PROBLEMS

11586. Proposed by Takis Konstantopoulos, Uppsala University, Uppsala, Sweden. Let $A_0$, $B_0$, and $C_0$ be noncollinear points in the plane. Let $p$ be a line that meets lines $B_0C_0$, $C_0A_0$, and $A_0B_0$ at $A^*$, $B^*$, and $C^*$ respectively. For $n \geq 1$, let $A_n$ be the intersection of $B^*B_{n-1}$ with $C^*C_{n-1}$, and define $B_n$, $C_n$ similarly. Show that all three sequences converge, and describe their respective limits.

11587. Proposed by Andrei Ciupan, Harvard University, Cambridge, MA, and Bozgan Francisc, UCLA, Los Angeles, CA. For which pairs $(a, b)$ of positive integers do there exist infinitely many positive integers $n$ such that $n^2$ divides $a^n + b^n$?

11588. Proposed by Taras Banakh, Ivan Franko National University of Lviv, Lviv, Ukraine, and Igor Protasov, Taras Shevchenko National University of Kyiv, Kyiv, Ukraine. Show that $\mathbb{R} - \{0\}$ can be partitioned into countably many subsets, each of which is linearly independent over $\mathbb{Q}$, if and only if the continuum hypothesis holds.

11589. Proposed by Catalin Barboianu, Infarom Publishing, Craiova, Romania. Let $P$ be a polynomial over $\mathbb{R}$ given by $P(x) = x^3 + a_2x^2 + a_1x + a_0$, with $a_1 > 0$. Show that $P$ has a least one zero between $-a_0/a_1$ and $-a_2$.

11590. Proposed by Khodakhast Bibak, University of Waterloo, Waterloo, Ontario, Canada. Let $m$ balls numbered 1 to $m$ each be painted with one of $n$ colors, with $n \geq 2$ and at least two balls of each color. For each positive integer $k$, let $P(k)$ be the number of ways to put these balls into urns numbered 1 through $k$ so that no urn is empty and no urn gets two or more balls of the same color. Prove that

$$\sum_{k=1}^{m} \frac{(-1)^k}{k} P(k) = 0.$$
Let $I_n$ be the set of all idempotent elements of $\mathbb{Z}/n\mathbb{Z}$. That is, $e \in I_n$ if and only if $e^2 \equiv e \pmod{n}$. Let $I_n^1 = I_n$, and for $k \geq 2$, let $I_n^k$ be the set of all sums of the form $u + v$ where $u \in I_n$, $v \in I_n^{k-1}$, and the addition is done modulo $n$. Determine, in terms of $n$, the least $k$ such that $I_n^k = \mathbb{Z}/n\mathbb{Z}$.

**SOLUTIONS**

A Telescoping Sum of Floors

Let $I_n$ be the set of all idempotent elements of $\mathbb{Z}/n\mathbb{Z}$. That is, $e \in I_n$ if and only if $e^2 \equiv e \pmod{n}$. Let $I_n^1 = I_n$, and for $k \geq 2$, let $I_n^k$ be the set of all sums of the form $u + v$ where $u \in I_n$, $v \in I_n^{k-1}$, and the addition is done modulo $n$. Determine, in terms of $n$, the least $k$ such that $I_n^k = \mathbb{Z}/n\mathbb{Z}$.

**SOLUTIONS**

A Telescoping Sum of Floors

11444 [2009, 548]. Proposed by Marian Tetiva, National College “Gheorghe Roșca Codreanu,” Bălțățeți, Romania. Let $k$ and $s$ be positive integers with $s \leq k$. Let $f(n) = n - s \lfloor n/k \rfloor$. For $j \geq 0$, let $f^j$ denote the $j$-fold composition of $f$, taking $f^0$ to be the identity function. Show that

$$S(n) = \sum_{j=0}^{\infty} \left\lfloor \frac{f^{j}(n)}{k} \right\rfloor = \sum_{j=0}^{\infty} \left\lfloor \frac{n_j}{k} \right\rfloor = \sum_{j=0}^{\infty} a_j,$$

where $q = \min\{k - 1, n\}$.

**Solution by Robin Chapman, University of Exeter, Exeter, U.K.** We prove that the formula holds for nonnegative $n$. The formula as stated fails for negative $n$; we correct it. For $j \geq 0$, let $n_j = f^{j}(n)$, $a_j = \lfloor n_j/k \rfloor$, and

$$S(n) = \sum_{j=0}^{\infty} \left\lfloor \frac{f^{j}(n)}{k} \right\rfloor = \sum_{j=0}^{\infty} \left\lfloor \frac{n_j}{k} \right\rfloor = \sum_{j=0}^{\infty} a_j.$$

Consider first the case of nonnegative $n$. Clearly $f(n) = n$ if $0 \leq n \leq k - 1$ and $f(n) < n$ if $n \geq k$. Also, $f(n) = n - s \lfloor n/k \rfloor \geq n - sn/k \geq 0$. Hence $\{n_j\}_{j \geq 0}$ is non-increasing and reaches its integer limit. Since $a_j = (n_j - n_{j+1})/s$, the sum $S(n)$ thus has only finitely many nonzero terms. That is, there exists $N$ such that

$$S(n) = \sum_{j=0}^{N} a_j = \frac{1}{s} \sum_{j=0}^{N} (n_j - n_{j+1}) = \frac{n_0 - n_{N+1}}{s}.$$

If $0 \leq n < k$, then $n_j = n$ for all $j$ and $S(n) = 0$. If $n \geq k$, then $f(n) \geq (1 - s/k)n \geq (1 - s/k)k \geq k - s$, and $n_j \geq k - s$ for all $j$. Hence $n_{N+1}$ is the unique integer $m$ with $k - s \leq m < k$ that is congruent to $n$ modulo $s$. That is, $m = n - s \lfloor (n - k + s)/s \rfloor$. Thus for $n \geq k$,

$$S(n) = \lfloor (n - k + s)/s \rfloor = - \lfloor (k - 1 - n)/s \rfloor = - \lfloor (q - n)/s \rfloor.$$

If $0 \leq n < k$, then $q = n$ and again $S(n) = 0 = - \lfloor (q - n)/s \rfloor$.

We now prove that $S(n) = \lfloor n/s \rfloor$ when $n$ is a negative integer. In this case, $n < f(n) < n - s(n/k - 1) = n(1 - s/k) + s \leq s$. Thus $\{n_j\}_{j \geq 0}$ increases until it reaches a value $m$ between 0 and $s - 1$, after which it is stationary. Hence $m = n - s \lfloor n/s \rfloor$, and $S(n) = (n - m)/s = \lfloor n/s \rfloor$. 

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**A Positive Sequence**

11445 [2009, 548]. Proposed by H. A. ShahAli, Tehran, Iran. Given $a, b, c > 0$ with $b^2 > 4ac$, let $\langle \lambda_n \rangle$ be a sequence of real numbers, with $\lambda_0 > 0$ and $c \lambda_1 > b \lambda_0$. Let $u_0 = c \lambda_0, u_1 = c \lambda_1 - b \lambda_0$, and for $n \geq 2$ let $u_n = a \lambda_{n-2} - b \lambda_{n-1} + c \lambda_n$. Show that if $u_n > 0$ for all $n \geq 0$, then $\lambda_n > 0$ for all $n \geq 0$.

Solution I by J. C. Linders, Eindhoven, The Netherlands. Since $u_0 > 0$ and $u_1 > 0$, both $\lambda_0$ and $\lambda_1$ are positive. We show by induction on $n$ that

$$c \lambda_n > \frac{n + 1}{2n} b \lambda_{n-1} \quad \text{and} \quad \lambda_n > 0$$

for $n \geq 1$. Since $u_1 > 0$, this holds for $n = 1$. In general, $u_{n+1} > 0$ and the induction hypothesis imply for $n \geq 1$ that

$$c \lambda_{n+1} > b \lambda_n - a \lambda_{n-1} > b \lambda_n - a \frac{2n}{n + 1} \cdot \frac{c}{b} \lambda_n = \left(1 - \frac{2nac}{b^2(n+1)}\right) b \lambda_n$$

$$> \left(1 - \frac{n}{2(n+1)}\right) b \lambda_n = \frac{n + 2}{2(n+1)} b \lambda_n,$$

where the last inequality follows from $b^2 > 4ac$ and $\lambda_n > 0$. This proves the two inequalities in the claim for $n + 1$.

Solution II by David Beckwith, Sag Harbor, NY. Define generating functions by letting $U(x) = \sum_{n=0}^{\infty} u_n x^n$ and $\Lambda(x) = \sum_{n=0}^{\infty} \lambda_n x^n$. The recursion yields $U(x) = (ax^2 - bx + c) \Lambda(x)$. The conditions on $a, b,$ and $c$ imply $ax^2 - bx + c = a(x - \rho_+)(x - \rho_-)$, where $\rho_+$ and $\rho_-$ are the real and positive roots of $ax^2 - bx + c$. Thus

$$\Lambda(x) = \left(\sum_{n=0}^{\infty} \rho_+^n x^n\right) \left(\sum_{n=0}^{\infty} \frac{1}{\rho_-^n} x^n\right) \left(\sum_{n=0}^{\infty} u_n x^n\right).$$

Since the product of three power series with all positive coefficients is a power series with all positive coefficients, it follows that $\lambda_n > 0$ for all $n$.

Editorial comment. From the proofs above, the claim also holds when $b^2 = 4ac$. O. P. Lossers showed also that the condition $a > 0$ is superfluous.


**Matrices Whose Products Are All Different**

11446 [2009, 647]. Proposed by Christopher Hillar, Mathematical Research Sciences Institute, Berkeley, CA, and Lionel Levine, Massachusetts Institute of Technology,
Cambridge, MA. Prove or disprove: there exist $2 \times 2$ symmetric integer matrices $A$ and $B$ such that no element of the multiplicative semigroup generated by $A$ and $B$ can be written in two different ways. (Thus, $A$, $B$, $AA$, $AB$, $BA$, $BB$, $AAA$, $AAB$, ... are all different.)

**Solution by Reiner Martin, Bad Soden-Neuenhain, Germany.** Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$  

No element of the multiplicative group generated by $A$ and $B$ can be written in two ways: When $v$ is a column vector with two entries, both positive, the first entry of $Av$ is larger than the second, and the first entry of $Bv$ is smaller than the second. Therefore, when two products in $A$ and $B$ are equal, the first factor in the two products is the same. Since $A$ and $B$ are invertible, the products of the remaining factors must be the same. The claim follows by induction on the number of factors in the product.

Also solved by V. D. Blondel, R. Chapman (U. K.), C. Curtis, C. Delorme (France), O. Geupel (Germany), J. Grivaux (France), A. Ilić (Serbia), O. P. Lossers (Netherlands), V. S. Miller, R. Stong, J. V. Tejedor (Spain), A. Wyn-jones (U. K.), BSI Problems Group (Germany), Microsoft Research Problems Group, and the proposers.

A Sufficient Condition for a Division Ring

11451 [2009, 648]. **Proposed by Greg Oman, Otterbein College, Westerville, OH.** Let $k$ and $n$ be positive integers, with $k > 1$. Let $R$ be a ring, not assumed to have an identity, with the following properties:

(i) There is an element of $R$ that is not nilpotent.

(ii) If $x_1, \ldots, x_k$ are nonzero elements of $R$, then $\sum_{j=1}^{k} x_j^n = 0$.

Show that $R$ is a division ring, that is, the nonzero elements of $R$ form a group under multiplication.

**Solution by the NSA Problems Group, Fort Meade, MD.** Take $a, x \in R$ with $a$ a non-nilpotent and $x$ nonzero. With all $x_j$ set to $x$ in (ii), we obtain $kx^n = 0$. With $x_i = x$ for $i < k$ and $x_k = a$, we obtain $(k-1)x^n + a^n = 0$. Hence, $x^n = a^n$ for every nonzero $x \in R$. Let $e = a^n$; setting $x = a^2$ yields $e^2 = e$. Furthermore, $x^n = e \neq 0$ shows that $R$ has no nonzero nilpotent elements.

We claim that $e$ is the identity in $R$. First, $ex = a^n x = x^n x = x a^n = x e$. Next, expand $(x - ex)^n$ by the binomial theorem, which applies since $e$ commutes with every element of $R$. We obtain

$$(x - ex)^n = \sum_{j=0}^{n} \binom{n}{j} x^j (-ex)^{n-j} = x^n + \sum_{j=0}^{n-1} \binom{n}{j} x^j e (-x)^{n-j}$$

$$= x^n + e \sum_{j=0}^{n} \binom{n}{j} x^j (-x)^{n-j} - ex^n = x^n + e(x - x)^n - ex^n = 0.$$  

Hence $x - ex$ is nilpotent and must be 0, so $x = ex$ and $e$ must be the identity. Finally, $x^n = e$ implies $x^{n-1} = x^{-1}$, and we conclude that $R$ is a division ring.

Note that $x^{n+1} = x$ for $x \in R$, so a well-known theorem of Jacobson implies that $R$ is commutative. Hence $R$ is a field. Since $x^n = 1$ has at most $n$ solutions in any field, $R$ has at most $n$ elements; thus it is a finite field whose characteristic divides $k$. 

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Editorial comment. Several other readers also showed that the conditions of the problem imply that $R$ is a finite field. Jacobson’s “$x^{n(x)} = x$” theorem appears in N. Jacobson, *Structure of Rings*, AMS Colloq. Pub., vol. 37, AMS, 1956, p. 217, as well as in Lam’s *A First Course in Noncommutative Rings*, 2nd ed., and other graduate algebra texts.


### Permutation Flipping

**11452 [2009, 648]. Proposed by Donald E. Knuth, Stanford University, Stanford, CA.** Say that the permutations $a_1 \cdots a_k a_{k+1} \cdots a_n$ and $a_k \cdots a_1 a_{k+1} \cdots a_n$ are equivalent when $k = n$ or when $a_{k+1}$ exceeds all of $a_1, \ldots, a_k$. Also say that two permutations are equivalent whenever they can be obtained from each other by a sequence of such flips. For example, $321 \equiv 132 \equiv 231 \equiv 123 \equiv 213 \equiv 312 \equiv 132$.

Show that the number of equivalence classes is equal to the Euler secant-and-tangent number for all $n$. (The $n$th secant-and-tangent number counts the “up-down” permutations of length $n$, namely the permutations like 25341 that alternately rise and fall beginning with a rise.)

**Solution by Robin Chapman, University of Exeter, UK.** We consider permutations of any totally ordered $n$-set; let $E_n$ be the number of equivalence classes. We shall establish a recurrence for $(E_n)_{n \geq 0}$. Set $E_0 = 1$. For a word $a$, let $\bar{a}$ denote its reversal.

If $a$ is a permutation of a totally ordered $n$-set $A$ with largest letter $\alpha$, then $a = b\alpha c$, where $b$ and $c$ are permutations of complementary subsets $B$ and $C$ of $A - \{\alpha\}$. No flip can change the unordered pair $\{B, C\}$ (the sets can be exchanged and may be empty). Thus all permutations equivalent to $a$ have the form $b'\alpha c'$ or $c'\alpha b'$, where $b' \equiv b$ and $c' \equiv c$. Conversely, any such permutation is equivalent to $a$: the presence of $\alpha$ allows transforming the part before $\alpha$ into anything in its equivalence class, and thus $b\alpha c \equiv b'\alpha c' \equiv \overline{c'\alpha b'} \equiv \overline{c'\alpha b'} \equiv b'\alpha c'$. The equivalence class of $b\alpha c$ is $\langle b'\alpha c, c'\alpha b' : b' \equiv b, c' \equiv c \rangle$.

To count the equivalence classes of permutations of $A$, we choose a partition of $A - \{\alpha\}$ into sets $B$ and $C$ of sizes $k$ and $n - k - 1$ and populate the portions of the permutation before and after $\alpha$ with equivalence classes on those sets. Summing over $k$ counts each equivalence class twice, since $B$ and $C$ can be switched. For $n \geq 2$,

$$2E_n = \sum_{k=0}^{n-1} \binom{n-1}{k} E_k E_{n-k-1}.$$ 

It is well known that the number of up-down permutations satisfies the same recurrence and initial condition; see, for example, the solution to Exercise 7.41 in Graham, Knuth, and Patashnik’s *Concrete Mathematics*, Addison-Wesley, 1989. Thus, by induction, the two sequences are the same.

**Editorial comment.** The origin of the name for the numbers in this sequence is that its exponential generating function is $\sec x + \tan x$.

A Simplicial Complex Sum

11453 [2009, 746]. Proposed by Richard Stanley, Massachusetts Institute of Technology, Cambridge, MA. Let $\Delta$ be a finite collection of sets such that if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$. Fix $k \geq 0$. Suppose that every $F$ in $\Delta$ (including $F = \emptyset$) with $|F| \leq k$ satisfies

$$\sum_{G \in \Delta, G \supseteq F} (-1)^{|G|} = 0.$$ 

Show that $|\Delta|$ is divisible by $2^{k+1}$.

Solution I by Richard Bagby, New Mexico State University, Las Cruces, New Mexico. First we show that

$$|\Delta| = \sum_{F \in \Delta} \sum_{G \in \Delta, G \supseteq F} (-1)^{|G|}.$$ 

Indeed,

$$\sum_{F, G \in \Delta \atop F \supseteq G} 2^{|F|} (-1)^{|G| - |F|} = \sum_{G \in \Delta} \left( \sum_{F \supseteq G} 2^{|F|} (-1)^{|G| - |F|} \right)$$

$$= \sum_{G \in \Delta} \sum_{j=0}^{|G|} \binom{|G|}{j} 2^j (-1)^{|G| - j}$$

$$= \sum_{G \in \Delta} (2 - 1)^{|G|} = |\Delta|.$$ 

Interchanging the order of summation yields

$$|\Delta| = \sum_{F \in \Delta} (-2)^{|F|} \sum_{G \in \Delta, G \supseteq F} (-1)^{|G|}.$$ 

Now the contribution to the outer sum from each set $F$ is either 0 (for $|F| \leq k$) or divisible by $2^{k+1}$ (for $|F| > k$).

Solution II by Richard Stong, Center for Communication Research, San Diego, CA. Let $P(x) = \sum_{G \in \Delta} x^{|G|}$; note that $P$ is a polynomial with integer coefficients. For $m \leq k$,

$$\frac{(-1)^m}{m!} P^{(m)}(-1) = \sum_{G \in \Delta} \binom{|G|}{m} (-1)^{|G|} = \sum_{F \in \Delta \atop |F| = m} \sum_{G \in \Delta, G \supseteq F} (-1)^{|G|} = 0.$$ 

Hence $-1$ is a zero of $P$ with multiplicity at least $k + 1$, and we can write $P(x) = (x + 1)^{k+1} Q(x)$ for some polynomial $Q$ with integer coefficients. Setting $x = 1$ yields $|\Delta| = P(1) = 2^{k+1} Q(1)$; hence $|\Delta|$ is a multiple of $2^{k+1}$.

Comment by the proposer. This result is the combinatorial analogue of a much deeper topological result of G. Kalai in *Computational Commutative Algebra and Combinatorics*, Adv. Stud. Pure Math., vol. 33, Math. Soc. Japan, 2002, 121–163 (Theorem 4.2), a special case of which can be stated as follows. Let $\Delta$ be a finite simplicial complex. Suppose that for any face $F$ of dimension at most $k - 1$ (including the empty face of dimension $-1$), the link of $F$ (i.e., the set of all $G \in \Delta$ such that $F \cap G = \emptyset$ and
$F \cup G \in \Delta$ is acyclic (that is, has vanishing reduced homology). Letting $f_i$ denote the number of $i$-dimensional faces of $\Delta$, there exists a simplicial complex $\Gamma$, with $g_i$ faces having dimension $i$, such that

$$\sum_{i \geq -1} f_i x^i = (1 + x)^{k+1} \sum_{i \geq -1} g_i x^i.$$ 

The present problem does not follow from Kalai’s result, since the hypotheses here concern only Euler characteristics, while Kalai’s result concerns homology groups.


**An Orientation Game**

11454 [2009, 746]. *Proposed by Azer Kerimov, Bilkent University, Ankara, Turkey.*

Alice and Bob play a game based on a 2-connected graph $G$ with $n$ vertices, where $n > 2$. Alice selects two vertices $u$ and $v$. Bob then orients up to $2n - 3$ of the edges. Alice then orients the remaining edges and selects some edge $e$, which may have been oriented by her or by Bob. If the oriented graph contains a path from $u$ to $v$ through $e$, then Bob wins; otherwise, Alice wins. Prove that Bob has a winning strategy, while if he is granted only $2n - 4$ edges to orient, on some graphs he does not. (A graph is 2-connected if it has at least three vertices and each subgraph obtained by deleting one vertex is connected.)

*Solution by Michelle Delcourt (student), Georgia Institute of Technology, Atlanta, GA.*

We show first that orienting $2n - 4$ edges does not guarantee a win for Bob. Let $G$ consist of two vertices adjacent to each other and to the remaining $n - 2$ vertices; $G$ has $2n - 3$ edges. Alice chooses the high-degree vertices as $u$ and $v$. Since Bob only orients $2n - 4$ edges, some edge remains unoriented. Alice selects this edge as $e$ and orients it into $u$ and/or away from $v$. No path from $u$ to $v$ passes through $e$.

Now allow Bob to orient $2n - 3$ edges. Bob produces a special vertex ordering and edge partition and uses them to orient at most $2n - 3$ edges. Let $u$ and $v$ be the vertices chosen by Alice. Whitney’s theorem for 2-connected graphs states that there are two paths from $u$ to $v$ with no shared internal vertices. Thus $u$ and $v$ lie on a cycle; let $C$ be a shortest cycle containing them. Order its vertices by starting with $u$, then listing the internal vertices of one path from $u$ to $v$ along $C$, then listing the internal vertices of the other such path, then ending with $v$. Bob orients each edge of $C$ from its earlier endpoint to its later endpoint, producing two oriented paths from $u$ to $v$.

Bob now iteratively decomposes the rest of $G$ into paths $P_1, \ldots, P_r$. Let $G_0 = C$. Suppose that $G_{i-1}$ has been defined, with a linear order on its vertices. If $G_{i-1} \neq G$, then there is a path joining distinct vertices of $G_{i-1}$ whose edges and internal vertices are not in $G_{i-1}$ (again by Whitney’s theorem). Among all such paths, consider those whose earlier endpoint is earliest in the ordering of $V(G_{i-1})$; among these, consider those whose later endpoint is latest in the ordering; among these, let $P_i$ be a shortest such path. Let $G_i = G_{i-1} \cup P_i$. Insert the internal vertices of $P_i$ in the vertex ordering between its endpoints, ordered so that each new vertex has a neighbor occurring earlier and a neighbor occurring later in the ordering. Bob orients $P_i$ if its length is at least 2, in that case orienting each edge from its earlier endpoint to its later endpoint. Bob leaves $P_i$ unoriented if it has only one edge.

The number of edges of $P_i$ oriented by Bob is at most twice the number of vertices added by $P_i$. The number of edges oriented in $C$ is $|V(C)|$, and $|V(C)| \leq 2|V(C)| - 3$ since cycles have at least three vertices. Hence Bob orients at most $2n - 3$ edges. The
orientation produced by Bob explicitly has a path from $u$ to $v$ through each oriented edge (and a unique such path through each vertex outside $\{u, v\}$).

To prove that this partial orientation wins for Bob, it suffices to show that for any edge $xy$ added as a path of length 1 (hence not oriented by Bob), there are disjoint paths oriented by Bob from $u$ to $x$ and from $y$ to $v$. This is immediate when $x$ precedes $y$ in the ordering, so we may assume that $x$ is later than $y$. 

It suffices prove that these two paths exist in the first $G_i$ that contains $x$ and $y$, since they remain (oriented) as the rest of the decomposition is added. The claim holds when $i = 0$ since $C$ was chosen to be a shortest cycle through $u$ and $v$; thus $x$ and $y$ lie in distinct $u, v$-paths on $C$ (and do not equal $u$ or $v$).

For $i > 0$, vertices $x$ and $y$ cannot both be added by $P_i$, since then there would be a shorter path joining its endpoints that would be added instead. Let $a$ and $b$ be the first and last vertices of $P_i$ in the ordering. If $x$ is added by $P_i$, then it suffices to show that the $u, x$-path created then by Bob contains no vertex of the $y, v$-path in $G_{i-1}$. If it does, then $y$ is earlier than $a$ in the ordering, and the path that starts with $yx$ and continues along $P_i$ to $b$ would be chosen in preference to $P_i$. Similarly, if $y$ is added by $P_i$, then it suffices to show that the $y, v$-path created then by Bob contains no vertex of the $u, x$-path in $G_{i-1}$. If it does, then $x$ is later than $b$ in the ordering, and the path that follows $P_i$ from $a$ to $y$ and finishes with $yx$ would be chosen in preference to $P_i$.

**Editorial comment.** The list $C, P_1, \ldots, P_t$ is an example of an ear decomposition of $G$. The vertex ordering is an example of an $s, t$-numbering with the source $s$ being $u$ and the terminus $t$ being $v$; the condition is that each vertex outside $\{s, t\}$ has an earlier neighbor and a later neighbor in the ordering.

Also solved by D. Beckwith, J. Simons (U. K.), R. Stong, S. Xiao (Canada), and the proposer.

### Our Gamma Inequality Flops

**11474 [2010, 86].** *Proposed by Cezar Lupu, student, University of Bucharest, Bucharest, Romania, and Valentin Vornicu, Aops-MathLinks forum, San Diego, CA.* (Corrected) Show that when $x$, $y$, and $z$ are greater than 1,

$$\Gamma(x)^{x^2+2xyz} \Gamma(y)^{y^2+2zyx} \Gamma(z)^{z^2+2xyz} \geq (\Gamma(x)\Gamma(y)\Gamma(z))^{xy+yz+zx}.$$  

**Solution by Richard Stong, Center for Communications Research, San Diego, CA.** When $x = y$, the inequality becomes $\Gamma(z)^{(z-x)^2} \geq 1$, which fails if $1 < z < 2$.

For $x, y, z > 2$, though, it follows from the fact that $\Gamma(x)$ is increasing on $[2, \infty)$ and $\Gamma(2) = 1$. Indeed: without loss we may assume $2 \leq x \leq y \leq z$. The desired inequality rearranges to

$$(y - x)(z - x) \log \Gamma(x) - (y - x)(z - y) \log \Gamma(y) + (z - x)(z - y) \log \Gamma(z) \geq 0.$$

The first term is nonnegative and the third term is greater than or equal to the second; hence this inequality holds.

**Editorial comment.** The corrected version of the problem, shown above, appeared in the April, 2010, issue of the MONTHLY.

Also solved by P. P. Dályay (Hungary), O. Kouba (Syria), O. P. Lossers (Netherlands), M. Muldoon (Canada), GCHQ Problem Solving Group (U. K.), and the Microsoft Research Problems Group.