1. Prove from the definition of a convergent sequence that if \( a_n \to L \) and \( b_n \to M \), then \( a_n + b_n \to L + M \).

**Solution.** This is theorem 14.5(a) in the book.

2. (a) Prove that every infinite subset of a countable set is countable.

(b) Prove that every set that contains an uncountable set is uncountable.

**Solution.** This is problem 13.7 on homework 6.

3. (a) State the definition of a Cauchy sequence and the Cauchy convergence criterion.

(b) State the definition of a subsequence and the Bolzano–Weierstrass Theorem.

**Solution.** These are Definition 14.12, Theorem 14.19, and Theorem 14.17 in the book.

4. Let \( a, b \in \mathbb{N} \).

(a) State the definition of \( \gcd(a, b) \).

(b) Prove that the set of integer combinations of \( a \) and \( b \) equals the set of integer multiples of \( \gcd(a, b) \).

**Solution.** a) \( \gcd(a, b) \) is the largest natural number that divides both \( a \) and \( b \).

b) This is Theorem 6.12 in the book.

5. Prove that a bounded sequence need not be Cauchy by providing a counterexample and a formal \( \epsilon \)-style argument showing that your sequence is not a Cauchy sequence.

**Solution.** Consider \( a_n = (-1)^n \) and take \( \epsilon = 1/2 \) and set \( m = n + 1 \). Then for all \( N \), if \( n, m \geq N \) we have

\[
|a_n - a_m| = |a_n - a_{n+1}| = |2| \geq 1/2 = \epsilon,
\]

so the sequence is not Cauchy.

6. The set of **irrational numbers** is \( \mathbb{R} \setminus \mathbb{Q} \). Prove that the irrational numbers are uncountable.

**Solution.** We argue by contradiction, so assume \( \mathbb{R} \setminus \mathbb{Q} \) is countable, and let \( f : \mathbb{N} \to \mathbb{R} \setminus \mathbb{Q} \) be a bijection. Since \( \mathbb{Q} \) is countable, there is also a bijection \( g : \mathbb{N} \to \mathbb{Q} \). We use these bijections to define a new bijection, \( h : \mathbb{N} \to \mathbb{R} \), by

\[
h(n) = \begin{cases} 
  f(n/2) & n \text{ even} \\
  g((n+1)/2) & n \text{ odd}
\end{cases}
\]

Since \( h \) has an inverse \( h^{-1} : \mathbb{R} \to \mathbb{N} \) defined by

\[
h^{-1}(x) = \begin{cases} 
  2f^{-1}(x) & x \in \mathbb{R} \setminus \mathbb{Q} \\
  2g^{-1}(x) - 1 & x \in \mathbb{Q}
\end{cases}
\]

\( h \) is indeed a bijection as claimed. But since \( \mathbb{R} \) is uncountable, there cannot exist a bijection \( h : \mathbb{N} \to \mathbb{R} \), which gives a contradiction to \( \mathbb{R} \setminus \mathbb{Q} \) being countable. Hence, \( \mathbb{R} \setminus \mathbb{Q} \) is uncountable.
7. (a) State the comparison test for series.
   Solution. See Proposition 14.29 in the book.

(b) Prove by induction that $3^n - 1 \geq 2^n$ when $n \geq 1$.
   Solution. For a base case, we have $3^1 - 1 = 2 \geq 2^1$. For the inductive step, assume that $3^n - 1 \geq 2^n$ up to some $n$. Then
   \[3^{n+1} - 1 = 3(3^n - 1) + 2 > 3(3^n - 1) > 2(3^n - 1) \geq 2 \cdot 2^n = 2^{n+1}\]
   where the last greater than or equal to sign uses the inductive hypothesis. This completes the induction.

(c) Prove that \[\sum_{n=1}^{\infty} \frac{1}{3^n - 1}\] converges.
   Solution. By the comparison test, we have
   \[0 \leq \frac{1}{3^n - 1} \leq \frac{1}{2^n}\]
   and since the series \[\sum_{n=1}^{\infty} \frac{1}{2^n}\] converges (since it is a geometric series) the series \[\sum_{n=1}^{\infty} \frac{1}{3^n - 1}\] also converges by the comparison test.

8. For the following questions, decide whether the statement is true or false and provide a brief proof or counterexample justifying your choice.

(a) If \(\lim_{n \to \infty} a_n\) does not exist, then \(\sum_{n=1}^{\infty} a_n\) is divergent.
   Solution. TRUE. The contrapositive of the statement reads “If \(\sum_{n=1}^{\infty} a_n\) is convergent, then \(\lim_{n \to \infty} a_n\) exists” is true, since this limit must exist and equal zero.

(b) If \(\lim_{n \to \infty} a_n = 1\) and \(\lim_{n \to \infty} b_n\) does not exist, then \(\lim_{n \to \infty} a_n b_n\) does not exist.
   Solution. TRUE. We argue by contradiction. Suppose \(\lim_{n \to \infty} a_n b_n = L\). Then since \(1/a_n \to 1/1 = 1\), we have that
   \[L = \lim_{n \to \infty} a_n b_n \lim_{n \to \infty} \left(1/a_n\right) = \lim_{n \to \infty} (a_n b_n/a_n) = \lim_{n \to \infty} b_n,\]
   but since \(\lim_{n \to \infty} b_n\) does not exist, this is a contradiction.
   Note: It would be wrong to start out by writing
   \[\lim_{n \to \infty} a_n b_n = \lim_{n \to \infty} a_n \lim_{n \to \infty} b_n\]
   since this formula only holds with both limits on the right exist.

(c) If \(\lim_{n \to \infty} (a_{n+k} - a_n) = 0\) for every \(k \in \mathbb{N}\), then \(\{a_n\}\) converges.
   Solution. FALSE. The sequence \(a_n = \sqrt{n}\) provides a counterexample since
   \[\lim_{n \to \infty} (\sqrt{n + k} - \sqrt{n}) = \lim_{n \to \infty} \frac{k}{\sqrt{n + k} + \sqrt{n}} = 0\]
   for every \(k \in \mathbb{N}\).
(d) If $\sum_{k=1}^{\infty} a_k^2$ diverges, then $\sum_{k=1}^{\infty} a_k$ diverges.

**Solution.** FALSE. A counterexample is $a_k = (-1)^k / \sqrt{k}$. Then the series $\sum_{k=1}^{\infty} a_k^2$ is the harmonic series (which diverges) while $\sum_{k=1}^{\infty} a_k$ converges.

9. (a) Compute the gcd of 78 and 90 using the Euclidean algorithm.

**Solution.** We did this one in review: the Euclidean algorithm for this pair is

$\text{gcd}(78, 90) = \text{gcd}(90 - 78, 78) = \text{gcd}(12, 78) = \text{gcd}(78 - 6 \cdot 12, 12) = \text{gcd}(6, 12) = \text{gcd}(12 - 2 \cdot 6, 6) = \text{gcd}(0, 6) = 6.$

(b) Compute the gcd of 78 and 90 using prime factorization. (HINT: 78 = 6 · 13.)

**Solution.** We have 90 = 9 · 10 = 2 · 3² · 5 = 6 · 15 and 78 = 6 · 13. Since 13 and 15 are relatively prime, we have $\text{gcd}(90, 78) = 6$.

10. (a) Carefully explain how an infinite decimal expansion of a real number defines a Cauchy sequence whose limit is the real number given by the decimal expansion.

**Solution.** An infinite decimal expansion of a real number $x \in [0, 1]$ defines a sequence

$$x_k = \sum_{n=1}^{k} a_n \frac{1}{10^n}$$

with $x_k \to x$. This sequence is Cauchy since for all $m \geq n \geq N$ we have

$$|x_m - x_n| < \frac{1}{10^{n-1}} \leq \frac{1}{10^{N-1}}$$

(b) Give a careful proof (using the definition of a convergent sequence) that .999· · · = 1.

**Solution.** The infinite decimal .\bar{9} is the limit of the sequence

$$x_k = \sum_{n=1}^{k} \frac{9}{10^n}.$$ We have $1 - x_k = \frac{1}{10^k}$. Hence, for any given $\epsilon > 0$, there exists an $N$ such that for all $n \geq N$,

$$|1 - x_n| = \frac{1}{10^n} \leq \frac{1}{10^N} < \epsilon$$

and hence the limit of the sequence $x_k$ is 1. This shows .\bar{9} = 1.