Discrete Bernoulli Convolutions
Taking the Convoluted out of Bernoulli Convolutions

Michelle Delcourt

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A special thanks to:

Neil J. Calkin, Julia Davis, Zebediah Engberg, Jobby Jacob, and Kevin James,

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Motivation
A Bernoulli convolution for $0 < q < 1$ is the convolution

$$\mu_q(X) = b(X) * b(X/q) * b(X/q^2) * ...$$

where $b$ is the discrete Bernoulli measure concentrated at 1 and −1 each with weight $\frac{1}{2}$.

In 1935 Jessen and Wintner showed that $\mu_q$ is continuous for any $q$. 
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Alternatively, for $0 < q < 1$, consider the functional equation

$$F(t) = \frac{1}{2} F\left(\frac{t - 1}{q}\right) + \frac{1}{2} F\left(\frac{t + 1}{q}\right)$$

for $t$ on the interval $I_q := \left[\frac{-1}{1-q}, \frac{1}{1-q}\right]$.

There is a unique bounded solution $F_q(t)$, the distribution function of $\mu_q$, $F_q(t) = \mu_q([-\infty, t])$. 
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There is a unique bounded solution $F_q(t)$, the distribution function of $\mu_q$, $F_q(t) = \mu_q([-\infty, t])$. 
Jessen and Wintner showed that $F_q(t)$ is either absolutely continuous or purely singular. The major question is:

Which values of $q$ make the solution $F_q(t)$ absolutely continuous?

For $q = \frac{1}{2}$, $F_q(t)$ is absolutely continuous.

When $0 < q < \frac{1}{2}$, Kershner and Wintner have shown that $F_q(t)$ is always singular. For these values of $q$, the solution $F_q(t)$ is an example of a so called *Cantor function*, a function that is constant almost everywhere.
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Devil’s staircase

From www.mathworld.com, a plot of the Devil’s staircase:
The case when $q > \frac{1}{2}$ is much harder and more interesting.

In 1939 Erdős showed that if $q$ is of the form $q = \frac{1}{\theta}$ with $\theta$ a *Pisot number*, then $F_q(t)$ is again singular.

A *Pisot number* is an algebraic integer greater than 1 in absolute value, whose conjugates are all less than 1 in absolute value.

For example, the golden ratio $\tau = \frac{(1+\sqrt{5})}{2}$ is a Pisot number.
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Surprisingly, no actual example of such a \( q \) is known.

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If $F_q(t)$ is absolutely continuous, then one may consider its derivative $f_q(t) := F'_q(t)$,

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The existence of an absolutely continuous solution $F_q(t)$ is equivalent to the existence of an $L^1(I_q)$ solution $f_q(t)$ to $f(t)$. 
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Motivated by work of Girgensohn in 2007, for $q = \frac{2}{3}$, Calkin shifted the interval $I_q = [-3, 3]$ to $[0, 1]$ for simplicity and considered transform $T : L^1([0, 1]) \rightarrow L^1([0, 1])$ where

$$T : f(x) \mapsto \frac{3}{4} f \left( \frac{3x}{2} \right) + \frac{3}{4} f \left( \frac{3x - 1}{2} \right).$$

Start with an arbitrary initial function $f^0(t) \in L^1(I_q)$ and iterate the transform $T$ to gain a sequence of functions $f^0, f^1, f^2, \ldots$. 
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Bernoulli Convolutions

$q = 2/3$

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Intuitively, this transform takes two scaled copies of $f(x)$: one on the interval $[0, \frac{2}{3}]$ and the other on $[\frac{1}{3}, 1]$, and adds them.

The scaling factor of $\frac{1}{2q} = \frac{3}{4}$ gives us that

$$\int_0^1 f(x) \, dx = \int_0^1 Tf(x) \, dx.$$ 

In this setting, the question to be answered is: starting with the function $f^0(x) = 1$, does the iteration determined by this transform converge to a bounded function?
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Instead of viewing $T$ as a transform on $[0, 1]$, we consider a combinatorial analogue.

Consider the two maps $\text{dup}_n,\, \text{shf}_n : \mathbb{R}^n \rightarrow \mathbb{R}^{3n}$ defined by

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\text{dup}_n : (a_1, a_2, \ldots, a_{n-1}, a_n) \mapsto (a_1, a_1, a_2, a_2, \ldots, a_{n-1}, a_{n-1}, a_n, a_n, 0, \ldots, 0)
$$

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\text{shf}_n : (a_1, a_2, \ldots, a_{n-1}, a_n) \mapsto (0, \ldots, 0, a_1, a_1, a_2, a_2, \ldots, a_{n-1}, a_{n-1}, a_n, a_n).
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The combinatorial analogue of $T$ on $[0, 1]$ with $f^0(x) = 1$ is provided by the sequences

$$B_0 = (1) \text{ and }$$

$$B_{n+1} = \text{dup}_n(B_n) + \text{shf}_n(B_n).$$
A Useful Property

The fact that $B_n$ has a total of $3^n$ terms follows directly from the definition of dup$_n$ and shf$_n$.

The average value of $B_n$, $\mu(B_n) = \left(\frac{4}{3}\right)^n$.

The first few maximum values of $B_n$, $m_n$, are 1, 2, 3, 4, 6, 8, 11, 14, 18, 25, 33, 43, 56, 75, 99, 131, 176, 232, ... 

Does $m_n$ also grow like $\left(\frac{4}{3}\right)^n$?

If $m_n = O\left(\left(\frac{4}{3}\right)^n\right)$, then $F_q(t)$ is absolutely continuous at $q = \frac{2}{3}$. 

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Index versus $B_5$ entry
Index versus $B_7$ entry
Index versus $B_9$ entry
Index versus $B_{11}$ entry
Polynomial Approach
Translating DSA as a Polynomial Recursion

Consider the polynomial $p_n(x) := b_0 + b_1 x + ... + b_t x^t$ where $B_n = (b_0, b_1, ..., b_t)$ is the Bernoulli sequence on level $n$ where $t = 3^n - 1$.

We see that the duplication $b_0, b_0, b_1, b_1, ..., b_r, b_r$ corresponds to the polynomial $(1 + x)p_n(x^2)$. Shifting the sequence $3^n$ places to the right corresponds to multiplication by $x^{3^n}$.

Thus, for $p_0(x) = 1$ we have the recurrence

$$p_{n+1}(x) = (1 + x)p_n(x^2) \left(1 + x^{3^n}\right).$$
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Thus, for $p_0(x) = 1$ we have the recurrence

$$p_{n+1}(x) = (1 + x)p_n(x^2) \left(1 + x^{3^n}\right).$$
This formula allows us to explicitly solve for $p_n(x)$.

Theorem

The polynomials $p_n(x)$ satisfy

$$p_n(x) = \prod_{i=0}^{n-1} \left( 1 + x^{2^i} \right) \prod_{j=0}^{n-1} \left( 1 + x^{2^{n-1}(3/2)^j} \right).$$

This follows by induction.
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By factoring $p_n$ in a clever way, we can put a bound on how fast the coefficients grow with the level $n$.

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The maximum values obeys $m_n = O((\sqrt{2})^n)$. 

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$$s_n(x) = \prod_{1 \leq j \leq n-1, j \text{ odd}} \left(1 + x^{2^{n-1} \left(\frac{3}{2}\right)^j}\right)$$
$$r_n(x) = \prod_{1 \leq j \leq n-1, j \text{ even}} \left(1 + x^{2^{n-1} \left(\frac{3}{2}\right)^j}\right) = \prod_{j=1}^{\lfloor (n-1)/2 \rfloor} \left(1 + x^{2^{n-1} \left(\frac{9}{4}\right)^j}\right).$$
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We see that

\[ p_n(x) = q_n(x) \left( 1 + x^{2^{n-1}} \right) r_n(x)s_n(x). \]

Consider the polynomial \( q_n(x)r_n(x) \). Because \( 9/4 > 2 \), we have distinct powers of \( x \) when we expand \( q_n(x)r_n(x) \).

In other words, the coefficients are all either 0 or 1. Hence the coefficients of \( q_n(x)r_n(x)(1 + x^{2^{n-1}}) \) are all either 0, 1, or 2.
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In particular, the coefficients are bounded. On the other hand, there are at most $n/2$ terms in the product defining $s_n(x)$.

Hence there are at most $2^{n/2}$ nonzero terms in the polynomial $s_n(x)$ since we have 2 choices from each term in the product.

Therefore the coefficients of $p_n(x)$ are all $O(2^{n/2}) = O((\sqrt{2})^n)$. 
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Better Bound
Seeing that our sequence on level $n$ has length $3^n$, we naturally index it by the first $3^n$ nonnegative integers.

In certain circumstances, it is advantageous to normalize the indexing in such a way that each index is on the interval $[0, 1]$.

To this end, we can simply take the image of $k \in \{0, 1, 2, ..., 3^n - 1\}$ under the map $k \mapsto k/3^n$. 
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To this end, we can simply take the image of $k \in \{0, 1, 2, \ldots, 3^n - 1\}$ under the map $k \mapsto k/3^n$. 
Let $g_n(x)$ denote the $n^{th}$ level Bernoulli sequence where now $x \in [0, 1]$. In other words,

$$g_n \left( \frac{k}{3^n} \right) = b_k \quad \text{for} \quad k = 0, 1, \ldots, 3^n - 1.$$ 

For a subset $S \subset [0, 1]$, we define

$$\Gamma_n(S) = \max_{x \in S} g_n(x)$$

where $\overline{S} = S \cap \left\{ 0, \frac{1}{3^n}, \frac{2}{3^n}, \ldots, \frac{3^n - 1}{3^n} \right\}$. 
Let $g_n(x)$ denote the $n^{th}$ level Bernoulli sequence where now $x \in [0, 1]$. In other words,

$$g_n\left(\frac{k}{3^n}\right) = b_k \quad \text{for } k = 0, 1, \ldots, 3^n - 1.$$ 

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We now walk through an example to demonstrate our algorithm.

Each entry on level $n$ can be written as a sum of entries of previous levels. In this particular example we write each entry on level $n$ as a sum of entries on level $n - 3$.

We break up the interval $[0, 1]$ into subintervals of length $1/81$. Let’s see what we get.
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Pullback diagram
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\[
\begin{align*}
&\text{level } n - 3 \\
&\text{level } n - 2 \\
&\text{level } n - 1 \\
&\text{level } n
\end{align*}
\]
### Motivation

Polynomial Approach

Better Bound

Conclusion

### Notation

An Example

---

**Largest real root**

<table>
<thead>
<tr>
<th>Interval</th>
<th>Polynomial</th>
<th>Largest real root</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$x^n - 2x^{n-3} - x^{n-9}$</td>
<td>1.301688030…</td>
</tr>
<tr>
<td>2</td>
<td>$x^n - x^{n-3} - x^{n-4} - x^{n-7}$</td>
<td>1.288452726…</td>
</tr>
<tr>
<td>3</td>
<td>$x^n - x^{n-3} - x^{n-4} - x^{n-6}$</td>
<td>1.304077155…</td>
</tr>
<tr>
<td>4</td>
<td>$x^n - x^{n-3} - x^{n-4} - x^{n-5} - x^{n-9}$</td>
<td>1.349240712…</td>
</tr>
<tr>
<td>5</td>
<td>$x^n - x^{n-3} - x^{n-7} - 2x^{n-5}$</td>
<td>1.342242489…</td>
</tr>
<tr>
<td>6</td>
<td>$x^n - x^{n-3} - x^{n-4} - x^{n-5} - x^{n-6}$</td>
<td>1.380277569…</td>
</tr>
<tr>
<td>7</td>
<td>$x^n - x^{n-3} - x^{n-4} - x^{n-5} - x^{n-6}$</td>
<td>1.380277569…</td>
</tr>
<tr>
<td>8</td>
<td>$x^n - x^{n-3} - x^{n-4} - x^{n-5} - x^{n-7}$</td>
<td>1.366811194…</td>
</tr>
<tr>
<td>9</td>
<td>$x^n - x^{n-3} - 2x^{n-4} - x^{n-9}$</td>
<td>1.375394454…</td>
</tr>
<tr>
<td>10</td>
<td>$x^n - x^{n-3} - 2x^{n-4}$</td>
<td>1.353209964…</td>
</tr>
<tr>
<td>11</td>
<td>$x^n - x^{n-3} - 2x^{n-4}$</td>
<td>1.353209964…</td>
</tr>
<tr>
<td>12</td>
<td>$x^n - 2x^{n-3} - x^{n-5}$</td>
<td>1.363964602…</td>
</tr>
<tr>
<td>13</td>
<td>$x^n - 2x^{n-3} - x^{n-5} - x^{n-9}$</td>
<td>1.385877646…</td>
</tr>
<tr>
<td>14</td>
<td>$x^n - 2x^{n-3} - x^{n-6} - x^{n-7}$</td>
<td>1.383834352…</td>
</tr>
</tbody>
</table>
1.33997599527 ... 

This example gives us the bound $m_n = \Theta((1.385877646 \ldots)^n)$.

Continuing this process by pulling back 25 levels for $n = 33$, we see that $1.33997599527 \ldots$ is the largest real root of the polynomial

$$X^{33} - 752X^8 - 520X^7 - 319X^6 - 231X^5 - 141X^4 - 101X^3 - 54X^2 - 50X - 83,$$

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Conclusion
The maximums satisfy $m_n = O((1.33997599527 \ldots)^n)$.

We conjecture that $m_n = O\left(\left(\frac{4}{3}\right)^n\right)$, and therefore $F_q(t)$ is absolutely continuous at $q = \frac{2}{3}$.
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Further Questions

1. Can our bound improvement algorithm be pushed to further lower the bound given more computational power?

2. Is it possible to conclusively prove our conjecture?

3. Furthermore, is there an explicit formula to describe $m_n$ for any arbitrary level?

4. What else can be said regarding the global behavior of the Bernoulli sequence $B_n$?
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Thank you for listening!