Rainbow Copies of $C_4$ in Edge-Colored Hypercubes

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July 10, 2014
Definitions
For a graph $G$, an edge coloring

$$\varphi : E(G) \rightarrow \{1, 2, \ldots\}$$

is called **monochromatic** if all edges receive the same color.
Rainbow Coloring

For a graph $G$, an edge coloring

$$\varphi : E(G) \rightarrow \{1, 2, \ldots\}$$

is called **rainbow** if no two edges receive the same color.
Let $Q_d$ have vertices corresponding elements of $\{0, 1\}^d$ and put edges between elements of Hamming distance 1.
Motivation
Rainbow Variants

Many people have studied the maximum number of rainbow subgraphs of a certain type in hypercubes.

- $C_4$ (Faudree, Gyárfás, Lesniak, and Schelp)
- Cycles (Mubayi and Stading)
- $Q_3$ (Mubayi and Stading)
We were motivated by the work of Faudree, Gyárfás, Lesniak, and Schelp published in 1993.

**Theorem (Faudree, Gyárfás, Lesniak, and Schelp)**

If \( d \in \mathbb{N} \) with \( 4 \leq d \) and \( d \neq 5 \), then there is a \( d \)-edge-coloring of \( Q_d \) such that every \( C_4 \) is rainbow.
$d = 4$
Faudree, Gyárfás, Lesniak, and Schelp claim that there is no 5-edge-coloring of \( Q_5 \) where every copy of \( C_4 \) is rainbow.

Using a computer, we find that the maximum number of rainbow copies of \( C_4 \) in a 5-edge-coloring of \( Q_5 \) is 73 out of the 80 total copies of \( C_4 \).
Faudree, Gyárfás, Lesniak, and Schelp claim that there is no 5-edge-coloring of $Q_5$ where every copy of $C_4$ is rainbow.

Using a computer, we find that the maximum number of rainbow copies of $C_4$ in a 5-edge-coloring of $Q_5$ is 73 out of the 80 total copies of $C_4$. 
We would like to understand this case better.

Perhaps the reason for this unusual behavior is the ratio between number of edges and the total copies of $C_4$.

The number of edges of $Q_5$ is

$$d2^{d-1} = 80,$$

exactly equal to the total copies of $C_4$ in $Q_5$

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Theorem (Balogh, D., Lidický, Palmer, 2013+)

Fix \( k, d \in \mathbb{N} \) such that \( 4 \leq k < d \) and \( k \neq 5 \). Then the maximum number of rainbow copies of \( C_4 \) in a \( k \)-edge-coloring of \( Q_d \) is

\[
2^{d-2} \left[ \binom{d}{2} - k \binom{a}{2} - ba \right]
\]

where \( d = ka + b \) with \( a \in \mathbb{N} \) and \( b \in \{0, 1, 2, \ldots, k - 1\} \).
Assume that $Q_d$ is edge-colored with colors 

$$[k] = \{1, \ldots, k\}$$

such that the number of rainbow copies of $C_4$ is maximized.

A vertex in $Q_d$, say $v$, has $\binom{d}{2}$ incident copies of $C_4$.

In the set of $t_i$ edges of color $i \in [k]$ which are incident to $v$, none of the $\binom{t_i}{2}$ possible pairs can be in a rainbow copy of $C_4$. 
Assume that $Q_d$ is edge-colored with colors $[k] = \{1, \ldots, k\}$ such that the number of rainbow copies of $C_4$ is maximized.

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In the set of $t_i$ edges of color $i \in [k]$ which are incident to $v$, none of the $\binom{t_i}{2}$ possible pairs can be in a rainbow copy of $C_4$. 
If the color classes are as equal as possible and

\[ t_1 + \ldots + t_k = d = ka + b, \]

then there are at most

\[
\binom{d}{2} - \sum_{i \in [k]} \binom{t_i}{2} \leq \binom{d}{2} - (k - b) \binom{a}{2} - b \binom{a + 1}{2} \\
= \binom{d}{2} - k \binom{a}{2} + b \binom{a}{2} - b \binom{a + 1}{2} \\
= \binom{d}{2} - k \binom{a}{2} - ba
\]

rainbow copies of \( C_4 \) at \( v \).
Upper bound

Summing up this for each of the $2^d$ vertices of $Q_d$, we overcount by a factor of four.

Thus, the maximum number of rainbow copies of $C_4$ in a $k$-edge-coloring of $Q_d$ is at most

$$2^{d-2} \left[ \binom{d}{2} - k \binom{a}{2} - ba \right],$$

as desired.
Lower bound

We would like to use edge-coloring of $Q_k$ to color edges of $Q_d$.

Now we give a construction using a “blow-up technique”.

Thinking of vertices of $Q_d$ as elements of $\{0, 1\}^d$, we want to partition each string into $k$ “blocks” of consecutive binary digits of length either $a$ or $a + 1$.

We partition the first $(k - b)a$ binary digits into $(k - b)$ blocks of length $a$ and the last $b(a + 1)$ digits into $b$ blocks of length $a + 1$. 
We associate an element of \( \{0, 1\}^k \) with each vertex of \( Q_d \) by computing the sum of the terms in each block modulo 2. This process gives a map

\[
h : V(Q_d) \to V(Q_k).
\]

For example, consider \( d = 10 \) and \( k = 3 \):

\[
\begin{array}{ccc}
111 & 011 & 1011 \\
1 & 0 & 1
\end{array}
\]

and

\[
h(1110111011) = 101.
\]
Furthermore, $h$ preserves edges.

If $u, v \in V(Q_d)$ have Hamming distance 1, then $h(u)$ and $h(v)$ differ exactly in one block and have Hamming distance 1.

Again consider $d = 10$ and $k = 3$:

\[\begin{array}{ccc|ccc|ccc}
111 & 011 & 1011 & 111 & 011 & 1111 \\
1 & 0 & 1 & 1 & 0 & 0 \\
\end{array}\]
Faudree, Gyárfás, Lesniak, and Schelp showed there is a $k$-edge-coloring of $Q_k$, say $\varphi$, such that every $C_4$ is rainbow.

Color edges of $Q_d$ with the color of their image under $h$ in $Q_k$, i.e. the color of an edge $e$ in $Q_d$ is $\varphi(h(e))$.

Using this coloring, each vertex in $Q_d$ is incident to $d$ edges, $a$ edges of each of $k - b$ colors and $a + 1$ edges of each of the remaining $b$ colors.
Lower bound

We must check that for each vertex $v$ in $Q_d$, each pair of edges with different colors incident to $v$ is actually in a rainbow $C_4$.

Note that among the four vertices in any $C_4$ the maximum Hamming distance is 2.

Thus, all differences among elements of $\{0, 1\}^d$ of the four vertices of the $C_4$ occur in at most 2 blocks.
Lower bound

If all the differences occur in the same block, then the four edges of the $C_4$ are mapped to the same edge in $Q_k$, and thus, the $C_4$ is monochromatic in $Q_d$.

If the differences occur in 2 distinct blocks, then the four edges of the $C_4$ are mapped to a $C_4$ in $Q_k$, and thus, receive different colors in the coloring of $Q_d$. 
Further Directions
Upper Bound
For $k = 5$, flag algebra methods did not improve the upper bound obtained from our main result.

We actually suspect that the upper bound might be the correct order of magnitude for large $d$.

Lower Bound
For a lower bound, our blow-up method can be applied to a 5-edge-coloring of $Q_5$ with 73 rainbow copies of $C_4$.

This, however, is very far from our upper bound.
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This, however, is very far from our upper bound.
Let $G$ and $H$ be graphs and $|E(H)| \geq q \in \mathbb{N}$.

Denote the minimum number of colors required to edge-color $G$ such that the edges of every copy of $H$ in $G$ receive at least $q$ colors by

$$f(G, H, q).$$

In this context, Faudree, Gyárfás, Lesniak, and Schelp show

$$f(Q_d, C_4, |E(C_4)|) = f(Q_d, C_4, 4) = d,$$

for integer $4 \leq d$ with $d \neq 5$. 
Let $G$ and $H$ be graphs and $|E(H)| \geq q \in \mathbb{N}$.

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$$f(Q_d, C_4, |E(C_4)|) = f(Q_d, C_4, 4) = d,$$

for integer $4 \leq d$ with $d \neq 5$. 
Mubayi and Stading generalized this result.

They proved that there are positive constants, say \( c_1 \) and \( c_2 \), depending only on \( k \) such that

\[ c_1 d^{k/4} < f(Q_d, C_k, k) < c_2 d^{k/4} \]

for \( k \equiv 0 \pmod{4} \).
Mubayi and Stading generalized this result.

They proved that there are positive constants, say $c_1$ and $c_2$, depending only on $k$ such that

$$c_1 d^{k/4} < f(Q_d, C_k, k) < c_2 d^{k/4}$$

for $k \equiv 0 \pmod{4}$. 
Mubayi and Stading showed that

\[ f(Q_d, C_6, 6) = f(Q_d, Q_3, |E(Q_3)|) = f(Q_d, Q_3, 12). \]

They were able to show that for every \( \varepsilon > 0 \), there exists \( d_0 \) such that for \( d > d_0 \)

\[ d \leq f(Q_d, Q_3, 12) \leq d^{1+\varepsilon}. \]
Mubayi and Stading showed that
\[ f(Q_d, C_6, 6) = f(Q_d, Q_3, |E(Q_3)|) = f(Q_d, Q_3, 12). \]

They were able to show that for every \( \varepsilon > 0 \), there exists \( d_0 \) such that for \( d > d_0 \)
\[ d \leq f(Q_d, Q_3, 12) \leq d^{1+\varepsilon}. \]
**Problem**

*Determine the value of*

\[ f(Q_d, Q_\ell, |E(Q_\ell)|) = f(Q_d, Q_\ell, \ell 2^{\ell-1}) \]

*for \( \ell \geq 3. \)*

Perhaps a generalization of our blow-up technique could be used to determine the maximum number of rainbow copies of \( Q_\ell \) in a \( k \)-edge-coloring of \( Q_d \) in general.
Problem

Determine the value of

$$f(Q_d, Q_\ell, |E(Q_\ell)|) = f(Q_d, Q_\ell, \ell 2^{\ell-1})$$

for $\ell \geq 3$.

Perhaps a generalization of our blow-up technique could be used to determine the maximum number of rainbow copies of $Q_\ell$ in a $k$-edge-coloring of $Q_d$ in general.
Thank you for listening!