_k \le 2\lambda_k,$$

and in the case $n \le k$,

$$\frac{k}{k+n}\lambda_n + \frac{n}{k+n}\lambda_k \le \frac{k}{k+n}\lambda_k + \frac{n}{k+n}\lambda_k = \lambda_k.$$

□

Lemma 3.14. *Let $\mathbb{S}$ be a subsymmetric sequence space and $\sigma = (\sigma_n)_{n=1}^\infty$ be an ordered partition of $\mathbb{N}$. For all $A \subseteq \mathbb{N}$ and $f \in \mathbb{S}$ we have*

$$\|Q_\sigma(S_A(f))\|_{\mathbb{S}} \le 5\|Q_\sigma(f)\|_{\mathbb{S}} + 2\sum_{n=1}^\infty \frac{\Lambda_{|A_n|}}{\Lambda_{|\sigma_n|}} |\mathbf{v}_n^*(f)|,$$

where $A_n := A \cap \sigma_n$.

Proof. We write $\|\cdot\|$ rather than $\|\cdot\|_{\mathbb{S}}$ as there is no possibility of confusion in this proof. Put $B_n = \sigma_n \setminus A_n$. For every $f \in Q_\sigma(\mathbb{S})$ note that

$$\operatorname{Av}(f, A_n)|A_n| + \operatorname{Av}(f, B_n)|B_n| = 0$$

and, hence, $P_\sigma(S_A(f)) = (y_j)_{j=1}^\infty$, where

$$y_j = \begin{cases} \operatorname{Av}(f, A_n)\dfrac{|A_n|}{|\sigma_n|} & \text{if } j \in A_n \\ -\operatorname{Av}(f, B_n)\dfrac{|B_n|}{|\sigma_n|} & \text{if } j \in B_n. \end{cases}$$

Let P_τ be the averaging projection with respect to the partition $\tau = (A_n, B_n)_{n=1}^\infty$. We infer from Theorem 3.1 that $\|P_\tau\|_{\mathbb{S}\to\mathbb{S}} \le 4$. Taking into account the lattice structure of $\mathbb{S}$ we get

$$\|Q_\sigma(S_A(f))\| \le \|P_\sigma(S_A(f))\| + \|S_A(f)\| \le \|P_\tau(f)\| + \|f\| \le 5\|f\|.$$

Assume that $f \in P_\sigma(\mathbb{S})$. Pick $(b_n)_{n=1}^\infty$ such that $\mathbf{e}_j^*(f) = b_n$ if $j \in \sigma_n$. Then $Q_\sigma(S_A(f)) = (c_j)_{j=1}^\infty$, where

$$c_j = \begin{cases} b_n \dfrac{|B_n|}{|\sigma_n|} & \text{if } j \in A_n \\ -b_n \dfrac{|A_n|}{|\sigma_n|} & \text{if } j \in B_n. \end{cases}$$

Therefore, taking into account Lemma 3.13,

$$\begin{aligned} \|Q_\sigma(S_A(f))\| &\le \sum_{n=1}^\infty |b_n| \left(\frac{|B_n|}{|\sigma_n|} \Lambda_{|A_n|} + \frac{|A_n|}{|\sigma_n|} \Lambda_{|B_n|} \right) \\ &\le 2 \sum_{n=1}^\infty |b_n| \Lambda_{|A_n|} \\ &= 2 \sum_{n=1}^\infty \frac{\Lambda_{|A_n|}}{\Lambda_{|\sigma_n|}} |\mathbf{v}_n^*(f)|. \end{aligned}$$

We complete the proof by expressing any $f \in \mathbb{S}$ in the form $f_1 + f_2$ with $f_1 \in P_\sigma(\mathbb{S})$ and $f_2 \in Q_\sigma(\mathbb{S})$, and combining. □

Lemma 3.15. *Let $(N_n)_{n=1}^\infty$ a sequence of positive integers such that $\sum_{k=1}^n N_k \lesssim N_{n+1}$ for $n \in \mathbb{N}$, and let $(\lambda_n)_{n=1}^\infty$ be a sequence of positive scalars having the LRP. Assume that $(\lambda_n)_{n=1}^\infty$ and $(n/\lambda_n)_{n=1}^\infty$ are non-decreasing. Then*

$$\sup_{r \in \mathbb{N}} \sum_{n=r}^\infty \frac{\lambda_{N_r}}{\lambda_{N_n}} < \infty.$$

Proof. Since $(\lambda_n)_{n=1}^\infty$ has the LRP, appealing to [1, Theorem 2.12] we claim the existence of $0 < \alpha < 1$ and $0 < C_1 < \infty$ such that

$$\frac{\lambda_n}{n^\alpha} \le C_1 \frac{\lambda_m}{m^\alpha} \text{ for } n \le m.$$

Let $C_2 > 1$ be such that $M_r := \sum_{n=1}^r N_n \le C_2 N_r$ for all $r \in \mathbb{N}$. Then, if $t = C_2/(1 + C_2)$, we have $M_{r+1} \le t M_r$ for $r \in \mathbb{N}$. Hence

$$\begin{aligned} \sum_{n=r}^\infty \frac{\lambda_{N_r}}{\lambda_{N_n}} \le \sum_{j=n}^\infty \frac{M_n}{N_n} \frac{\lambda_{N_r}}{\lambda_{M_n}} \le C_2 \sum_{n=r}^\infty \frac{\lambda_{M_r}}{\lambda_{M_n}} \le C_1 C_2 \sum_{n=r}^\infty \left(\frac{M_r}{M_n} \right)^\alpha \\ \le C_1 C_2 \sum_{n=r}^\infty t^{\alpha(n-r)} = C_1 C_2 \frac{1}{1 - t^\alpha}, \end{aligned}$$

as desired. □

Lemma 3.16. *Assume that all the hypotheses of Theorem 3.9 hold and that either* $\mathbb{S} = \ell_1$ *or* $(\Lambda_n)_{n=1}^\infty$ *has both LRP and the URP. Then there exists a constant* C_a *such that whenever* A *and* r *are such that* $A \subset \cup_{n=r}^\infty \sigma_n$ *and* $|A| \le M_r$*, we have* $\|S_A(f)\|_{\mathcal{B},\mathbb{S},\sigma} \le C_a \|f\|_{\mathcal{B},\mathbb{S},\sigma}$ *for all* $f \in c_{00}$.

Proof. Without loss of generality assume that $\mathcal{B} = (\mathbf{x}_n)_{n=1}^\infty$ is normalized and bimonotone. Put $A_n = A \cap \sigma_n$. By assumption there is $r \in \mathbb{N}$ such that $A_n = \emptyset$ for $1 \le n \le r-1$ and, then,

$$\left\| \sum_{n=1}^\infty \mathbf{v}_n^*(S_A(f))\, \mathbf{x}_n \right\|_{\mathbb{X}} \le \sum_{n=1}^\infty |\mathbf{v}_n^*(S_A(f))| = \sum_{n=r}^\infty |\mathbf{v}_n^*(S_{A_n}(f))|.$$

In the case when $(\Lambda_n)_{n=1}^\infty$ has the URP, we infer from inequality (3.9) and [1, Theorem 2.12] that $(\Lambda_n^*)_{n=1}^\infty$ has the LRP. Using the fact that $(\Lambda_n^*)_{n=1}^\infty$ is doubling and Lemma 3.15 gives

$$C_1 := \sup_r \frac{\Lambda^*_{M_r}}{\Lambda^*_{|\sigma_r|}} < \infty, \quad C_2 = \sup_r \sum_{n=r}^\infty \frac{\Lambda^*_{|\sigma_r|}}{\Lambda^*_{|\sigma_n|}} < \infty.$$

Then, by Lemma 3.11, Lemma 3.3 and Lemma 3.4,

$$\begin{aligned}
\sum_{n=r}^\infty |\mathbf{v}_n^*(S_{A_n}(f))| &\le 2 \sum_{n=r}^\infty \frac{\Lambda^*_{|A_n|}}{\Lambda^*_{|\sigma_n|}} \|S_{\sigma_n}(f)\|_{\mathbb{S}} \\
&\le 2 \sum_{n=r}^\infty \frac{\Lambda^*_{M_r}}{\Lambda^*_{|\sigma_n|}} \|S_{\sigma_n}(f)\|_{\mathcal{B},\mathbb{S},\sigma} \\
&\le 2C_1 \|f\|_{\mathcal{B},\mathbb{S},\sigma} \sum_{n=r}^\infty \frac{\Lambda^*_{|\sigma_r|}}{\Lambda^*_{|\sigma_n|}} \\
&\le 2C_1 C_2 \|f\|_{\mathcal{B},\mathbb{S},\sigma}.
\end{aligned}$$

In the case when $\mathbb{S} = \ell_1$, Lemma 3.12 gives

$$\begin{aligned}
\sum_{n=r}^\infty |\mathbf{v}_n^*(S_{A_n}(f))| &\le \sum_{n=r}^\infty \frac{\Lambda_{|A_n|}}{\Lambda_{|\sigma_n|}} |\mathbf{v}_n^*(f)| + \sum_{n=r}^\infty \sum_{j \in \sigma_n} |a_j - \operatorname{Av}(f, \sigma_n)| \\
&\le \|Q_\sigma(f)\|_1 + \sum_{n=r}^\infty \frac{\Lambda_{|A_n|}}{\Lambda_{|\sigma_n|}} |\mathbf{v}_n^*(f)|
\end{aligned}$$

The sequence $(\Lambda_n)_{n=1}^\infty$ is also doubling, and our hypothesis always gives that it has the LRP. Hence, appealing again to Lemma 3.15,

$$C_3 := \sup_r \frac{\Lambda_{M_r}}{\Lambda_{|\sigma_r|}} < \infty, \quad C_4 = \sup_r \sum_{n=r}^\infty \frac{\Lambda_{|\sigma_r|}}{\Lambda_{|\sigma_n|}} < \infty.$$

Consequently,

$$\begin{aligned}\sum_{n=1}^\infty \frac{\Lambda_{|A_n|}}{\Lambda_{|\sigma_n|}} |\mathbf{v}_n^*(f)| &\le \sum_{n=r}^\infty \frac{\Lambda_{M_r}}{\Lambda_{|\sigma_n|}} |\mathbf{v}_n^*(f)| \\ &\le C_3 \sum_{n=r}^\infty \frac{\Lambda_{|\sigma_r|}}{\Lambda_{|\sigma_n|}} |\mathbf{v}_n^*(f)| \\ &\le C_3 C_4 \left\| \sum_{n=1}^\infty \mathbf{v}_n^*(f)\, \mathbf{x}_n \right\|_{\mathbb{X}}.\end{aligned}$$

Using Lemma 3.14 and combining we get the desired result with $C_a = 2C_1C_2 + \max\{5, 2C_3C_4\}$ in the case when (Λ_n) has both the LRP and the URP, and $C_a = \max\{6, 3C_1C_2\}$ when $\mathbb{S} = \ell_1$. □

Theorem 3.17 (cf. [16, Theorem 7.1])**.** *Assume that all the hypotheses of Theorem 3.9 hold and that either* $\mathbb{S} = \ell_1$ *or* $(\Lambda_n)_{n=1}^\infty$ *has both the LRP and the URP. Then the unit-vector system is an almost greedy basis for* $\mathbb{Y}[\mathcal{B}, \mathbb{S}, \sigma]$ *with fundamental function equivalent to* $(\Lambda_n)_{n=1}^\infty$.

Proof. Assume that $\mathcal{B}$ is bi-monotone. By Corollary 3.10, which asserts that for all $f \in c_{00}$ and some positive constants C_1, C_2,

$$\frac{1}{C_1} \|f\|_{d_1^\infty(\mathbf{w})} \le \|f\|_{\mathcal{B},\mathbb{S},\sigma} \le C_2 \|f\|_{d_1(\mathbf{w})},$$

it suffices to prove that $\mathcal{E}$ is a quasi-greedy basis for $\mathbb{Y} := \mathbb{Y}[\mathcal{B}, \mathbb{S}, \sigma]$. Let C_a be as in Lemma 3.16. By Theorem 3.6 (a),

$$C_b := \sup_{m\in\mathbb{N}} \| \operatorname{Id}_{\mathbb{F}^\mathbb{N}} - S_m \|_{\mathbb{Y}\to\mathbb{Y}} < \infty.$$

Let $f = (a_j)_{n=1}^\infty \in c_{00}$ and let $F \subseteq \mathbb{N}$ be a non-empty set such that $|a_j| \le |a_k|$ whenever $k \in F$ and $j \in \mathbb{N} \setminus F$. Denote $m = |F|$ and pick $r \in \mathbb{N}$ such that $m \in \sigma_r$. Let $A = [1, m] \setminus F$ and $B = F \cap [m+1, \infty)$. We have

$$F \cup A = \{1, \dots, m\} \cup B, \quad F \cap A = \{1, \dots, m\} \cap B = \emptyset.$$

Therefore $s := |A| = |B| \le m \le M_r$, $B \subseteq \cup_{n=r}^\infty \sigma_n$ and

$$S_F(f) = S_m(f) - S_A(f) + S_B(f).$$

We infer that $\|S_B(f)\|_{\mathcal{B},\mathbb{S},\sigma} \le C_a \|f\|_{\mathcal{B},\mathbb{S},\sigma}$ and that, if $(a_n^*)_{n=1}^\infty$ is the non-increasing rearrangement of f, $|a_j| \le a_s^*$ for all $j \in A$. Then,

$$\|S_A(f)\|_{\mathcal{B},\mathbb{S},\sigma} \le C_2 \max_{j\in A} |a_j| \Lambda_s \le a_s^* \Lambda_r \le C_1C_2 \|f\|_{\mathcal{B},\mathbb{S},\sigma}.$$

Combining we get

$$\|f - S_F(f)\|_{\mathcal{B},\mathbb{S},\sigma} \le \|f - S_k(f)\|_{\mathcal{B},\mathbb{S},\sigma} + \|S_A(f)\|_{\mathcal{B},\mathbb{S},\sigma} + \|S_B(f)\|_{\mathcal{B},\mathbb{S},\sigma}$$

$$\le (C_b + C_a + C_1C_2)\|f\|_{\mathcal{B},\mathbb{S},\sigma}.$$

That is, the unit vector system is $(C_b + C_a + C_1C_2)$-quasi-greedy. ☐

Remark 3.18. Note that (3.11) implies

$$\log\left(\sum_{n=1}^{r}|\sigma_n|\right) \gtrsim r, \quad r \in \mathbb{N}. \tag{3.12}$$

This exponential growth is essentially optimal. Indeed, Theorem 1.1(a) implies that when the DKK-method is applied to a basis $\mathcal{B}$ for a Banach space $\mathbb{X}$ with $L_m[\mathcal{B},\mathbb{X}] \approx m$ (such as the summing basis of c_0) it can only produce a Banach space $\mathbb{Y}[\mathcal{B},\mathbb{S},\sigma]$ for which the unit-vector basis is quasi-greedy when (3.12) holds.

4. Banach spaces having quasi-greedy bases with large conditionality constants

The conductive thread of this section is the search for results that will allow us to include the spaces $Z_{p,q}$, $B_{p,q}$, and $D_{p,q}$ (see Section 1) in the list of Banach spaces possessing highly conditional quasi-greedy bases. We recall that the matrix spaces $Z_{p,q}$ are isomorphic to Besov spaces over Euclidean spaces (see, e.g., [2]) and that the mixed-norm spaces $B_{p,q}$ are isomorphic to Besov spaces over the unit interval (see, e.g., [3, Appendix 4.2]).

Apart from the trivial cases, namely

$$D_{q,q} \approx Z_{q,q} \approx B_{q,q} \approx \ell_q, \quad 1 \le q < \infty,$$

and the case

$$\ell_q \approx B_{2,q}, \quad 1 < q < \infty, \tag{4.1}$$

all the above-mentioned spaces are mutually non-isomorphic (see [3]). The isomorphism in (4.1) was obtained by Pełczyński in [35] by combining the uniform complemented embeddings

$$\ell_2^n \lesssim_c \ell_p^{2^n} \text{ for } n \in \mathbb{N}, \text{ if } 1 < p < \infty, \tag{4.2}$$

(which can be obtained as a consequence of the boundedness of the Rademacher projections in L_p) with the Pełczyński decomposition technique (see, e.g., [8, Theorem 2.2.3]). Another well-known consequence of Pełczyński decomposition technique, is that for any unbounded sequence of integers $(d_n)_{n=1}^\infty$ we have

$$B_{p,q} \approx (\oplus_{n=1}^\infty \ell_p^{d_n})_q, \quad p \in [1,\infty],\ q \in \{0\} \cup [1,\infty), \tag{4.3}$$

(see, e.g., [3, Appendix 4.1].)

Theorem 4.1. *Let $\mathbb{X}$ be a Banach space with a basis $\mathcal{B}$ and suppose that either $\mathbb{S} = \ell_1$ or $\mathbb{S}$ is a subsymmetric sequence space with nontrivial type. Assume that $\mathbb{S} \lesssim_c \mathbb{X}$ and that $L_m[\mathcal{B}] \gtrsim \delta(m)$ for $m \in \mathbb{N}$ for some doubling non-decreasing function $\delta \colon [0,\infty) \to [0,\infty)$. Then there is an almost greedy basis $\mathcal{B}_\kappa$ for $\mathbb{X}$ with fundamental function equivalent to $(\| \sum_{j=1}^n \mathbf{e}_j \|_{\mathbb{S}})_{n=1}^\infty$, and with $L_m[\mathcal{B}_\kappa] \gtrsim \delta(\log m)$ for $m \in \mathbb{N}$.*

Proof. For each $n \in \mathbb{N}$ put $\sigma_n = [2^{n-1}, 2^n - 1]$, and let $\mathbf{v}_n$ be defined as in (3.4). Notice that $\sigma = (\sigma_n)_{n=1}^\infty$ verifies both (3.11) and (3.7) and that $\mathcal{V} = (\mathbf{v}_n)_{n=1}^\infty$ is an unconditional basis for $P_\sigma[\mathbb{S}]$. Then, by [17, Proposition 4.1], Lemma 2.2, Theorem 3.17 and Proposition 3.8, the unit-vector system is an almost greedy basis as desired for $\mathbb{Y} := \mathbb{Y}[\mathcal{V} \oplus \mathcal{B}, \mathbb{S}, \sigma]$. Let $\mathbb{Z}$ such that $\mathbb{X} \approx \mathbb{S} \oplus \mathbb{Z}$. The chain of isomorphisms

$$\mathbb{Y} \approx Q_\sigma(\mathbb{S}) \oplus P_\sigma(\mathbb{S}) \oplus \mathbb{X} \approx \mathbb{S} \oplus \mathbb{S} \oplus \mathbb{Z} \approx \mathbb{S} \oplus \mathbb{Z} \approx \mathbb{X}$$

(see Remark 3.2) completes the proof. □

Remark 4.2. As the attentive reader may have noticed, the only information we use about the subsymmetric space $\mathbb{S}$ in the proof above in the case when $\mathbb{S} \neq \ell_1$ is that the fundamental function of its canonical basis has both the LRP and URP. Thus, Theorem 4.1 holds replacing the hypothesis "$\mathbb{S}$ has nontrivial type" with the hypothesis "the fundamental function of the subsymmetric basis of $\mathbb{S}$ has both the LRP and the URP."

In order to effectively use Theorem 4.1 we need to ensure the existence of bases $\mathcal{B}$ with large conditionality constants $(L_m[\mathcal{B}])_{m=1}^\infty$ in subsymmetric sequence spaces. We start by recalling the following result by Garrigós and Wojtaszczyk, which, in our language, reads as follows.

Theorem 4.3 (cf. [24, Proposition 3.10])**.** *For each $0 < a < 1$ there is a basis $\mathcal{B}$ in ℓ_2 with $L_m[\mathcal{B}] \gtrsim m^a$ for $m \in \mathbb{N}$.*

Proposition 4.4. *For each $0 < a < 1$ and each $1 < q < \infty$ there is a basis $\mathcal{B}$ in ℓ_q with $L_m[\mathcal{B}] \gtrsim m^a$ for $m \in \mathbb{N}$.*

Proof. Apply Theorem 4.3 for picking a basis $\mathcal{B}_0 = (\mathbf{x}_j)_{j=1}^\infty$ for ℓ_2 with $L_m[\mathcal{B}_0] \gtrsim m^a$ for $m \in \mathbb{N}$. By Lemma 2.3, $\mathcal{B} = \bigoplus_{n=1}^\infty (\mathbf{x}_j)_{j=1}^{2^n}$ is a basis for $\mathbb{X} = (\bigoplus \ell_2^{(2^n)}[\mathcal{B}_0])_p$ with $L_m[\mathcal{B}] \gtrsim m^a$. Since any N-dimensional Hilbert space is isometric to ℓ_2^N, the isomorphisms (4.1) and (4.3) yield $\mathbb{X} \approx (\bigoplus \ell_2^{2^n})_q \approx B_{2,q} \approx \ell_q$. □

Our next result improves Corollary 3.13 from [24], where it is shown the existence of quasi-greedy bases as conditional as possible in ℓ_q,

$1 < q < \infty$. The main improvement consists of building, for $q \neq 2$, almost greedy bases instead of quasi-greedy ones.

Theorem 4.5 (cf. [24, Theorem 1.2, Corollary 3.12 and Corollary 3.13])**.** *Let $\mathbb{X}$ be a Banach space with a basis and $1 < q < \infty$. If $\ell_q \lesssim_c \mathbb{X}$ then for any $0 < a < 1$ the space $\mathbb{X}$ has an almost greedy basis $\mathcal{B}_\kappa$ with fundamental function equivalent to $(m^{1/q})_{m=1}^\infty$ and with $L_m[\mathcal{B}_\kappa] \gtrsim (\log m)^a$ for $m \in \mathbb{N}$.*

Proof. By Lemma 2.2 and Proposition 4.4, $\ell_p \oplus \mathbb{X}$ has a basis $\mathcal{B}$ with $L_m[\mathcal{B}] \gtrsim m^a$ for $m \in \mathbb{N}$. Then, Theorem 4.1 gives a basis as desired for $\ell_q \oplus \mathbb{X}$. Finally, since $\ell_q \oplus \ell_q \approx \ell_q$, we infer that $\ell_q \oplus \mathbb{X} \approx \mathbb{X}$. □

Example 4.6.

(i) Theorem 4.5 applies to L_q and to ℓ_q for $1 < q < \infty$. More generally, Theorem 4.5 yields that if $q \in (1, \infty)$ any separable $\mathcal{L}_q$-space $\mathbb{X}$ has almost greedy bases as conditional as possible whose fundamental function is equivalent to $(m^{1/q})_{m=1}^\infty$ (see [31, Proposition 7.3] and [27, Theorem 5.1]). Moreover, if $\mathbb{X}$ is not isomorphic to ℓ_q it also has almost greedy bases as conditional as possible whose fundamental function is equivalent to $(m^{1/2})_{m=1}^\infty$ (see [26, Corollary 1] and [29, Corollary 1]).

(ii) Let $1 < p, q < \infty$. Theorem 4.5 applies to the superreflexive spaces $Z_{p,q}$, $B_{p,q}$ and $D_{p,q}$.

(iii) Theorem 4.5 also applies to Lorentz sequence spaces. In fact, appealing also to Theorem 4.1 we claim that, given $\mathbf{w} = (w_n)_{n=1}^\infty$ non-increasing, $1 < q < \infty$, and $\epsilon > 0$, the space $d_q(\mathbf{w})$ has almost greedy bases $\mathcal{B}_\kappa$ with $L_m[\mathcal{B}_\kappa] \gtrsim (\log m)^{1-\epsilon}$ and with fundamental function equivalent either to $(m^{1/q})_{m=1}^\infty$ or to $(W_m^{1/q})_{n=1}^\infty$, where $W_m = \sum_{n=1}^m w_n$. In the case when $(W_m)_{m=1}^\infty$ has the LRP, the space $d_q(\mathbf{w})$ is superreflexive (see [9, Theorem 1]) and, then, the estimate of $L_m[\mathcal{B}_\kappa]$ is sharp. In particular, the classical Lorentz sequence spaces $\ell_{p,q}$ for $1 < p, q < \infty$ have almost greedy bases as conditional as possible with fundamental function equivalent either to $(m^{1/q})_{m=1}^\infty$ or to $(m^{1/p})_{m=1}^\infty$.

Before returning to our main theme of almost greedy bases for concrete spaces let us present a more abstract application of Theorem 4.1.

Theorem 4.7. *Let $\mathbb{S}$ be a subsymmetric sequence space with nontrivial type. Then, for every $0 < a < 1$, $\mathbb{S}$ contains an almost greedy basic sequence $\mathcal{B}_\kappa = (\mathbf{x}_n)_{n=1}^\infty$ such that $L_m[\mathcal{B}_\kappa] \gtrsim (\log m)^a$ for $m \in \mathbb{N}$.*

Proof. Combining the Mazur construction of basic sequences (see e.g., [7, Proposition 3.1]) with Dvoretzky's theorem on the finite representability of ℓ_2 in all infinite-dimensional Banach spaces [22] it follows that every infinite-dimensional Banach space contains a basic sequence $\mathcal{B} = (\mathbf{x}_j)_{j=1}^\infty$ such that $[\mathbf{x}_j]_{j=2^n}^{j=2^{n+1}-1}$ is 2-isomorphic to $\ell_2^{2^n}$ for all $n \ge 1$. Moreover, Theorem 4.3 allows us to select the basis vectors $[\mathbf{x}_j]_{j=2^n}^{j=2^{n+1}-1}$ in such a way that we may ensure that $L_m[\mathcal{B}] \gtrsim m^a$ for $m \in \mathbb{N}$. Choose such a basic sequence $\mathcal{B}$ inside $\mathbb{S}$ and let $\mathbb{X}$ be its closed linear span. Setting $\sigma = ([2^{n-1}, 2^n - 1])_{n=1}^\infty$, by Theorem 4.1 the unit-vector basis $\mathcal{E}$ is an almost greedy basis for $\mathbb{Y} = \mathbb{Y}[\mathcal{B}, \mathbb{S}, \sigma]$ satisfying $L_m[\mathcal{E}, \mathbb{Y}] \gtrsim (\log m)^a$ for $m \in \mathbb{N}$. Finally, by Theorem 3.6 (c) and Remark 3.2,

$$\mathbb{Y}[\mathcal{B}, \mathbb{S}, \sigma] \approx Q_\sigma(\mathbb{S}) \oplus \mathbb{X} \subseteq \mathbb{S} \oplus \mathbb{S} \approx \mathbb{S}.$$

□

Remark 4.8. It follows from Theorem 4.7 that if $\mathbb{X}$ has nontrivial type then every spreading model for $\mathbb{X}$ generated by a weakly null sequence contains a basic sequence satisfying the conclusion of Theorem 4.7. This follows because spreading models of $\mathbb{X}$ are finitely represented in $\mathbb{X}$ (and hence have nontrivial type) and those generated by weakly null sequences are subsymmetric.

Example 4.6 shows that Theorem 4.1 is strong enough for a wide class of superreflexive Banach spaces. When dealing with non-superreflexive Banach spaces we need to combine the DKK-method with other techniques. To that end, we recover some ideas from [7]. Recall that a basis $(\mathbf{x}_j)_{j=1}^\infty$ is said to be of *type P* if

$$\sup_k \left\| \sum_{j=1}^k \mathbf{x}_j \right\| < \infty \quad \text{and} \quad \inf_j \|\mathbf{x}_j\| > 0.$$

Notice that both the unit-vector system in c_0 and the *difference system* $\mathcal{D} = (\mathbf{d}_j)_{j=1}^\infty$ in ℓ_1, defined by

$$\mathbf{d}_j = \mathbf{e}_j - \mathbf{e}_{j-1} \text{ (with the convention } \mathbf{e}_0 = 0),$$

are bases of type P. It is known that the *summing system* $\mathcal{S} = (\mathbf{s}_j)_{j=1}^\infty$, given by

$$\mathbf{s}_j = \sum_{k=1}^j \mathbf{e}_j.$$

is a conditional basis for c_0. Most proofs of this fact (see, e.g., [8, Example 3.1.2]) give the following.

Lemma 4.9. *The summing system $\mathcal{S}$ is a basis for c_0 with $L_m[\mathcal{S}, c_0] \approx m$ for $m \in \mathbb{N}$, and $c_0^{(N)}[\mathcal{S}] = \ell_\infty^N$ for all $N \in \mathbb{N}$.*

By duality, the difference system is a conditional basis for ℓ_1. Indeed, we have the following.

Lemma 4.10. *The difference system $\mathcal{D}$ is a basis of ℓ_1 with $L_m[\mathcal{D}, \ell_1] \approx m$ for $m \in \mathbb{N}$, and $\ell_1^{(N)}[\mathcal{D}] = \ell_1^N$ for all $N \in \mathbb{N}$.*

Lemmas 4.9 and 4.10 exhibit the fact that c_0 and ℓ_1 have bases as conditional as possible. The following lemma shows that Banach spaces with a basis of type P follow the pattern of c_0 and ℓ_1.

Lemma 4.11 (see [7]). *Let $\mathcal{B}_0$ be a basis of type P of a Banach space $\mathbb{X}$. Then there is a basis $\mathcal{B}$ for $\mathbb{X}$ such that $L_m[\mathcal{B}] \approx m$ for $m \in \mathbb{N}$ and $\mathbb{X}^{(2^n-2)}[\mathcal{B}] = \mathbb{X}^{(2^n-2)}[\mathcal{B}_0]$ for all $n \geq 2$.*

Proof. The proof of Theorem 3.3 from [7] gives the result, although is not explicitly stated there. □

Theorem 4.12. *Suppose $\mathbb{X}$ is a Banach space with a basis of type P and let $(\mathbb{S}, \|\cdot\|_{\mathbb{S}})$ be a subsymmetric Banach space. Assume that $\mathbb{S} \lesssim_c \mathbb{X}$ and that $\mathbb{S}$ has nontrivial type. Then $\mathbb{X}$ has an almost greedy basis $\mathcal{B}_\kappa$ whose fundamental function is equivalent to $(\|\sum_{j=1}^m \mathbf{e}_j\|_{\mathbb{S}})_{m=1}^\infty$ and such that $L_m[\mathcal{B}_\kappa] \approx \log m$ for $m \geq 2$.*

Proof. Just combine Lemma 4.11 with Theorem 4.1. □

Example 4.13.

(i) By Proposition 2.4 and [7, Proposition 2.10] (which states that the unit-vector system is a basis of type P for Pisier-Xu spaces), Theorem 4.12 applies to $\mathcal{W}_{p,q}^0$ for $1 < p, q < \infty$.

(ii) The unit-vector system is a shrinking basis of type P for the James space $\mathcal{J}^{(p)}$. It is also known that $\ell_p \lesssim_c \mathcal{J}^{(p)}$. Actually, the linear operator L defined in (2.1) is bounded from ℓ_p into $\mathcal{J}^{(p)}$ and the linear operator T defined in (2.2) is bounded from $\mathcal{J}^{(p)}$ into ℓ_p (see also [14, Lemma 3.2]). So, Theorem 4.12 applies both to $\mathcal{J}^{(p)}$ and $(\mathcal{J}^{(p)})^*$.

Theorem 4.14. *Suppose $\mathbb{X}$ is a Banach space with a basis and that $(\mathbb{S}, \|\cdot\|_{\mathbb{S}})$ be a subsymmetric Banach space. Assume that $\mathbb{S} \lesssim_c \mathbb{X}$, that $\mathbb{S}$ has nontrivial type, and that either $\ell_1 \lesssim_c \mathbb{X}$ or $c_0 \lesssim_c \mathbb{X}$. Then $\mathbb{X}$ has an almost greedy basis $\mathcal{B}_\kappa$ whose fundamental function is equivalent to $(\|\sum_{j=1}^m \mathbf{e}_j\|_{\mathbb{S}})_{m=1}^\infty$ and such that $L_m[\mathcal{B}_\kappa] \approx \log m$ for $m \geq 2$.*

Theorem 4.15. *Let $\mathbb{X}$ be a Banach space with a basis. If $\ell_1 \lesssim_c \mathbb{X}$ then $\mathbb{X}$ has an almost greedy basis $\mathcal{B}_\kappa$ with fundamental function equivalent to $(m)_{m=1}^\infty$ and $L_m[\mathcal{B}_\kappa] \approx \log m$ for $m \ge 2$.*

Proof of Theorems 4.14 and 4.15. Since $\ell_1 \oplus \ell_1 \approx \ell_1$ and $c_0 \oplus c_0 \approx c_0$, we infer from Lemma 4.10 (or Lemma 4.9) and Lemma 2.2 that $\mathbb{X}$ has a basis $\mathcal{B}$ with $L_m[\mathcal{B}] \approx m$ for $m \in \mathbb{N}$. Appealing to Theorem 4.1 (we put $\mathbb{S} = \ell_1$ when proving Theorem 4.15) concludes the proof. □

Example 4.16.

(i) Given $1 < p < \infty$, Theorem 4.14 applies to the spaces $D_{p,0}$, $D_{p,1}$, $Z_{1,p}$, $Z_{p,1}$, $Z_{p,0}$, $Z_{0,p}$, and the fundamental function of the bases we obtain is $(m^{1/p})_{m=1}^\infty$.

(ii) Theorem 4.14 also applies to the Hardy space H_1 and its predual VMO. Indeed, these spaces have complemented Hilbertian subspaces (see, e.g., [34]) and so the fundamental functions of the almost greedy bases we obtain are equivalent to $(m^{1/2})_{m=1}^\infty$.

Example 4.17.

(i) The list of Banach spaces for which Theorem 4.15 applies includes $D_{1,p}$, $Z_{p,1}$ and $Z_{1,p}$ for $p \in \{0\} \cup (1,\infty)$, $B_{p,1}$ for $p \in (1,\infty]$, ℓ_1, L_1, the Hardy space H_1, and the Lorentz sequence spaces $d_1(\mathbf{w})$ for $\mathbf{w}$ decreasing.

(ii) By invoking [31, Proposition 7.3] and [27, Theorem 5.1], Theorem 4.15 applies to any separable $\mathscr{L}_1$-space. In this case, since $\mathscr{L}_1$-spaces are GT-spaces (see [31, Theorem 4.1]), the democracy of the bases we obtain is satisfied automatically due to [20, Theorem 4.2].

At this moment, the only reflexive spaces $D_{p,q}$, $Z_{p,q}$ or $B_{p,q}$ not yet in our list of Banach spaces with an almost greedy basis as conditional as possible are Besov spaces $B_{1,q}$ and $B_{\infty,q}$ for $1 < q < \infty$.

Theorem 4.18. *Suppose $\mathbb{X}$ is a Banach space with a basis so that either $B_{\infty,q} \lesssim_c \mathbb{X}$ or $B_{1,q} \lesssim_c \mathbb{X}$ for some $1 < q < \infty$. Then $\mathbb{X}$ has an almost greedy basis $\mathcal{B}_\kappa$ with fundamental function equivalent to $(m^{1/q})_{m=1}^\infty$ and such that $L_m[\mathcal{B}_\kappa] \approx \log m$ for $m \ge 2$.*

Proof. Let $p \in \{1,\infty\}$. Since $\ell_q \lesssim_c \mathbb{X}$ and $\ell_q \oplus B_{p,q} \approx B_{p,q}$, $\mathbb{X}$ has an almost greedy basis whose fundamental function is equivalent to $(m^{1/q})_{m=1}^\infty$, and we have $\mathbb{X} \approx B_{p,q} \oplus \mathbb{X}$. Hence, taking into account Lemma 2.2, it suffices to prove the result for $\mathbb{X} = B_{p,q}$. For $p = 1$, let $\mathcal{B}$ be the difference basis in ℓ_1, whereas for $p = \infty$, let $\mathcal{B}$ be the summing basis of c_0. Let $\sigma = ([2^{n-1}, 2^n - 1])_{n=1}^\infty$. By Lemma 2.3 and

Theorem 4.1, the sequence $\bigoplus_{n=1}^\infty (\mathbf{e}_j)_{j=1}^{2^n-1}$ is an almost greedy basis as desired for $\mathbb{Z} = (\bigoplus_{n=1}^\infty \mathbb{Y}^{(2^n-1)}[\mathcal{B}, \ell_q, \sigma])_q$. By Theorem 3.6 (c),

$$\mathbb{Z} \approx \left(\bigoplus_{n=1}^\infty Q_\sigma(\ell_q^{2^n-1}) \oplus \ell_p^n \right)_q \approx \mathbb{V} \oplus B_{p,q},$$

where

$$\mathbb{V} = \left(\bigoplus_{n=1}^\infty Q_\sigma(\ell_q^{2^n-1}) \right)_q .$$

Notice that $\mathbb{V} \lesssim_c (\bigoplus_{n=1}^\infty \ell_q^{2^n-1})_q \approx \ell_q$. Moreover, appealing to (4.3) gives $\ell_q(B_{p,q}) \approx B_{p,q}$. Hence, applying Pełczyński's decomposition technique yields $\mathbb{Z} \approx B_{p,q}$. □

Remark 4.19. By Propositions 4.1 and 4.4 from [17], the bases obtained in Theorems 4.5, 4.12, 4.14 and 4.18 are bi-democratic. Then (see [17, Theorem 5.4]) their dual bases also are almost greedy.

Now, in order to complete our study, it remains to deal with non-reflexive Besov spaces $B_{p,0}$. First, we realize that, in this case, it is hopeless to try to obtain almost greedy bases.

Theorem 4.20. *Let $1 \le p < \infty$. Then $B_{p,0}$ has no superdemocratic basis. In particular, $B_{p,0}$ has no almost greedy basis.*

This will follow immediately from our next Proposition, taking into account that $B_{p,0}$ and c_0 are not isomorphic.

Proposition 4.21. *Let $(\mathbb{X}_n)_{n=1}^\infty$ be a sequence a finite-dimensional Banach spaces, and $\mathcal{B}$ be a superdemocratic basic sequence in $\mathbb{X} = (\oplus_{n=1}^\infty \mathbb{X}_n)_q$ for some $q \in \{0\} \cup [1, \infty)$. Then*

(a) *If $q \neq 0$ the fundamental function of $\mathcal{B}$ is equivalent to $(n^{1/q})_{n=1}^\infty$.*
(b) *If $q = 0$ then $\mathcal{B}$ is equivalent to the unit vector basis of c_0.*

Proof. Given $N \in \mathbb{N}$, let $P_N \colon \mathbb{X} \to \mathbb{X}$ be the canonical projection onto the first N coordinates. With the convention $P_0 = 0$, put also $P_{M,N} = P_N - P_M$ and $P_{M,N}^c = \operatorname{Id}_{\mathbb{X}} - P_{M,N}$ for $0 \le M \le N$. Let $\mathcal{B} = (\mathbf{x}_j)_{j=1}^\infty$ be a superdemocratic basic sequence in $\mathbb{X}$. Since the set $\{P_N(\mathbf{x}_j) \colon j \in \mathbb{N}\}$ is compact, we have that for every $N \in \mathbb{N}$ every subsequence of $\mathcal{B}$ possesses a further subsequence, say $(\mathbf{z}_j)_{j=1}^\infty$, such that $(P_N(\mathbf{z}_j))_{j=1}^\infty$ converges, and so $\lim_j P_N(\mathbf{z}_{2j}) - P_N(\mathbf{z}_{2j-1}) = 0$. Fix a sequence $(\varepsilon_k)_{k=1}^\infty$ of positive scalars. Applying the gliding-hump technique, we infer that there are increasing sequences $(j_k)_{k=1}^\infty$ and $(N_k)_{k=1}^\infty$ of positive integers such that

$$\left\| P_{N_{k-1},N_k}^c (\mathbf{x}_{j_{2k}} - \mathbf{x}_{j_{2k-1}}) \right\| \le \varepsilon_k,$$

Then, by the principle of small perturbations (see [8, Theorem 1.3.9]), if we choose $(\varepsilon_k)_{k=1}^\infty$ conveniently and we put

$$\mathbf{y}_k = P_{N_{k-1},N_k}(\mathbf{x}_{j_{2k}} - \mathbf{x}_{j_{2k-1}}), \quad k \in \mathbb{N},$$

the basic sequences $(\mathbf{y}_k)_{k=1}^\infty$ and $(\mathbf{x}_{j_{2k}} - \mathbf{x}_{j_{2k-1}})_{k=1}^\infty$ are equivalent. In particular, $(\mathbf{y}_k)_{k=1}^\infty$ is semi-normalized and then, since it is disjointly supported, it is equivalent to the canonical basis of ℓ_q (c_0 in the case when $q = 0$). Therefore

$$\left\| \sum_{k=1}^m \mathbf{x}_{j_{2k}} - \sum_{k=1}^m \mathbf{x}_{j_{2k-1}} \right\| \approx \left\| \sum_{k=1}^m \mathbf{y}_k \right\| \approx m^{1/q}, \quad m \in \mathbb{N}$$

(with the usual modification when $q = 0$). Since $\phi_m[\mathcal{B}] \le \phi_{2m}[\mathcal{B}] \le 2\phi_m[\mathcal{B}]$, the superdemocracy of $\mathcal{B}$ yields $\phi_m[\mathcal{B}] \approx m^{1/q}$ for $m \in \mathbb{N}$ in the case when $q \ge 1$ and $\phi_m[\mathcal{B}] \approx 1$ for $m \in \mathbb{N}$ in the case when $q = 0$. The proof of (a) is over, and from here (b) is straightforward. □

Theorem 4.22. *Let $\mathbb{X}$ be a Banach space with a quasi-greedy basis. Assume that $B_{p,0} \lesssim_c \mathbb{X}$ for some $1 \le p < \infty$, then $\mathbb{X}$ has a quasi-greedy basis $\mathcal{B}_\kappa$ with $L_m[\mathcal{B}_\kappa] \approx \log m$ for $m \ge 2$.*

Proof. The relations (4.2) and (4.3) give $B_{2,0} \lesssim_c B_{p,0}$ for $1 < p < \infty$. Moreover, since $B_{p,0} \approx B_{p,0} \oplus B_{p,0}$ we have $B_{p,0} \oplus \mathbb{X} \approx \mathbb{X}$. Then, by Lemma 2.2, it suffices show to the existence of a quasi-greedy basis as conditional as possible in the space $B_{p,0}$ in the case when $p \in \{1, 2\}$. Let $\sigma = ([2^{n-1}, 2^n - 1])_{n=1}^\infty$ and $\mathcal{S}$ be the summing basis of c_0. Combining Lemma 4.9, Lemma 2.3 and Theorem 4.1 we obtain that $\mathcal{B}_\kappa = (\bigoplus_{n=1}^\infty (\mathbf{e}_j)_{j=1}^{2^n-1})_0$ is a quasi-greedy basis for the space

$$\mathbb{Z} := \left(\bigoplus_{n=1}^\infty \mathbb{Y}^{(2^n-1)}[\mathcal{S}, \ell_p, \sigma] \right)_0$$

such that $L_m[\mathcal{B}_\kappa] \approx \log m$ for $m \ge 2$. By Theorem 3.6 (c), if we put

$$\mathbb{V} = \left(\bigoplus_{n=1}^\infty Q_\sigma(\ell_p^{2^n-1}) \right)_0,$$

we have

$$\mathbb{Z} \approx \mathbb{V} \oplus (\oplus_{n=1}^\infty \ell_\infty^n)_0 \approx \mathbb{V} \oplus c_0 \approx \mathbb{V}.$$

In the case when $p = 2$ we have $Q_\sigma(\ell_q^{2^n-1}) \approx \ell_2^{2^n-n-1}$ for $n \in \mathbb{N}$. Assume that $p = 1$ and let $B = \mathbb{N} \setminus \{2^n - 1 \colon n \in \mathbb{N}\}$. The coordinate projection S_B restricts to an isomorphism from $Q_\sigma(\ell_1)$ onto

$$\mathbb{W} = \{(a_n)_{n=1}^\infty \in \ell_1 \colon a_{2^n-1} = 0 \text{ for all } n \in \mathbb{N}\}.$$

Hence, $Q_\sigma(\ell_1^{2^n-1}) \approx \ell_1^{2^n-n-1}$ for $n \in \mathbb{N}$. In both cases, taking into account (4.3), we have $\mathbb{V} \approx (\bigoplus_{n=1}^\infty \ell_p^{2^n-n-1}))_0 \approx B_{p,0}$. $\square$

References

[1] F. Albiac and J. L. Ansorena, *Lorentz spaces and embeddings induced by almost greedy bases in Banach spaces*, Constr. Approx. **43** (2016), no. 2, 197–215.

[2] ______, *The isomorphic classification of Besov spaces over* $\mathbb{R}^d$ *revisited*, Banach J. Math. Anal. **10** (2016), no. 1, 108–119.

[3] ______, *Isomorphic classification of mixed sequence spaces and of Besov spaces over* $[0,1]^d$, Math. Nachr. **290** (2017), no. 8-9, 1177–1186.

[4] ______, *Characterization of 1-almost greedy bases*, Rev. Mat. Complut. **30** (2017), no. 1, 13–24.

[5] F. Albiac, J. L. Ansorena, S. J. Dilworth, and Denka Kutzarova, *Banach spaces with a unique greedy basis*, J. Approx. Theory **210** (2016), 80–102.

[6] F. Albiac, J. L. Ansorena, G. Garrigós, E. Hernández, and M. Raja, *Conditionality constants of quasi-greedy bases in super-reflexive Banach spaces*, Studia Math. **227** (2015), no. 2, 133–140.

[7] F. Albiac, J. L. Ansorena, and P. Wojtaszczyk, *Conditionality constants of quasi-greedy bases in non-superreflexive Banach space*, Constr. Approx., DOI 10.1007/s00365-017-9399-x.

[8] F. Albiac and N. J. Kalton, *Topics in Banach space theory, 2nd revised and updated edition*, Graduate Texts in Mathematics, vol. 233, Springer International Publishing, 2016.

[9] Z. Altshuler, *Uniform convexity in Lorentz sequence spaces*, Israel J. Math. **20** (1975), no. 3-4, 260–274.

[10] G. D. Allen, *Duals of Lorentz spaces*, Pacific J. Math. **177** (1978), 287–291.

[11] J. L. Ansorena, *A note on subsymmetric renormings of Banach spaces*, Quaest. Math. **41** (2018), no. 5, 615–628.

[12] P. Berná, O. Blasco, G. Garrigós, E. Hernández, and T. Oikhberg, *Embeddings and Lebesgue-type inequalities for the greedy algorithm in Banach spaces*, Constr. Approx., DOI 10.1007/s00365-018-9415-9.

[13] C. Bennett and R. Sharpley, *Interpolation of operators*, Pure and Applied Mathematics, vol. 129, Academic Press, Inc., Boston, MA, 1988.

[14] A. Bird, G. Jameson, and N. Laustsen, *The Giesy-James theorem for general index p, with an application to operator ideals on the pth James space*, J. Operator Theory **70** (2013), no. 1, 291–307.

[15] M.J. Carro, J.A. Raposo, and J. Soria, *Recent developments in the theory of Lorentz spaces and weighted inequalities*, Mem. Amer. Math. Soc. **187** (2007), no. 877, xii+128.

[16] S. J. Dilworth, N. J. Kalton, and D. Kutzarova, *On the existence of almost greedy bases in Banach spaces*, Studia Math. **159** (2003), no. 1, 67–101.

[17] S. J. Dilworth, N. J. Kalton, D. Kutzarova, and V. N. Temlyakov, *The thresholding greedy algorithm, greedy bases, and duality*, Constr. Approx. **19** (2003), no. 4, 575–597.

[18] S. J. Dilworth and D. Mitra, *A conditional quasi-greedy basis of* l_1, Studia Math. **144** (2001), no. 1, 95–100.

[19] S. J. Dilworth, D. Kutzarova, and P. Wojtaszczyk, *On approximate* l_1 *systems in Banach spaces*, J. Approx. Theory **114** (2002), no. 2, 214–241.

[20] S. J. Dilworth, M. Soto-Bajo, and V. N. Temlyakov, *Quasi-greedy bases and Lebesgue-type inequalities*, Studia Math. **211** (2012), no. 1, 41–69.

[21] D. L. Donoho, *Unconditional bases are optimal bases for data compression and for statistical estimation*, Appl. Comput. Harmon. Anal. **1** (1993), no. 1, 100–115.

[22] A. Dvoretzky, *Some results on convex bodies and Banach spaces*, Proc. Internat. Sympos. Linear Spaces (Jerusalem, 1960) (1961), 123-160.

[23] G. Garrigós, E. Hernández, and T. Oikhberg, *Lebesgue-type inequalities for quasi-greedy bases*, Constr. Approx. **38** (2013), no. 3, 447–470.

[24] G. Garrigós and P. Wojtaszczyk, *Conditional quasi-greedy bases in Hilbert and Banach spaces*, Indiana Univ. Math. J. **63** (2014), no. 4, 1017–1036.

[25] R. C. James, *Bases and reflexivity of Banach spaces*, Ann. of Math. (2) **52** (1950), 518–527.

[26] W. B. Johnson and E. Odell, *Subspaces of L_p which embed into l_p*, Compositio Math. **28** (1974), 37–49.

[27] W. B. Johnson, H. P. Rosenthal, and M. Zippin, *On bases, finite dimensional decompositions and weaker structures in Banach spaces*, Israel J. Math. **9** (1971), 488–506.

[28] S. Gogyan, *An example of an almost greedy basis in $L^1(0,1)$*, Proc. Amer. Math. Soc. **138** (2010), no. 4, 1425–1432.

[29] M. I. Kadec and A. Pełczyński, *Bases, lacunary sequences and complemented subspaces in the spaces L_p*, Studia Math. **21** (1961/1962), 161–176.

[30] S. V. Konyagin and V. N. Temlyakov, *A remark on greedy approximation in Banach spaces*, East J. Approx. **5** (1999), no. 3, 365–379.

[31] J. Lindenstrauss and A. Pełczyński, *Absolutely summing operators in $\mathcal{L}_p$-spaces and their applications*, Studia Math. **29** (1968), 275–326.

[32] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces. I*, Springer-Verlag, Berlin-New York, 1977. Sequence spaces; Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 92.

[33] ______, *On Orlicz sequence spaces II*, Israel J. Math. **11** (1972), 355–379.

[34] P. F. X. Müller and G. Schechtman, *On complemented subspaces of H^1 and* VMO, Geometric aspects of functional analysis (1987–88), Lecture Notes in Math., vol. 1376, Springer, Berlin, 1989, pp. 113–125.

[35] A. Pełczyński, *Projections in certain Banach spaces*, Studia Math. **19** (1960), 209–228.

[36] G. Pisier and Q. H. Xu, *Random series in the real interpolation spaces between the spaces v_p*, Geometrical aspects of functional analysis (1985/86), Lecture Notes in Math., vol. 1267, Springer, Berlin, 1987, pp. 185–209.

[37] I. Singer, *Bases in Banach spaces. I*, Springer-Verlag, New York-Berlin, 1970. Die Grundlehren der mathematischen Wissenschaften, Band 154.

[38] V. N. Temlyakov, M. Yang, and P. Ye, *Lebesgue-type inequalities for greedy approximation with respect to quasi-greedy bases*, East J. Approx. **17** (2011), no. 2, 203–214.

[39] ______, *Greedy approximation with regard to non-greedy bases*, Adv. Comput. Math. **34** (2011), no. 3, 319–337.

[40] P. Wojtaszczyk, *Greedy algorithm for general biorthogonal systems*, J. Approx. Theory **107** (2000), no. 2, 293–314.

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