Math 147 Practice Midterm Exam

Name: ________________________________  SUID#: ___________

- Complete the following problems. In order to receive full credit, please show all of your work and justify your answers. Unless stated otherwise, you may use any result proved in class or the text, but be sure to clearly state the result before using it, and to verify that all hypotheses are satisfied.

- Please check that your copy of this exam contains 10 numbered pages and is correctly stapled.

- This is a closed-book, closed-notes exam. No electronic devices, including cellphones, headphones, or calculation aids, will be permitted for any reason.

- **You have 2 hours.** The organizer will signal the times between which you are permitted to be writing, including anything on this cover sheet, and to have the exam booklet open. During these times, the exam and all papers must remain in the testing room. When you are finished, you must hand your exam paper to a member of teaching staff.

- If you need extra room for your answers, use the back sides of each page. If you must use extra paper, use only that provided by teaching staff; make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.

- Please sign the following:

  "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

  **Signature: ________________________________**
The following boxes are strictly for grading purposes. Please do not mark.

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1. (30 points) Show that the subspace $O(n) \subset M_n(\mathbb{R}) = \mathbb{R}^{n^2}$ of orthogonal matrices is a manifold.
[HINT: Consider the function $A \mapsto AA^T$ and show that the identity matrix is a regular value.]
2. (30 points) Prove the Stack of Records Theorem:

Suppose that $y$ is a regular value of $f : X \to Y$ where $X$ is compact and has the same dimension as $Y$. Show that $f^{-1}(y)$ is a finite set $\{x_1, \ldots, x_n\}$, and that there exists a neighborhood $U$ of $y \in Y$ such that $f^{-1}(U)$ is a disjoint union $V_1 \cup \cdots \cup V_n$ where $V_i$ is an open neighborhood of $x_i$ and $f$ maps each $V_i$ diffeomorphically onto $U$. 
[HINT: Pick disjoint neighborhoods $W_i$ of $x_i$ that are mapped diffeomorphically under $f$ to $V_i$. Show that $f(X - \cup W_i)$ is compact and does not contain $y$, and take $U = V_1 \cap \cdots \cap V_n - f(X - \cup W_i)$.]
3. (30 points) The unit sphere bundle of a smooth $m$-manifold $M \subset \mathbb{R}^k$ is the set of unit tangent vectors in $TM$,

$$SM := \{(x, v) \mid x \in M, v \in TM_x, |v|^2 = 1\}.$$ 

Show that $SM$ is a smooth manifold of dimension $2m - 1$. Also show that $TM$ is compact whenever $M$ is compact.
[HINT: Let $f : TM \to \mathbb{R}$ be defined by $f(x, v) = |v|^2$. Show that 1 is a regular value.]
4. (30 points) Suppose $f$ and $g$ are smooth maps $M \to S^2$ that satisfy $|f(x) - g(x)| < 2$ for each $x \in M$. Show that $f$ and $g$ are smoothly homotopic.
[HINT: The assumed inequality shows that $f(x)$ and $g(x)$ are not antipodal, and hence there is a \textit{unique} great circle connecting $f(x)$ to $g(x)$. Use these paths to define a smooth homotopy.]
5. (30 points) Show that the boundary of a $n$-dimensional manifold with boundary is an $(n-1)$-dimensional manifold.