THE WITTEN GENUS, AFTER KEVIN COSTELLO

These are notes from the Witten Genus Seminar that took place at the Max Planck Institute in Bonn, Fall 2010. Any errors in what follows are mine.

1. Introduction, Daniel Berwick Evans

In a series of two papers [Cos10a, Cos10b], Kevin Costello gives a construction of the Witten genus via his notion of a quantum field theory. The goal of this seminar will be to develop enough machinery to understand the proof of this theorem, and try to grasp some of it’s implications for physically-motivated mathematics.

1.1. Genera.

Definition 1.1. Given a ring $R$, a genus with values in $R$ is a ring homomorphism, $\Omega^G \otimes \mathbb{Q} \to R$, where $\Omega^G$ is the $G$-bordism ring.

For example, the $\hat{A}$-genus and $L$-genus are ring maps from $\Omega\otimes\mathbb{Q} \to \mathbb{Q}$. The Atiyah-Singer theorem shows that $\hat{A}$ can be refined to a genus $\Omega^{Spin} \otimes \mathbb{Q} \to \mathbb{Z}$. The $L$-genus (or signature) is defined on $\Omega \otimes \mathbb{Q}$, and the Todd genus is defined on the complex cobordism category.

We can define genera via multiplicative sequences, which we won’t discuss here, but see e.g. [HBJ92]. Essentially, they are certain power series in characteristic numbers, and so may be evaluated on a manifold. An old theorem of Thom guarantees this will be a cobordism invariant, and certain properties of the power series force the invariant to give a ring homomorphism. One way to define the Witten genus is by some multiplicative sequence:

$$\exp \left( \sum_{k \geq 2} \frac{2G_{2k}(\tau)z^{2k}}{(2k)!} \right),$$

where $G$ is a multiple of some Eisenstein series. This will give a ring homomorphism from $\Omega^{Spin} \otimes \mathbb{Q}$ to the ring of modular forms (hence the parameter $\tau$ in the above formula).

The problem with this formula is that it doesn’t tell us anything about geometry. The Atiyah-Singer theorem does precisely this for $\hat{A}$: we realize this genus as the index of the Dirac operator. This formulation led to the celebrated result:

Theorem 1.2. If $M$ admits a metric of positive scalar curvature, then $\hat{A}$ vanishes.

We may wish to generalize this to other genera. Indeed, this is the content of the Stolz conjecture: there are conditions under which a positive Ricci metric is expected to force the Witten genus to vanish (though a precise formulation is somewhat subtle). Proving such statements will be impossible until
we have a better understanding of the geometry that underlies the Witten genus.

1.2. Witten’s Idea from Supersymmetric Sigma Models. Witten’s original definition is arguably geometric, if one thinks the language of sigma models in physics is related to geometry. We will give a very rough idea of how this works to motivate what will follow, but be warned that there will be no attempts made (yet) to define physical ideas.

So, a sigma model is a field theory whose classical space of fields is

\[ \mathcal{F} := \text{Maps}(E, X) \]

where \( E \) is (say) some elliptic curve with some super geometric structure, and \( X \) is a Riemannian manifold. We think of \( \mathcal{F} \) as being some infinite dimensional supermanifold. Then we define some action function on this space, which is roughly the energy of the map:

\[ S(\Phi) = \int_{E} ||\Phi||^2 d\mu \]

where the measure is defined in terms of the conformal structure on \( E \) and the Riemannian structure on \( X \). This (more or less) describes the classical theory.

Now if we wish to quantize this theory, Feynman tells us to make sense of the (asymptotics of the) integral:

\[ Z_X(q) \sim \int_{\mathcal{F}} e^{-S} d\Phi \]

where \( d\Phi \) is some undefined measure. Kevin makes sense out of some related integral in his construction of the Witten genus.

1.3. The Statement of Costello’s Result.

**Theorem 1.3.** Let \( X \) be a Kähler manifold. Then

1. The obstruction to quantization of holomorphic Chern-Simons theory is

\[ ch_2(TX) \in H^2(X, \Omega_{cl}^2(X)) \]

where \( ch_2 \) is the degree 2 part of the Chern character in \( \Omega_{cl}^2(X) \), the sheaf of closed holomorphic 2-forms on \( X \). Moreover, the simplicial set of quantizations of this theory is quasi-isomorphic to trivializations of this class.

2. If \( E \) is an elliptic curve with holomorphic volume element \( \omega \), then there is a quasi-isomorphism of Bellinson-Drinfeld algebras in quasi-coherent sheaves on \( X_{dR} \times \mathbb{C}[\hbar] \):

\[ \{\text{Global quantum observables}\} \cong (\Omega^{-\bullet}(\mathcal{T}^*X)[\hbar], \hbar L_{\pi} + \hbar \{\log \text{Wit}(X, E, \omega), -\} \}

where \( L_{\pi} = [d, \iota_\pi] \), where \( \pi \) is the Poisson tensor on \( \mathcal{T}^*X \) and \( \{-, -\} \) is the Poisson bracket.

We will need to spend a significant amount of time and effort to understand the statement of the theorem. There will be several talks devoted to various topics alluded to above. For now we will attempt to give a little clarification:
Holomorphic Chern-Simons theory is related to the AKSZ sigma model whose fields are maps $\Phi : \mathbb{C} \rightarrow T^*X$ and Lagrangian is $dz \wedge \Phi^*\alpha$ where $\alpha$ is the canonical 1-form on $T^*X$. However, Kevin uses some form of “Koszul duality” to turn this sigma model into a sort of gauge theory, albeit one whose gauge algebra is a curved $L_\infty$-algebra; $L_\infty$-algebras are homotopy Lie algebras: the Jacobi identity only holds up to higher coherence relations.

The Chern class above arises as an obstruction to finding BV quantizations of this gauge theory (or equivalently, AKSZ model): one must find solutions to the so-called quantum master equation (QME), and there are topological obstructions to doing this. If these obstructions vanish, one must still choose a solution to the QME, and the theorem states the the space of such choices is given by trivializations of $ch_2(TX)$.

Bellinson-Drinfeld algebras are much like Batalin-Vilkovisky (BV) algebras, but with slightly different grading conventions. In both cases (roughly), we have a graded algebra with a (graded) differential and a (graded) bracket satisfying certain compatibility relations.

“Global observables” of the quantum theory will be global sections of a particular cosheaf (or more precisely, a factorization algebra) gotten from the above quantization. This cosheaf is a deformation of the “classical observables,” which is roughly the Poisson algebra of $T^*X$, a commutative algebra with bracket and differential. The QME gives a deformation of this structure to a noncommutative algebra.

Let me just say a few more words about what a “field theory” is here. Kevin’s notion of quantum field theory all comes down to computing the Feynman integral perturbatively, which means we treat $\hbar$ as a formal parameter and expand everything in question in power series in $\hbar$. We start with a (linear) space of fields (think of sections of a vector bundle) and an action functional. The critical points of this action functional are the classical solutions, which we think of as some subvariety in the space of fields. The perturbative integral is supported on a formal neighborhood of these classical solutions. In the best of cases, we choose an isolated solution and (using heat kernel techniques) we try to understand the usual physicists Feynman diagram description of this integral.

However, often these diagrams themselves diverge. In order to extract finite answers, we must make a choice: a renormalization scheme. Morally, this chooses a splitting of our functions into “singular” and “smooth.” Once we’ve made this choice, we construct “counter-terms” that cancel the divergences and the pertubative integrals all make sense.

If we have gauge symmetry, things are somewhat more complicated: there may not be any choice that allows us to quantize. To manage this problem, Kevin uses the BV formalism which allows us to understand the salient problems in terms of homological algebra. We’ll spend at least a few talks understanding theories from this perspective.

So roughly our work is divided into two parts that are somewhat entangled:
(1) We need to understand Kevin’s notion of field theory. He has an entire book devoted to this \cite{Costello10c}, and we will certainly not try to digest all of it in this seminar. It will turn out that holomorphic Chern-Simons does not suffer serious issues of divergences typical of QFTs: “all the counter-terms vanish,” which is to say that the renormalization scheme is in some sense trivial. This will simplify our life considerably. However, we do need to understand effective field theory, the renormalization group equation, BV theories, and BV quantization. I’m not sure yet how much we should learn about factorization algebras, though secretly that’s what we’re constructing in part 2 of the theorem. For more on this point, see Kevin Costello and Owen Gwilliam’s wiki, \cite{CG10}, which can be found on Kevin’s website.

(2) We need to know enough formal and derived geometry to follow the intuition behind Kevin’s constructions. One way to motivate this problem is that perturbative quantization needs a linear space of fields. The AKSZ sigma model has a nonlinear space of fields, and from that we need to construct some “derived mapping space” that contains the information of the formal neighborhood of the constant maps inside the entire mapping space. Derived geometry is also implicit in the BV construction, and will help us understand better what exactly BV “is doing” from a more geometric perspective.

2. Effective BV Theories, Arturo Prat Waldron

Mostly this is a summary of chapter 5 of Costello’s book. We’ll start with a brief reminder of what effective means, and then we’ll spend most of the time on what BV means.

2.1. Motivation for Effective Theories. In field theories we normally have a (infinite dimensional vector) space of fields $V$, and we want to measure observables, $O \in \mathcal{O}(V)$ (or rather their expectation values $\langle O \rangle$, where

$$\mathcal{O}(V) := \prod_{n=0}^{\infty} ((V^\otimes n)^*)^S_n.$$ 

Normally $V$ would be something like the sections of a vector bundle over “space time.” We have a way of measuring observables in terms of a certain integral we will need to make sense of. To formulate this we need an action functional, $S \in \mathcal{O}(V)$, and we want to compute

$$\langle O \rangle = \int_{\phi \in V} e^{-S(\phi)/\hbar} D\phi \in \mathbb{R}[[\hbar]].$$

For example, a scalar field theory on $M$ has as fields the vector space $C^\infty M$, where $M$ is a Riemannian manifold. We take $S = q + I$ where $q$ is a quadratic function and $I$ is called the “interaction.” We typically have

$$q(\phi) = -\langle \phi, (D_0^2 + m^2)\phi \rangle,$$

and $I \in \mathcal{O}_{loc}(V)$.

However, this usually can’t be made sensible. In some sense, we’re being too greedy by asking to know the physics at all length scales simultaneously.
A solution to this problem (due to Wilson initially, and made mathematically rigorous by Costello) is to introduce a length scale $L > 0$. At each of these length scales we have an interaction $I[L]$. If we were physicists, our lab would be able to probe certain length scales, so we would only need to know $I[L]$ for the fixed $L$. If some other physicist comes along and build a bigger, more expensive lab she might be able to probe a smaller length scale, and our experiments should be related in some coherent way. This is the content of the renormalization group equation,

$$I[L] = W(P(\epsilon, L), I[\epsilon]), \quad 0 < \epsilon < L$$

This more or less tells us that our physical theory is consistent. We also think of $I[L]$ pertubatively now, so

$$I[L] = (q + I_0[L]) + \hbar I_1[L] + \hbar^2 I_2[L] + \cdots = \sum_i \hbar^i I_i[L].$$

2.2. Motivation from Gauge Theories. Let $G$ be a gauge group, and suppose that $G$ acts on the space of fields $V$. Let $S$ be a $G$-invariant action functional. Assume that $S$ has a critical point of at $0 \in V$, and $S(0) = 0$. Again, we want to make sense of

$$\langle O \rangle = \int_{V/G} O(\phi) e^{S/\hbar} D\phi.$$

BV is a recipe to get a well defined answer to this problem. This has several steps:

2.2.1. BRST Construction. The BRST construction linearizes the space, and considers $\mathfrak{g}[1] \oplus V$. Why on earth do we do this? Somehow this is a derived version of the quotient. To see this, consider functions on this space:

$$O(\mathfrak{g}[1] \oplus V) = \Lambda^* \mathfrak{g}^* \otimes O(V).$$

This comes equipped with the Chevelley-Eilenberg differential, and we find that for the resulting complex,

$$H^0(\mathfrak{g}, O(V)) = O(V)^g = O(V/G).$$

Now we have a nicer “space” over which to do the integral, and want to make a definition like

$$\int_{V/G} e^{S/\hbar} := \int_{\mathfrak{g}[1] \oplus V} e^{S/\hbar},$$

but we can’t compute the right hand side with Feynman diagrams because the quadratic part is highly degenerate. Even physicists get stuck here. There are various solutions to this that involve many choices. BV somehow avoids a lot of this by doing symplectic geometry.
2.2.2. **BV Construction.** So in BV we consider the larger space of fields:

\[ E := T^* [-1] (\mathfrak{g}[1] \oplus V) = \mathfrak{g}[1] \oplus V \oplus V^*[1] \oplus \mathfrak{g}^*[2]. \]

This is an odd symplectic manifold, and by this we mean there is an odd Poisson bracket, \( \{ , \} \), on the functions. Now we need to define an action on this space. We do this by pulling back \( S \) along the projection \( E \to \mathfrak{g}[1] \oplus V \). Since this is only a function on the base of the symplectic manifold, so we have that \( \{ S, S \} = 0 \).

The Chevelley-Eilenberg differential \( d_{CE} \) induces an odd vector field \( X \) on \( \mathfrak{g}[1] \oplus V \). We extend this to \( E \) (the easiest way to define this is probably just at the function level: \( X \) is zero on ghosts and antighosts, or something). Since \( d_{CE}^2 = 0 \), we have that \( [X, X] = 2X^2 = 0 \). We have an exact symplectic manifold here, and \( X \) preserves the symplectic structure. So we can choose a Hamiltonian function \( h_X \) with \( \{ h_X, h_X \} = 0 \). We specify \( h_X \) uniquely by saying it vanishes at the origin of \( V \).

Moreover, \( XS = 0 \) since \( d_{CE} S = 0 \) originally. From this we find that \( \{ h_X, S \} = 0 \). With all of this we can define the BV action,

\[ S_{BV} = S + h_X, \]

and it satisfies the “classical master equation”:

\[ \{ S + h_X, S + h_X \} = 0. \]

2.2.3. **“Computing” the Feynman Integral.** Now we want to get back to doing our Feynman integral. So we need a splitting of \( S_{BV} \) into a quadratic and interacting part:

\[ S_{BV}(e) = \frac{1}{2} (e, Qe) + I(e) \]

for \( e \in E \), and \( I \) has degree greater than or equal to 3. We are again assuming that \( S \) has a critical point of at \( 0 \in V \), and \( S(0) = 0 \), so the linear term in the action disappears.

Now from the classical master equation, we have that \( Q : E \to E \) is odd and skew adjoint for the symplectic pairing, \( Q^2 = 0 \) and \( I \) satisfies

\[ QI + \frac{1}{2} \{ I, I \} = 0. \]

Now we wish to make the definition

\[ \int_{V/G} e^{S/\hbar} := \int_{L} e^{S_{BV}/\hbar}, \]

where \( L \subset E \) is a Lagrangian submanifold such that the quadratic part of \( S_{BV} \) is non-degenerate on \( L \). We need to check this is well-defined. First note that \( S_{BV}|_{\mathfrak{g}[1] \oplus V} = S \). Next, can we even choose these Lagrangians? The answer is yes, if \( H^*(E, Q) = 0 \).

Finally, we want this integral to be independent of \( L \). But for this, the classical master equation is not enough, so we must impose the quantum master equation. Essentially, this computation is some version of Stokes theorem. Let’s sketch this briefly.
Let $N$ be some manifold (think $N = g[1] \oplus V$). We’re looking at $E = T^*[1]N$. Then
\[ O(E) = \Gamma(\Lambda^*TN) = \chi^*(N), \]
the polyvector fields on $N$. Let’s choose $\omega \in \Omega^n(N)$, a volume form. Then we have
\[ \chi^k(N) \xrightarrow{\Delta} \omega \downarrow \cong \chi^{k-1}(N) \]
\[ \Omega^{n-k}(N) \xrightarrow{d} \Omega^{n-k+1}(N). \]
So then if $\Delta(e^{I_S BV/\hbar}) = 0$ (which is the quantum master equation) the integral over $L$ only depends on the homology class of $L \subset E$.

**Remark 2.1.** One such class is to take conormal bundles to submanifolds of $E$, and then we can do things quite explicitly.

The QME is equivalent to
\[ (Q + \hbar \Delta) e^{I/h} = 0 \iff QI + \frac{1}{2} \{I, I\} + \hbar \Delta I = 0. \]

From the BV construction, we automatically get interactions that satisfy the classical master equation, but in general $\Delta I_{BV} \neq 0$. So now the idea is to modify the interaction $I_{BV} \rightarrow I' \in O(E)[[\hbar]]$ to satisfy the QME, where we do this recursively in powers of $\hbar$:
\[ I' = I + \sum_{i \geq 1} \hbar^i I'_i. \]

There is no guarantee that this will converge, so in a way we are forced to think of $\hbar$ as a formal parameter. This is the price we must pay for working perturbatively, but at least we can make some definition of these integrals.

### 2.3. Effective BV Theories

Now we want to combine these two ingredients, which more or less amounts to doing BV in families. Now assume $M$ to be compact.

**Definition 2.2.** A free theory in the BV formalism is described by the following data:

1. A $\mathbb{Z}$-graded vector bundle $E \rightarrow M$, $\mathcal{E} = \Gamma(E)$,
2. An odd, skew-symmetric vector bundle over $M$,
   \[ \langle \ , \ \rangle : E \otimes E \rightarrow \text{Den}(M), \]
   fiberwise non-degenerate. This induces a pairing on sections,
   \[ \langle \ , \ \rangle : \mathcal{E} \otimes \mathcal{E} \rightarrow \mathbb{R} \]
   by integration.
3. $Q : \mathcal{E} \rightarrow \mathcal{E}$, an order 1 differential operator of cohomological degree 1, such that $Q^2 = 0$ and is skew adjoint for the pairing.
Let’s quickly look at a partial example of this to connect this definition with our previous intuition. In Chern-Simons, $V = \Omega^1(M) \otimes g$ (with degree 0). Then $g = \Omega^0(M) \otimes g$, in degree -1. Then

$$E = \Omega^*(M) \otimes g[1],$$

a $\mathbb{Z}$-graded bundle on $M$. We have the usual $Q = d_{dR} \otimes d_{CE}$ here.

**Definition 2.3.** A gauge fixing operator on a free BV theory $E$ is an odd operator $Q_{GF} : E \to E$ such that

1. $Q_{GF}$ degree -1, $(Q_{GF})^2 = 0$ and is self-adjoint for $\langle \ , \ \rangle$.
2. $D := [Q, Q_{GF}]$ is a generalized Laplacian, in the sense that the principle symbol is the metric times identity.

Given a free BV theory, we want to define an interacting theory. So we need a “renormalized version” of the BV Laplacian, $\Delta_L$ and of the BV bracket, $\{\ , \ \}_L$. We don’t have the measure that we used in the previous argument to define $\Delta_L$ (since now we are in infinite dimensions), so we have to be slick.

**Definition 2.4.** An effective theory on a free BV theory is a collection of effective interactions of scale $L$,

$$I[L] \subset \mathcal{O}^{+,ev}(E)[[\hbar]].$$

such that

1. the renormalization group equation is satisfied

$$I[L] = W(P(\epsilon, L), I[\epsilon])$$

is satisfied.
2. the interaction is local, which roughly means as $L \to 0$, $I$ becomes a local function.
3. the quantum master equation is satisfied,

$$\Delta_L(e^{S[L]/\hbar}) = 0,$$

which holds if and only if

$$(Q + \hbar \Delta_L)e^{I[L]/\hbar} = 0.$$

A priori, we need to check both the RGE and the QME for all length scales, but the magic is that if the RGE is satisfied, we need only check the QME for a single length scale, and then is will be satisfied at all length scales.

To define the propagator, we take the kernel of $e^{-\epsilon D}$ ($D$ is trace class since it is a generalized Laplacian on a compact manifold), $K_\ell \in E \otimes E$ for $\ell > 0$, which means that

$$K_\ell \star e = e^{-\epsilon D}(e), \quad e \in E.$$

Then

$$P(\epsilon, L) := \int_{\ell=\epsilon}^{\ell=L} (Q_{GF} \otimes 1)K_\ell d\ell,$$

and

$$\Delta_L := -\partial_{K_\ell}$$
which is defined as contraction with $K_\ell$, $\Delta_L : \mathcal{O}(\mathcal{E}) \to \mathcal{O}(\mathcal{E})$, where
\[
\mathcal{O}(\mathcal{E}) := \prod_{n=0}^{\infty}((\mathcal{E}^\otimes n)^*)^{S_n}.
\]
Then
\[
[\partial_{P(\epsilon,L)}, Q] = \Delta_L - \Delta_\epsilon,
\]
and we define
\[
W(P(\epsilon, L), I) := \hbar \log(e^{\hbar \partial_{P(\epsilon,L)} I/\hbar})
\]
So far, we defined this for a compact manifold, but in general we will want to define sheaves of these theories, so we need to define them on open sets, which presents many technical issues.

Remark 2.5. The “global observables” of the quantum theory will be given by the dga
\[
(\mathcal{O}(\mathcal{E})[[\hbar]], Q + \{I, -\} + \hbar \Delta).
\]
I say “roughly” because there is a length scale $L$ dependence one needs to make sense of the differential here in the infinite dimensional setting. However, the above complexes will be quasi-isomorphic for any $L$, so I didn’t include it in the notation. By the way, the above is also sometimes called the “factorization algebra” associated to the QFT. It turns out that this algebra (and some operadic structure on it) completely determines the QFT.

3. Examples of BV Theories, Arturo Prat Waldron

Let’s recall a little about what happened last time, and clarify some things.

We have a space $V$, a (vector) space of fields, with a gauge group $G$ acting on it. We have a classical action $S$, which is a $G$-invariant function on $V$. We’d like to make sense out of
\[
\int_{V/G} e^{-S/\hbar}.
\]
We started by forming $M = V \oplus g[1]$, so $\mathcal{O}(M) = \Lambda^* g^* \otimes \mathcal{O}(V)$, where (the grading on the exterior algebra is positive). We have an odd vector field $X$ on $M$, with $X^2 = 0$, which gives $M$ the structure of an NQ-manifold.

Now consider
\[
\mathcal{E} = T^*[-1]M = g[1] \oplus V \oplus V^*[-1] \oplus g^*[-2].
\]
As an (odd) cotangent bundle, this graded manifold has an (odd) symplectic structure, that gives us an odd Poisson bracket, $\{\cdot, \cdot\}$ on its functions. We call this Poisson structure a P-structure for short.

We extend our original function $S \in \mathcal{O}(V)$ to a function on $E$, and we find that
\[
\{S, S\} = 0,
\]
which is called the classical master equation. Then
\[
\mathcal{O}(\mathcal{E}) = \mathcal{O}(M) \otimes \mathcal{O}(\text{odd fiber}),
\]
and we have the action of $X \otimes 1 + 1 \otimes X^*$ on this, giving an odd vector field on $\mathcal{E}$, that also squares to zero. Since $X$ comes from the base of the odd
cotangent bundle, $L_X \omega = 0$ (or we can check this directly from the formula). Thus there is a hamiltonian function for this vector field, denoted $S_X$ with

$$dS_X = \iota_X \omega,$$

and $X^2 = 0$ implies

$$\{S_X, S_X\} = 0,$$

and $XS = 0$ implies that $\{S, S\} = 0$. Then we define

$$S_{BV} = S + S_X.$$

By construction, this satisfies the classical master equation.

Now, the quantum master equation is gotten from a volume form $\mathcal{E}$, called $\mu$. Then $h \in \mathcal{O}(\mathcal{E})$, $\Delta h = \text{div}(X_h)$, and the quantum master equation is $\Delta(e^{S/h}) = 0$, but usually $\Delta$ is ill-defined in infinite dimensions because $\mu$ isn’t defined.

Now (thinking naively in coordinates),

$$S_X(g, v, v^*, g^*) = \frac{1}{2} \langle [g, g], g^* \rangle + \langle g(v), v^* \rangle,$$

this gives an explicit formula for $S_X$.

### 3.1. Example: Perturbative Chern-Simons Theory.

We’re about to brush some infinite dimensional technicalities under the rug here, so beware. However, all these issues are dealt with in Kevin’s book, so look there if you are in doubt.

Let $M^3$ be a compact 3-manifold, and $\mathfrak{g}$ be the Lie algebra of a compact Lie group with symmetric bi-invariant inner product $\langle -,- \rangle_\mathfrak{g}$. Now define

$$V = \Omega^1(M) \otimes \mathfrak{g}, \quad \mathfrak{g}_\ast = \Omega^0(M) \otimes \mathfrak{g}, \quad V^\ast \cong \Omega^2(M) \otimes \mathfrak{g}, \quad \mathfrak{g}_\ast^\ast \cong \Omega^3(M) \otimes \mathfrak{g}.$$ 

Then the graded manifold $\mathcal{E}$ from before is $\Omega^\ast(M) \otimes \mathfrak{g}$.

Let $e_i = \omega_i \otimes g_i \in \Omega^\ast(M) \otimes \mathfrak{g}$,

$$\langle e_i, e_j \rangle = \int_M \omega_i \wedge \omega_j \langle g_i, g_j \rangle_\mathfrak{g}$$

and

$$[e_i, e_j] := \omega_i \wedge \omega_j [g_i, g_j].$$

Now let $e = (e_0, e_1, e_2, e_3)$, were $e_i$ is an $i$-form. We have

$$S_{BV}(e) = \frac{1}{2} \langle e, de \rangle + \frac{1}{6} \langle e, [e, e] \rangle,$$

where notice

$$S_{cl}(e) = \frac{1}{2} \langle e_1, de_1 \rangle + \frac{1}{6} \langle e_1, [e_1, e_1] \rangle.$$

Now for $X \in \mathfrak{g}$, we have the action

$$X \cdot A = dX + [X, A].$$
3.2. AKSZ Method.

**Definition 3.1.** A P-structure on a graded manifold is a degree $-1$ symplectic structure. A P-structure on a supermanifold is an odd symplectic structure.

A P-structure thus gives rise to a Gerstenhaber bracket, $\{ -, - \}$ which defines the classical master equation.

**Proposition 3.2.** Any P-structure on a graded manifold will be isomorphic to the canonical one on $T^*[−1]N_0$.

**Definition 3.3.** A Q-structure on a graded manifold is a degree $+1$ derivation on its functions, i.e. a degree $+1$ vector field.

For example if we have a P-manifold and $S$ satisfies the CME, then we can define $Q := \{ S, - \}$ and this will give a Q-structure (one can verify that $Q$ is indeed of degree $+1$). Moreover, $L_Q \omega = 0$ (which follows from Cartan).

**Definition 3.4.** If we have an odd symplectic structure $\omega$ and a Q-structure, $Q$, if $L_Q \omega = 0$, we say we have a PQ-structure.

As a prototypical example of a Q-manifold, consider $N = [−1]TN_0$ and $Q = \text{deRham}$.

**Definition 3.5.** A measure on $N^{n|m}$ is a section $\mu$ of the Berezinian line bundle.

This gives a linear map $C^\infty(N) \to \mathbb{R}$, $f \mapsto \int_N f \mu$.

**Definition 3.6.** A measure is called non-degenerate if the pairing

$$(f, g) := \int_N fg \mu$$

is non-degenerate.

**Definition 3.7.** Let $D \in \chi(N)$. We say $\mu$ is $D$-invariant if $\int_N (Df) \mu = 0$ for all $f \in C^\infty(N)$.

For example, let $N = \pi TN_0$. Then $C^\infty N = \Omega^* N_0$ and we define

$$\int_N f \mu = \int_{N_0} f.$$ 

If we let $D = d$ the deRham $d$, then $\mu$ is $d$-invariant by Stokes theorem (if $N_0$ is orientable and without boundary).

Let $N, L$ be supermanifolds, and $\mu$ be a measure on $N$. We get the push-forward map on chains

$$\mu_* : \Omega^*(N \times L) \to \Omega^*(L)$$

and $\omega \in \Omega^k(N \times L)$,

$$(\mu_* \omega)(z)(\lambda_1, \ldots, \lambda_k) = \int_N \omega(y, z)(\lambda_1, \ldots, \lambda_k) \mu(y).$$

Note $\mu_*$ lowers the ghost number by $n$. 
Now we can describe the AKSZ construction. The goal is to put a QP-structure on $\text{Maps}(N, M)$ so that we can do BV there. To do this we have to add structure to the source and target. So let the source manifold be given by the data $(N, D, \mu)$ where $D$ is a Q-structure and $\mu$ is a $D$-invariant measure of degree $-n$. Let the target manifold be $(M, Q, \omega)$ where $Q$ is a Q-structure and $\omega$ is a symplectic form of degree $(n - 1)$.

Then the Q-structure on the mapping space is determined by the fact that the action of $\text{Diff}(M)$ and $\text{Diff}(N)$ on the mapping space commute with one another. Let $\hat{D}$ (resp $\hat{Q}_M$) be the lift of the $D$-action (resp $Q$-action) to the mapping space. So then we can define a Q-structure by

$$Q = a\hat{D} + b\hat{Q}_M,$$

for any $a, b \in \mathbb{R}$.

The P-structure is determined by the evaluation map

$$ev : N \times Maps(N, M) \to M$$

and pushforward

$$\mu_* : \Omega^*(N \times Maps(N, M)) \to M \to \Omega^*(Maps(N, M)).$$

so we pull $\omega$ back by $ev$ and then pushforward along $\mu$. Non-degeneracy of the measure implies nondegeneracy of the resulting form, which we call $\mathcal{F}$. To see this, fix $f \in Maps(N, M)$ and consider the tangent space at that map. An element of that tangent space is a section of $f^*T M$. Then

$$\mathcal{F} (\phi_0, \phi_1) = \int_N \omega(\phi_0(x), \phi_1(x))\mu(x).$$

The last thing to check is compatibility with $\hat{Q}$. So we compute

$$L_{\hat{Q}_M} \mathcal{F} = L_{\hat{Q}_M} \mu_* ev^* \omega = \mu_* ev^* L_Q \omega,$$

which vanishes if $Q$ was compatible with $\omega$ to begin with. If $Q$ was actually hamiltonian with hamiltonian function $S$, then $\hat{Q}_M$ is also hamiltonian with hamiltonian function

$$\hat{\text{Sym}}_M := \mu_* ev^* S_M.$$

Compatibility with $\hat{D}$ follows from the fact that the measure is $D$-invariant.

**Lemma 3.8.** If we assume that $\omega$ is exact, $\omega = d\theta$, then $\hat{D}$ is Hamiltonian with Hamiltonian function $\hat{\text{Sym}}_N = \iota_{\hat{D}} \hat{\theta}$, where $\hat{\theta} = \mu_* ev^* \theta$.

**Proof.** $d\hat{\text{Sym}}_N = d\iota_D \hat{\theta} = \iota_D d\hat{\theta} + L_D \hat{\theta}$, but this last term vanishes as it is precisely $L_D \mu_* ev^* \theta$, and then we get

$$d\hat{\text{Sym}}_N = \iota_D d\mu_* ev^* \theta = \iota_D \mu_* ev^* d\theta = \iota_D \omega.$$ 

□

When we write $S = a\hat{\text{Sym}}_M + b\hat{\text{Sym}}_N$, the first term is typically called the interaction term and the second term is the kinetic term. Moreover, this $S$ automatically satisfies the classical master equation for the QP-structure on $Maps(N, M)$. 

3.3. **Back to Chern-Simons.** Let $N^3 = \pi T\Sigma^3$, $\Sigma^3$ oriented, compact. Let $D = d$, deRham, and $\mu$ the canonical measure.

Let $M = \pi g$, invariant inner product $\langle -, - \rangle$. From this we get $\omega : g[1] \oplus g[1] \to \mathbb{R}$, which is an odd symplectic structure. The $Q$ structure on $\pi g$ is just the Chevalley-Eilenberg differential, the dual to $[ -, - ] : g \otimes g \to g$, then extended by the Leibniz rule. The compatibility of these structures comes from the invariance of the metric.

Let $\theta_g(\delta g) = \frac{1}{3} \langle g, [g, \delta g] \rangle$, where $\delta g \in T_g g \cong g$. Then $\omega = d\theta$.

We have that $Q$ is hamiltonian with hamiltonian function $S_{g[1]}(g) = \frac{1}{3} \langle g, [g, g] \rangle$, and

$$dS_{g[1]}(g)(\delta g) = \langle \delta g, [g, g] \rangle.$$

3.4. **Advertisement for What Will Follow.** Kevin talks about holomorphic Chern-Simons theory. We can do this (in principle) for any Calabi-Yau $\Sigma$, whose sheaf of functions will be $\Omega^0(\Sigma)$, has $Q$-structure $\bar{\partial}$, and whose measure is a choice of holomorphic volume form $\mu \in \Omega^{\dim(\Sigma), 0}(\Sigma)$. In the previous notation, this is $T^{0,1}[-1] \Sigma$.

In Kevin’s construction, $\Sigma$ will be an elliptic curve.

The target will be $T^{0,1}[-1]T^*[1] X$ for $X$ a complex manifold, with $Q$-structure $\bar{\partial}$. In Kevin’s construction, $X$ will be compact Kähler.

Then the space of fields is

$$\underline{Maps}(T^{0,1}[-1] \Sigma, T^{0,1}[-1] T^*[1] X).$$

But this space of fields is nonlinear, so doesn’t fit into Kevin’s BV machine. So we want to take some derived mapping space, which is a formal neighborhood of the constant maps to $X$. The first step in doing this is replacing $X$ with a “curved $L\infty$ algebra” $g_X$. We’ll learn this next week.

4. **Curved $L\infty$-Algebras, Joan Milles**

We’ll cover two areas today: the geometry and the algebra of curved $L\infty$-algebras. At the end we’ll combine these two sides.

What we’re after is a good description of a the “space of fields” $Maps(M, X)$, where $M$ is a 1-dimensional complex manifold (a Riemann surface), and $X$ is a complex manifold. Last week we learned that we need the space of fields to be sections of some vector bundle over $M$ in order to employ the BV formalism. If $X$ is nonlinear we don’t get this, so we need a trick.

So first, we notice that perturbative integrals are supported in a (formal) neighborhood of classical solutions. In Kevin’s case with AKSZ, these are the constant maps. Thus, we need to have a good idea of what the formal neighborhood of a constant map is. We’ll see that in this case, our problem will be linearized.
So first, if we think about this problem near a single point \( x \in X \), the formal neighborhood is something like the completed symmetric powers of the holomorphic cotangent space \( T_xX \). But this only tells us about the theory near a single constant map. We really need to make this work in families of constant maps, in some sense. This is where we see curved \( L_\infty \)-algebras appear.

The plan for today is:
1. \( L_\infty \)-algebra and curved \( L_\infty \)-algebras.
2. Complex geometry
3. The curved \( L_\infty \)-algebra \( \mathfrak{g}_X \)

4.1. \( L_\infty \)-Algebras. Morally, an \( L_\infty \) algebra is a weakened version of a Lie algebra. So let’s (briefly) recall a concept from Carlo’s talk this morning.

**Definition 4.1.** An \( L_\infty \)-algebra \((V, d_V)\) is a dg vector space \( V \) endowed with a collection of maps
\[ \ell_n : \Lambda^n V \to V, \]
where are of degree \( 2 - n \), for \( n \geq 2 \), satisfying some relations:
\[ d_V \circ \ell_n \pm \ell_m \circ d_V = \partial_V(\ell_m) = \sum_{\begin{smallmatrix} p+q=m+1 \\ p, q > 0 \end{smallmatrix}} \sum_{\sigma} \pm \text{sgn}(\sigma)(\ell_p \circ \ell_q). \]

**Remark 4.2.** Equivalently, we can also look at \( \tilde{\ell} : S^n([1]V) \to [1]V \), and these maps will be degree 0. Also, we can represent the above relations using some trees. This makes the operad people happy.

There are equivalent definitions (Bruno calls the following the Rosetta Stone). (I wasn’t able to get all the diagrams down, but roughly there is some cool stuff going on with cofibrant replacements of algebras over the operad Lie, Koszul duality, twisting morphisms, representation theory, square zero coderivations... So it seems that \( L_\infty \)-algebras sit at the crossroads of many interesting ideas.)

If we want to work with derivations instead of coderivations, we can dualize, but when \( V \) is not finite dimensional, \((V \otimes V)^* \neq V^* \otimes V^*\), so we consider the completed tensor product and use continuous maps and find
\[ L_\infty - \text{alg} \leftrightarrow \text{Diff}(\widehat{\text{Sym}}^+(V[1]^*)) \hookrightarrow \text{Der}(\widehat{\text{Sym}}^+(V[1]^*)), \]
where Diff denotes differentials, the square-zero derivations.

**Example 4.3.** For a (non)example, let \( A = C^\infty(M) \) and consider graded derivations on \( A \),
\[ D(f \cdot g) = D(f) \cdot g \pm f \cdot D(g). \]
However, we never have that the composition of two vector fields is zero (unless we take the zero vector field), so we don’t get any \( L_\infty \) algebra structures here. We need to move to graded manifolds.

Over a point, take \( A = \widehat{\text{Sym}}^+(V[1]^*) = \mathcal{O}_Y \). Then we have an inclusion of \( L_\infty \) algebra structures into the derivations. The image are the homological
vector fields, or $Q$-structures, or whatever your favorite word for differential is.

We could also take graded manifolds (think smooth manifold endowed with sheaf of sections of a graded vector bundle) and play a similar game. Then $L_\infty$-structures are the same as $Q$-structures on the graded manifold, or odd, square zero derivations on its structure sheaf.

4.2. **Curved $L_\infty$-Algebras.** Let $A$ be a commutative dg algebra, $A^\#$ its underlying graded commutative algebra, and $I \subset A$ a nilpotent ideal. All tensors and duals in the following are with respect to $A^\#$, e.g.

$$V[1]^* := \text{Hom}_{A^\#-\text{Mod}}(V[1], A^\#).$$

First we will give a “local” version of the definition, then a “global” (i.e. sheafy) one.

**Definition 4.4.** A curved $L_\infty$-algebra $(V, d)$ over $A$ is a free $A^\#$ module $V$ and a square 0 cohomological degree 1 derivation

$$d : \hat{\text{Sym}}_{A^\#}(V[1]^*) \to \hat{\text{Sym}}_{A^\#}(V[1]^*)$$

such that $\text{Mod } I$, $d$ preserves the ideal of $\hat{\text{Sym}}_{A^\#}(V[1]^*)$ generated by $V$.

**Remark 4.5.** The “Taylor components” of $d$ are the maps

$$\ell_k : \Lambda^k(V) \to V.$$

These are of homological degree $2-k$, and satisfy certain identities analogous to the $L_\infty$ ones. In particular, notice that $\ell_0$ returns an element of $V$. If this element is nonzero, then $V$ will not be a differential graded module over $A$.

However, when we reduce mod $I$, we obtain an ordinary $L_\infty$ structure over the algebra $A/I$.

Notice that the condition in the definition implies $\ell_0 : A^\# \to V \otimes_A I \to V$.

**Remark 4.6.** Here there are really two new ingredients from an ordinary $L_\infty$-algebra. First, we have a relative version here: we’re considering $A^\#$-modules. Second, we’ve introduced $\ell_n : \Lambda A^\# V \to V$ and $\ell_0 : A^\# \to V$ such that $\ell_1(\ell_0) = 0$ and $\ell_1^2 = \ell_2(\ell_0, -) = [\ell_0, -]$.

The word “curved” comes from the nonzero $\ell_0$ component. Said equivalently, since $\ell_0^2 \neq 0$, $V$ is not a dg module over $A$. However, mod $I$, $V/I$ is a dg module over $A/I$.

We will use the notation, $C^\bullet(V) = (\hat{\text{Sym}}_{A^\#}(V[1]^*), d)$.

We need to generalize this even further to sheaves of curved $L_\infty$-algebras.

**Definition 4.7.** Let $A$ be a sheaf of commutative dg algebras. A curved $L_\infty$ algebra will be a locally free $A^\#$-module $V$ with a derivation as above, with the additional condition that $d$ makes $\hat{\text{Sym}}_{A^\#}(V[1]^*)$ into a sheaf of dg modules over the sheaf of dg algebras $A$.

**Example 4.8.** (The following is my guess!) One class of examples would be Lie algebra valued forms coming from a principle bundle with connection. These have $\ell_n = 0$ for $n \geq 3$. Then the curvature is precisely the curvature of the connection. The nilpotent ideal is the ideal of forms of positive degree.
The particular version we are interested has $A = \Omega_X^\bullet$, the complex valued holomorphic forms on a complex manifold and $I = \Omega_X^{>0}$.

**Definition 4.9.** A curved $L_\infty$-algebra over $\Omega_X^\bullet$ is a sheaf $g$ of graded $\Omega_X^\bullet$-modules on $X$ which is locally free of finite rank, equipped with the structure of a curved $L_\infty$-algebra over $\Omega_X^\bullet$.

Let $\Omega_{\Delta^n}$ be the polynomial forms on the $n$-simplex.

**Definition 4.10.** A homotopy between two curved $L_\infty$-algebras on a sheaf $V$ of $A^\bullet$-modules is a curved $L_\infty$-algebra on $V \otimes \Omega_{\Delta^1}$ over the sheaf $A \otimes \Omega_{\Delta^1}$, with $I' = I \otimes \Omega_{\Delta^1}$ such that the restriction to either end of $\Delta^1$ is either curved $L_\infty$-structure.

**Remark 4.11.** The above definition generalizes easily to provide a convenient description of the simplicial set of curved $L_\infty$-algebra structures on $V$.

**4.3. Complex Geometry.** Let $X$ be a complex manifold and $\Omega_X^\bullet$ be smooth forms on $X$. We give the tools to define a curved $L_\infty$-algebra $g_X$ which encodes the holomorphic geometry of $X$.

The $k$-jet of a smooth function on a manifold $X$ is an equivalence class of functions that agree up to order $k$. We can think about this via Taylor’s theorem. Then, in a way, we only remember the $k$th order polynomial that is “close” to a function at a particular point, though this idea isn’t a coordinate invariant one.

However, it turns out we can make this into a definition of jets that actually “glues well” to give a global bundle of $k$-jets, denoted $J^k(M, N)$ for smooth maps $M \to N$. This will be an affine bundle over $M$, whose fiber at a given point is isomorphic to the vector space described in the previous paragraph.

**Definition 4.12.** Let $J^{\text{hol}} := J^\infty(X, \mathbb{C})$, the bundle of infinite jets of holomorphic functions whose fiber at a point is like power series. We think of this as a $C^\infty X$ bundle. (See Carlo’s talk for a possible definition of this bundle).

This bundle has a flat connection, $\nabla$,

$$\nabla : \Gamma(X, J^{\text{hol}}) \to \Gamma(X, T^*X \otimes J^{\text{hol}}).$$

The flat sections on an open set $U$ are precisely the germs coming from a holomorphic function on $U$.

Now define

$$\Omega_X^\bullet(J^{\text{hol}}) := \Omega_X^\bullet \otimes_{C^\infty X} J^{\text{hol}} = \Gamma(X, J^{\text{hol}} \otimes_{C^\infty X} \Lambda^*T^*X).$$

Using the deRham $d$, we extend $\nabla$ to all $J^{\text{hol}}$-valued forms, and since $\nabla$ is flat this gives us a sheaf of dg commutative algebras.

Let $O_X^{\text{hol}}$ be the sheaf of holomorphic functions on $X$.

**Lemma 4.13.** There is a quasi-isomorphism, $O_X^{\text{hol}} \simeq \Omega_X^\bullet(J^{\text{hol}})$.

In this way, $\Omega_X^\bullet(J^{\text{hol}})$ encodes the holomorphic structure on $X$ completely.
4.4. The Curved $L_\infty$-algebra $g_X$.

**Lemma 4.14.** There exists a canonical (up to a contractible choice) curved $L_\infty$-algebra over $\mathcal{O}_X$ denoted $g_X$ such that

$$g_X \cong T^{1,0}_X[1] \otimes_{C^\infty_X} \mathcal{O}_X$$

as $\mathcal{O}_X$-modules. Moreover, there is a quasi-isomorphism of dg $\omega^\#_X$-algebras

$$C^\bullet(g_X) \cong \mathcal{O}_X^\text{hol} \cong \mathcal{O}_X^\text{hol}(\mathcal{J}^\text{hol})$$

We can define things like the tangent space to the curved $L_\infty$-algebra $g_X$ and do other “geometry.” We should think of $g_X$ as some kind of derived scheme with underlying space $X$.

**Proof of Lemma.** There is a decreasing filtration of subbundles on $\mathcal{J}^\text{hol}$ where the fiber $F^k\mathcal{J}^\text{hol}$ at $x \in X$ is the set of germs of holomorphic functions at $x$ which vanish to order $k$. We note that the connection does not respect this filtration. The connection gives a map

$$F^k\mathcal{J}^\text{hol} \rightarrow F^{k-1}\mathcal{J}^\text{hol} \otimes \Omega^1_X.$$ 

In particular, we have an isomorphism

$$F^1\mathcal{J}^\text{hol} / F^2\mathcal{J}^\text{hol} \cong (T^{1,0}_X)^*.$$ 

We want a splitting of $F^1\mathcal{J}^\text{hol} \rightarrow (T^{1,0}_X)^*$, denoted

$$\phi : (T^{1,0}_X)^* \rightarrow F^1\mathcal{J}^\text{hol}.$$ 

Explicitly,

$$\phi(dz^i) = 0 + \delta_{ij} t^j + a_{j_1j_2}^i(z)t^{j_1}t^{j_2} + \ldots$$

where $a_{j_1j_2}^i = a_{j_2j_1}^i$ is symmetry and $\delta_{ij}$ is the Kronecker delta. These coefficients behave like Christoffel symbols of a torsion-free connection (in particular, are not tensors). We also want a connection $\hat{\nabla}$ on $\text{Sym}(\mathcal{O}_X^\text{hol})$ such that

$$\nabla^{\text{Gr}} \circ \phi = \phi \circ \hat{\nabla}, \quad \hat{\nabla}^2 = 0,$$

and $\hat{\nabla}$ a $C^\infty$-connection that is compatible with the algebraic structure on $\text{Sym}(\mathcal{O}_X^\text{hol})$ so $\nabla \in \Omega^1_X \otimes \text{Der}(\text{Sym}(\mathcal{O}_X^\text{hol}))$. (It seems that this connection being derivation-valued equivalent to $\phi$ being multiplicative.) Here $\nabla^{\text{Gr}}$ is the Grothendieck connection. This connection $\hat{\nabla}$ is given by its restriction to $(T^{1,0}_X)^* \rightarrow \Omega^1_X \otimes \text{Sym}(T^{1,0}_X)^*$.

Explicitly, the Grothendieck connection is

$$\nabla^{\text{Gr}}(a_I(z)t^I) = \frac{\partial}{\partial z_j} a_I(z)dz^j \otimes t^I \pm a_I(z) \frac{\partial t^J}{\partial z_j} \otimes dz^j.$$ 

We find

$$\hat{\nabla} = \delta + d_{\text{DR}} + \nabla + A_2 + \ldots$$

where $\delta : (T^{1,0}_X)^* \rightarrow \Omega^1_X$, $d_{\text{DR}} : (T^{1,0}_X)^* \rightarrow \Omega^1_X \otimes (T^{1,0}_X)^*$, and we collect the higher terms $A_i$ into $A : (T^{1,0}_X)^* \rightarrow \Omega^1_X \otimes ((T^{1,0}_X)^*)^{\otimes 2}$. We notice that the condition

$$\nabla^{\text{Gr}} \circ \phi(dz^i) = \phi \circ \hat{\nabla}(dz^i),$$
shows that $\delta = dz^i \otimes \partial / \partial z_i$ and $a^i_{j,k}(z)$ are the Christoffel symbols of $\widetilde{\nabla}$, a torsion free connection on $X$ (that may have curvature!). We fix a connection $\widetilde{\nabla}$ on $\widehat{\text{Sym}}((T_X^{1,0})^*)$.

Now we look at the condition $\hat{\nabla}^2 = 0,$ and find that it gives

$$\delta^2 + \delta(\nabla) + \delta(A) + \nabla(A) + \nabla^2 + \frac{1}{2}[A, A] = 0$$

and notice that $\delta^2 = 0$ from its definition, and $\delta(\nabla) = 0$ since $\nabla$ is torsion-free.

Now, $\delta(A) : (T_X^{1,0})^* \to \Omega^2_X \otimes ((T_X^{1,0})^*)^2$.

For the curvature,

$$\nabla^2 : (T_X^{1,0})^* \to \Omega^2_X \otimes (T_X^{1,0})^*,$$

so $A_2$ is defined in terms of curvature, and by induction we can get the $A_i$ for $i > 2$. This gives the splitting we were after

$$\phi : (T_X^{1,0})^* \to F^1 \mathcal{J}^{\text{hol}}.$$  

Remark 4.15. This proof is a little weird. We really are building a particularly nice choice of splitting that plays well the Grothendieck connection and some yet-to-be-determined connection. But to do this we first suppose that you have $\phi$ and notice you get the first two terms for $\hat{\nabla}$. Now, finding $\phi$ and finding $\hat{\nabla}$ are equivalent problems. So then we find the other terms for $\hat{\nabla}$ by looking at the conditions we require. But in some sense the constructions of both $\phi$ and $\hat{\nabla}$ are simultaneous.

In turn, this map gives an isomorphism

$$\widehat{\text{Sym}}((T_X^{1,0})^*) \cong \mathcal{J}^{\text{hol}}.$$

So then we find that

$$\Omega^*_X \otimes C^{\infty}_X \mathcal{J}^{\text{hol}} \cong \Omega^*_X \otimes C^{\infty}_X \widehat{\text{Sym}}((T_X^{1,0})^*) \cong \widehat{\text{Sym}} \Omega^*_X ((T_X^{1,0}[−1] \otimes C^{\infty}_X \Omega^*_X)[1])$$

and define

$$\mathfrak{g}_X := T_X^{1,0}[−1] \otimes C^{\infty}_X \Omega^*_X.$$

We transfer the differential on $\Omega^*_X \otimes C^{\infty}_X \mathcal{J}^{\text{hol}}$ to $\widehat{\text{Sym}}(\mathfrak{g}_X)$ to get a curved $L_{\infty}$-algebra structure on $\mathfrak{g}_X$. We have

$$\ell_0 : \mathbb{C} \to (T_X^{1,0}[−1] \otimes C^{\infty}_X \Omega^*_X)_2 = (\mathfrak{g}_X)_2.$$

$$1 \mapsto \frac{\partial}{\partial z_i}[−1] \otimes dz_i,$$

where each part of the tensor product is of total degree 0. Note this is just the identity endomorphism. The splitting $\phi$ is equivalent to the data of a torsion free connection $\nabla$, so a homotopy is given by a path between the chosen connections.

Note that the bracket of this $L_{\infty}$ algebra is given by the curvature of $\nabla$. 


4.5. Peter’s Elaboration on Joan’s Talk. We want not only a splitting, but a multiplicative isomorphism of (smooth) algebra bundles:

\[ \widehat{\text{Sym}}((T^1_X)_*) \cong J^{\text{hol}} \]

and we can’t do this via partitions of unity. We accomplish this by looking at a derivation property with respect to the whole algebra bundle

\[ \hat{\nabla} \in \Omega^1_X \otimes \text{Der}(\widehat{\text{Sym}}((T^1_X)_*)) \]

Then once we have this fact, we can restrict to a linear subspace and do the above computation, and the flatness of \( \hat{\nabla} \) restricts possibilities dramatically, down to a choice of torsion free connection.

5. Formal Geometry and the Atiyah Class I, Carlo Rossi

5.1. Carlo’s Elaboration on Joan’s Talk. We were looking for a multiplicative isomorphism of smooth bundles between

\[ \widehat{\text{Sym}}((T^1_X)_*) \cong J^{\text{hol}}. \]

We will discuss why this is exactly the same thing as finding a smooth section of the bundle

\[ X^\text{aff} \xrightarrow{\pi} X, \]

where \( X^\text{aff} \) is some sort of infinite dimensional complex manifold that is some projective limit of finite dimensional ones. This space parametrizes infinite jets of holomorphic coordinate systems of \( X \), modulo the action of \( GL_d(\mathbb{C}) \).

In the language of formal geometry, we can think of this space as

\[ X^\text{aff} \cong X^\text{coord}/GL_d(\mathbb{C}), \]

where \( X^\text{coord} \) is the infinite dimensional manifold of coordinate systems, and as such is a space is more naturally defined and admits a canonical connection. Unfortunately, over a point it has a non-contractible fiber. If we form this quotient, this problem goes away. Moreover, the space \( X^\text{aff} \) has a nice universal property. The previous computation gives a hint at this already. First a digression on the holomorphic jet bundle. We may think of it as the formal completion of \( \Delta : X \to X \times X \). To give this a bit more meaning, we thinking of the sheaves \( \mathcal{O}_{X \times X} \to \mathcal{O}_X \). We note that \( \mathcal{O}_{X \times X} \cong \mathcal{O}_X \otimes \mathcal{O}_X \) and look at the kernel

\[ I = \ker(\Delta^* : \mathcal{O}_X \otimes \mathcal{O}_X \to \mathcal{O}_X), \]

and define

\[ J^{\text{hol}} := \lim_{\leftarrow} \mathcal{O}_{X \times X}/I^n. \]

Now let us look at this universal property of \( X^\text{aff} \xrightarrow{\pi} X \). We have the bundles \( \widehat{\text{Sym}}((T^1_X)_*) \to X \) and \( J^{\text{hol}} \to X \). The the universal property is

\[ \pi^* \widehat{\text{Sym}}((T^1_X)_*) \cong \pi^*(J^{\text{hol}}). \]

From this it is clear that a choice of section induces an isomorphism of the bundles.
From what we saw in the last talk, such an isomorphism is the same as a flat, multiplicative connection \( \tilde{\nabla} \) on \( Sym((T^{1,0}_X)^*) \), which in turn is equivalent to a curved \( L_\infty \)-structure on \( g_X \).

From this \( L_\infty \)-structure we have

\[
\text{id}_{T^{1,0}_X} = \ell_0 \in T^{1,0}_X[1] \otimes (T^{1,0}_X)^* = \text{End}(T^{1,0}_X)
\]

and

\[
\ell_1 = \tilde{\nabla} = \tilde{\nabla}^{1,0} + \tilde{\partial}
\]

and

\[
\ell_2 : T^{1,0}_X \otimes T^{1,0}_X \to T^{1,0}_X[-1] \otimes \Omega^2 C,
\]

We have that

\[
\delta(A) + R + \tilde{\nabla}(A) + \frac{1}{2}[A, A] = 0
\]

where \( R = \tilde{\nabla}^2 \) is the curvature of \( \tilde{\nabla} \). This has two components

\[
R = R^{2,0} + R^{1,1}.
\]

Now, \( R^{1,1} \) is a representative of the Atiyah class of the bundle \( T^{1,0}_X \).

We also note that the higher \( \ell_k \) really just give higher covariant derivatives of the curvature.

Just to give a hint about next time: the Atiyah class is some obstruction class to the existence of a global holomorphic connection on a holomorphic vector bundle, in that it maps holomorphic sections to holomorphic sections. This is what appears in this \( L_\infty \)-structure.

The novelty of Costello’s approach is he gives some kind of derived version of the Atiyah class.

\textbf{Remark 5.1.} The following are some remarks of mine that may or may not have any relation to what happens above.

Writing down Feynman diagrams requires a choice of affine structure: we need to split the action function into “kinetic” and “interaction” terms near a critical point. The kinetic part is some quadratic piece, and the interaction consists of the cubic and higher terms.

To actually do computations with Feynman diagrams, we must first choose this splitting. The claim typically made is that the actual computation is independent of this choice, but I have yet to see an actual proof of this anywhere.

Perhaps secretly this curved \( L_\infty \) structure is a choice of these affine coordinates. Then since the space of all such choices is contractible, perhaps we’ll finally have some proof that the Feynman diagram computations we do are actually independent of choices. To wit, we see that the kinetic and interaction parts of the holomorphic Chern-Simons Lagrangian are precisely related to the \( L_\infty \) structure that we choose here. However, I’m unsure if my interpretation of this is correct.
Let’s review a little. Joan introduced and discussed the curved $L_\infty$-structure. This came from the choice of isomorphism

$$\hat{\text{Sym}}((T^{1,0}_X)^*) \cong J^{hol}$$

for a complex manifold $X$ with the following properties:

1. $\Phi$ is $C^\infty$-linear;
2. $\Phi$ is an algebra morphism;
3. $\Phi$ is compatible with the two natural descending filtrations on the sheaves (given by the order);
4. the previous property allows us to consider the associated graded $\text{gr}(\Phi) : \text{Sym}^\bullet((T^{1,0}_X)^*) \rightarrow \text{Sym}^\bullet((T^{1,0}_X)^*)$

and we require that this is the identity.

We know that $J^{hol}$ admits a natural canonical (holomorphic!) connection, $\nabla^{gr}$, called the Grothendieck connection. This connection is compatible with the algebra structure.

As an aside, let’s get our hands on the filtration a bit more. Define $F^p D^{hol}$ as the holomorphic differential operators of degree at most $p$. Then

$$J^{\geq p, hol} := \{ \alpha \in \text{Hom}_{\mathcal{O}_X}(D^{hol}, \mathcal{O}_X) : \alpha|_{F^j D^{hol}} = 0, j \leq p \},$$

i.e. the functions that vanish to order at least $p$.

Now we define

$$\hat{\nabla} = \Phi \circ \nabla^{gr} \circ \Phi^{-1}.$$

Then we find

$$\hat{\nabla} = -\delta + \bar{\nabla} + A,$$

where $\bar{\nabla}$ is a torsion free connection on $T^{1,0}_X$ and

$$\delta = dz_i \frac{\partial}{\partial y_i}$$

where $y_i := dz_i$ is some derivative in the “fiber direction.” Now

$$A \in \Omega^1(X) \otimes \hat{\text{Sym}}^{\geq 1}((T^{1,0}_X)^*) \otimes T^{1,0}_X.$$

Notice we can rewrite

$$\hat{\nabla} = -\delta + \bar{\nabla} + A = d + \omega_{MC},$$

for $\omega_{MC}$ some Mauer-Cartan element. We notice that

$$\hat{\nabla}^2 = 0 \iff \delta(A) + \frac{1}{2}[A, A] + \bar{\nabla}(A) + \bar{\nabla}^2 = 0.$$

Using the fact that $\delta^2 = 0$ and $[\bar{\nabla}, d] = 0$.

We use the splitting

$$\hat{\nabla}^2 \in \Omega^{2,0} \otimes \Omega^{1,1} \otimes (T^{1,0}_X)^* \otimes T^{1,0}_X \cong \Omega^{2,0} \otimes \Omega^{1,1} \otimes \text{End}(T^{1,0}_X).$$

There are some explicit formulas that I didn’t copy down, but morally the “1,1” part of $A$ is precisely the Atiyah class; more precisely it’s some part of the 1,1 part.
6.1. The Atiyah Class.

**Definition 6.1.** The Atiyah class of $T^{1,0}_X$ is the obstruction to the existence of a global holomorphic connection on $T^{1,0}_X$.

Recall that a holomorphic connection maps holomorphic sections to holomorphic sections.

**Remark 6.2.** The above definition works for any holomorphic vector bundle over any complex manifold.

We can also define the Atiyah class in terms of the short exact sequence

$$0 \to \Omega^1_X \otimes \mathcal{O}_X \to J^1(T^{1,0}_X) \to T^{1,0}_X \to 0.$$ 

Then the Atiyah class is the extension class

$$at(T^{1,0}_X) \in \text{Ext}^1(T^{1,0}_X, \Omega^1_X \otimes T^{1,0}_X) \cong H^1(X, \Omega^1_X \otimes \Omega^1_X \otimes T^{1,0}_X) \cong \text{Hom}(T^{1,0}_X \otimes T^{1,0}_X, T^{1,0}_X).$$

This last isomorphism connects this with the above definition.

We can choose a Dolbeault representative of the Atiyah class: take any smooth connection on $T^{1,0}_X$ compatible with the holomorphic structure, (i.e. $\nabla = \nabla^{1,0} + \overline{\partial}$), and then $at(T^{1,0}_X)$ is determined by the 1,1 part of

$$R(\nabla) = R^{2,0}(\nabla) + R^{1,1}(\nabla).$$

Notice also that $\nabla^{1,0} = \partial + \Gamma^{1,0}$.

Summing up, the Atiyah class of $T^{1,0}_X$ appears in the curved $L_\infty$-structure on $T^{1,0}_X[-1] \otimes \Omega^*(X)$. This is important because the Atiyah class $at(T^{1,0}_X)$ defines a graded Lie algebra structure on sheaf cohomology. This really comes from the class, not a representative! Now let $\mathcal{A}$ be a sheaf of $\mathcal{O}_X$ algebras. Then we have a map

$$H^p(X, T^{1,0}_X \otimes \mathcal{A}) \otimes H^q(X, T^{1,0}_X \otimes \mathcal{A}) \to H^{p+q+1}(X, T^{1,0}_X \otimes \mathcal{A})$$

where the raise in degree comes from the fact that $at$ lives in $H^1(X, \text{Hom}(S^2(T^{1,0}_X), T^{1,0}_X))$, and it acts in the obvious way on the two copies of $T^{1,0}_X$ in the above formula.

One can show this is in fact a graded Lie algebra structure. The nontrivial part is showing that this actually satisfies Jacobi.

Now since there is a graded Lie algebra structure on cohomology, we might look for a weak Lie algebra (i.e. $L_\infty$) structure on the level of complexes. For example we may pick the Dolbeault complex for computing $H^\bullet(X, T^{1,0}_X \otimes \mathcal{A})$, so we’d have $\Omega^\bullet_{\overline{\partial}}(X, T^{1,0}_X \otimes \mathcal{A})$. Then since $at(T^{1,0}_X)$ raises the cohomological degree by 1, this is consistent with the fact that the $L_\infty$ structure is an element of $\Omega^0(X) \otimes \text{Hom}(S^2(T^{1,0}_X), T^{1,0}_X)$+ higher terms. In our example, these higher terms are given by higher covariant derivatives of the curvature and amounts to giving the 0,1 part of

$$A \in \Omega^{0,1} \otimes \Omega^{1,0} \otimes S^{\geq 2}(T^{1,0}_X)^* \otimes T^{1,0}_X.$$ 

This gives a (non-curved!) $L_\infty$-algebra. In fact, the differential comes from $\overline{\partial}$. Before we had $\ell_1 = \overline{\partial} + (\partial + \nabla^{1,1})$, and we are only seeing part of this in this $L_\infty$ lift of the Atiyah class.
6.2. Derived Mapping Spaces. First, we need some generalities on curved $L_{\infty}$ algebras. Let $(\overset{\sim}{\text{Sym}}_A(\mathfrak{g}[1]^*), d)$ be a curved $L_{\infty}$ algebra, so $d$ is a square zero differential. (Think $A = \Omega^*_*(X)$.) Let’s think of a generalized space (or derived scheme, or something) $B\mathfrak{g}$ whose ring of functions is the above curved $L_{\infty}$-algebra. Then vector fields are precisely derivations, 
\[ \chi(B\mathfrak{g}) := \text{Der}_A(\overset{\sim}{\text{Sym}}_A(\mathfrak{g}[1]^*)) \cong \overset{\sim}{\text{Sym}}_A(\mathfrak{g}[1]^*) \otimes \mathfrak{g}[1]. \]
Then 1-forms on $B\mathfrak{g}$ are just $\overset{\sim}{\text{Sym}}_A(\mathfrak{g}[1]^*) \otimes \mathfrak{g}[1]^*$. We can continue thinking along these lines to get $n$-forms.

We notice that vector fields and 1-forms admit natural differentials compatible with $d$. In terms of $L_{\infty}$ structures, this is equivalent to saying that $\mathfrak{g}[1]$ and $\mathfrak{g}[1]^*$ are $L_{\infty}$-modules over the $L_{\infty}$ algebra $\mathfrak{g}$.

Equivalently (dually), we must provide maps 
\[ \tilde{\ell}_n : \Lambda^n_A \mathfrak{g} \otimes \mathfrak{g}^* \rightarrow \mathfrak{g}^* \]
\[ \bar{\ell}_n : \Lambda^n_A \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}. \]
Then $\tilde{\ell}_n$ corresponds to 
\[ \overset{\sim}{\text{Sym}}_A(\mathfrak{g}[1]^*) \otimes \mathfrak{g}[1] \rightarrow \overset{\sim}{\text{Sym}}_A(\mathfrak{g}[1]^*) \otimes \mathfrak{g}[1]^* \otimes \mathfrak{g}[1] \otimes \mathfrak{g}[1] \]
and we have 
\[ \tilde{d}(x) = \pm \iota_x d(e_j^*) \otimes e_j \]
for $x \in \mathfrak{g}[1]$ and the $d$ is from the curved $L_{\infty}$ algebra.

We define a sort of connection 
\[ \overset{\sim}{\text{Sym}}_A(\mathfrak{g}[1]^*) \otimes \mathfrak{g}[1] \xrightarrow{\nabla} \overset{\sim}{\text{Sym}}_A(\mathfrak{g}[1]^*) \otimes \mathfrak{g}[1]^* \otimes \mathfrak{g}[1], \]
which is a map (using more suggestive notation) 
\[ \nabla : TB\mathfrak{g} \rightarrow \Omega^1_{B\mathfrak{g}} \otimes_{\mathcal{O}_{B\mathfrak{g}}} TB\mathfrak{g} \]
defined by 
\[ s \otimes x \xrightarrow{\nabla} (\iota_{e_j} s \otimes e_j^*) \otimes x \]
and then 
\[ \nabla(fs) = d_{dR}(f) \otimes s + f\nabla s. \]
We then define the Atiyah class as 
\[ at(TB\mathfrak{g}) = [\tilde{d}, \nabla], \]
which encodes the noncommutativity of the diagram 
\[ \begin{array}{ccc}
TB\mathfrak{g} & \xrightarrow{\nabla} & \Omega^1_{B\mathfrak{g}} \otimes_{\mathcal{O}_{B\mathfrak{g}}} TB\mathfrak{g} \\
\tilde{d} & \downarrow & \downarrow d \\
TB\mathfrak{g} & \xrightarrow{\nabla} & TB\mathfrak{g} \otimes_{\mathcal{O}_{B\mathfrak{g}}} \Omega^1_{B\mathfrak{g}}
\end{array} \] 
(1)
If we do this computation, we arrive at Costello’s formula in terms of $\ell_{n+2}$.

Remark 6.3. More precisely, the Taylor maps of the $L_{\infty}$-structure are encoded by the higher derivatives of the Atiyah class.
Remark 6.4. If we do all of this for an ordinary Lie algebra, we would get \( \ell_{n+2} = 0 \) for \( n > 0 \) and at \( n = 0 \) we just get the Lie bracket.

7. Remarks on Why the Derived Mapping Space Deserves This Name

As usual, these musing are my own opinions, and could be wildly inaccurate.

Consider \( \text{Maps}(Y, X) \). Then for a fixed \( \Phi \in \text{Maps}(Y, X) \), we have

\[
\Gamma(T_{\Phi} \text{Maps}(Y, X)) \cong \Gamma(Y, \Phi^*TX).
\]

So this tangent space roughly remembers the 1-jet of deformations of our map \( \Phi \). (This is also closely related to the space of fields for the supersymmetric sigma model.) We’d like to remember the entire jet space to encode all possible deformations, not just first order ones. So the thing to do would be to pullback not the tangent bundle of \( M \), but the infinite jet bundle. Kevin’s definition of derived mapping space does this, but only near constant maps, \( \Phi \).

But when we’re looking at some global section of the sheaf, \( \tilde{\text{O}}_{\text{Maps}(Y,X)}(X) \), we’re really looking at deformations of all the constant maps to \( X \). If we restrict to the stalk over a single point in \( X \), we recover something like the notion mentioned in the previous paragraph.

8. Putting the Pieces Together, Dmitri Pavlov

We’ll begin by reminding everyone where we are in this Witten genus story. Then we’ll look at holomorphic Chern-Simons as an effective BV theory, and look in analogy to normal Chern-Simons. Finally we’ll try to say what the obstruction complex is, what it does for us, and give an explicit description of this obstruction.

8.1. Where are we? We need to quantize holomorphic Chern-Simons theory. This happens in two steps:

1. Solving the renormalization group equation. This happens perturbatively in \( \hbar \), and it will turn out that

\[
I[L] = I_0[L] + \hbar I_1[L].
\]

We will learn this next week, in Qin’s talk.

2. We also need to solve the quantum master equation, \( \{S, S\} = -\hbar \Delta S \). A priori, this equation also depends upon \( L \) but we know that if the RGE is satisfied for all \( L \) then we only need to solve the QME for a fixed \( L \).

We can always solve the RGE given a local interaction \( I = I[0] \) by choosing a renormalization scheme. It will turn out that all the counterterms vanish for hCS, and we get the above from for \( I[L] \). Initially, this interaction could have had arbitrarily high terms in \( \hbar \). But we get very lucky in this case.
We can’t always solve the QME. There is an obstruction to doing this. Today we wish to explain hCS in the above context and say where this obstruction lives and what it is. Next week, we might actually get to do the computation.

8.2. A Friendly Reminder: Ordinary Chern-Simons as a BV Theory.

The data for Chern-Simons as a free BV theory is

1. A smooth manifold $T$, $\dim(T) = 3$.
2. Smooth section with compact support of some $\mathbb{Z}$-graded vector bundle $E \to T$,
   $$\mathcal{E} = \Gamma E = \Omega^\bullet T[1] \otimes \mathfrak{g}[1]$$
   with a degree 0 pairing on $\mathfrak{g}$.
3. A non-degenerate pairing $E \otimes E \to \text{Dens}(T)$ of degree -1, which induces a pairing $\mathcal{E} \otimes \mathcal{E} \to \mathbb{R}$. Here the pairing is induced by integration on $T$ of forms together with the pairing on $\mathfrak{g}[1]$.
4. $Q : \mathcal{E} \to \mathcal{E}$ of degree 1, skew-adjoint, that has $Q^2 = 0$, and here $Q = d$.

The above is the data of the free BV theory, and now we need some extra data to do the quantization:

1. Now if we define
   $$\mathcal{O}(\mathcal{E}) := \widehat{\text{Sym}}(\mathcal{E}^\ast) \supset \mathcal{O}_{\text{loc}}(\mathcal{E}),$$
   where here the dual on $\mathcal{E}$ denotes continuous dual for the topological vector space $\mathcal{E}$, and $\mathcal{O}_{\text{loc}}(\mathcal{E})$ denotes the local functionals. Then from this we can require an action functional
   $$S(\alpha) = \langle \alpha, Q\alpha \rangle + I_0[0](\alpha),$$
   were $I_0[0] \in \mathcal{O}_{\text{loc}}(\mathcal{E})$. For Chern-Simons,
   $$I_0[0](\alpha) = \frac{1}{3}(\alpha, [\alpha, \alpha]).$$

2. A gauge-fixing operator, $Q^{GF}$ that has $(Q^{GF})^2 = 0$ that has degree -1 and
   $$[Q, Q^{GF}] = \Delta.$$
   For Chern-Simons this is $d^\ast$, and $\Delta$ is the Hodge Laplacian on forms.

8.3. Holomorphic Chern-Simons Theory. So let’s now replicate the above list for this fancier theory, holomorphic Chern-Simons.

1. Let $\mathbb{T}$ be an elliptic curve with holomorphic volume element $\omega$.
2. $\mathcal{E} := \Omega^{0,\bullet}(\mathbb{T}) \otimes (\mathfrak{g}_X[1] \oplus \mathfrak{g}_X[-1])$. This is some sort of derived mapping space for maps $\mathbb{T} \to T^\ast X$ that are in the formal neighborhood of the constant maps to $X$, and compact Kähler manifold. We have a diagram
   $$\sigma^\ast \mathcal{O}_{\text{Maps}(\mathbb{T}, T^\ast X)} \to \mathcal{O}_{\text{Maps}(\mathbb{T}, T^\ast X)} \downarrow \leftarrow X \quad \hookrightarrow \quad T^\ast X.$$
where $\widehat{\text{Maps}}(\mathbb{T}, T^*X)$ is defined as a particular sheaf over $X$,
\[
\mathcal{O}_{\widehat{\text{Maps}}(\mathbb{T}, T^*X)} := C^\bullet(\Omega^0(\mathbb{T}) \otimes g_X)
\]
where $C^\bullet$ denotes the Chevalley-Eilenberg complex and $g_X$ is the unique (up to choice of torsion free connection) curved $L_\infty$-algebra associated to $X$. Further recall that the Chevalley-Eilenberg complex associated to $g_X$ has
\[
C^*_0(g_X) \cong J_{\text{hol}}(X).
\]
(3) Integration on $\mathbb{T}$ with respect to $\omega$ and the pairing on $\Omega(\mathbb{T})^0 \otimes g[1]$ with $\Omega(\mathbb{T})^0 \otimes g^*[−1]$

(4) $Q : \mathcal{E} \to \mathcal{E}$ is defined by $Q = \overline{\partial} + \ell_1$. Two remarks are in order. First, this squares to $[\ell_0, -]$, not to zero! However, it does square to zero mod nilpotents ($\Omega^{>0}(X)$), so this isn’t too serious. Second, this is only a kinetic part “in $X$,” which is to say we only record kinetic parts in the action when looking at deformations in $X$, not in the fiber direction of $T^*X$. This makes some sort of sense when thinking of the supersymmetric sigma model picture.

(5) The classical action is given by
\[
S(\alpha \oplus \beta) = \int_{\mathbb{T}} \omega \wedge \langle \overline{\partial} \alpha, \beta \rangle + \langle \ell_1 \alpha, \beta \rangle + I^0(\alpha, \beta)
\]
where
\[
I^0(\alpha, \beta) = \int_{\mathbb{T}} \omega \wedge \langle \ell_0, \beta \rangle + \sum_{k \geq 2} \frac{1}{k!} \langle \ell_k(\alpha \otimes^k, \beta) \rangle.
\]

(6) $Q^{GF} = \overline{\partial}^*$, so that $[Q, Q^{GF}] = \Delta$, the Doulbealt Laplacian. Here we use that $[\ell_1, \overline{\partial}] = 0$.

Remark 8.1. It is more or less clear that this construction will work for any curved $L_\infty$ algebra $g$, and indeed, Costello proves most of the following results at this level of generality in his paper. We have decided to stick to $g_X$ for concreteness.

8.4. The Obstruction. Now, for the above theory we have automatically that $I^0$ satisfies the classical master equation. This comes from the $L_\infty$ relations.

The quantum master equation is
\[
QI + \frac{1}{2}\{I, I\} + \hbar\Delta I = 0.
\]
After a computation that produces counterterms, we find that
\[
I[L] = I^0[L] + \hbar I_1[L]
\]
satisfies the RGE. This still isn’t quite a quantization of the theory. We need this to satisfy the QME. The obstruction is
\[
Ob = \Delta I^0 + QI_1 + \{I_0, I_1\}
\]
(We’ve dropped the dependence on $L$, but secretly we’ve fixed a length scale and are looking for QME solutions.)
From results in Kevin’s book, \[ Ob \in \mathcal{O}_{loc}(\Omega^{0,\bullet}(\mathbb{T}) \otimes g_X[1]). \]

First we find that there is a quasi-isomorphism between the above obstruction complex and \( \Omega^2_{cl}(Bg)[1] \). From chasing definitions that this is in turn isomorphic to \( \Omega^2_{cl}(X) \). There must be a fact from the theory of Kähler manifolds that ensures that \( \text{ch}_2 \) is actually holomorphic, but I don’t know it.

The map inducing the quasi-isomorphism goes \( \Omega^2_{cl}(Bg)[1] \to \mathcal{O}_{loc}(\Omega^{0,\bullet}(\mathbb{T}) \otimes g_X[1]) \).

In a computation we hope to examine next week, one can show that the image of a choice of representative \( \text{ch}_2 \) under this map gives a representative for the obstruction class (up to some non-zero multiple). It is in this sense that \( p_1(X) \) is the obstruction to quantization of holomorphic Chern-Simons.

9. Feynman Diagrams, and Some Computations, Qin Li

Recall that the space of fields for holomorphic Chern-Simons theory is \( \mathcal{E} = \Omega^{0,\bullet}(E) \otimes (g[1] \oplus g^*[-1]) \) where \( E \) is an elliptic curve or \( \mathbb{C} \). We have the action \( S(\alpha, \beta) \) with quadratic part
\[ \int_E \omega \wedge (\langle \partial \alpha, \beta \rangle + \ell_1 \alpha, \beta) \]
and (length scale \( \infty \)) interaction
\[ I_{hCS} := \int_E \omega \wedge \langle \ell_0, \beta \rangle + \sum_{k \geq 2} \frac{1}{k!} \langle \ell_k(\alpha^\otimes k), \beta \rangle, \]
where \( \omega \) is a holomorphic volume form and \( \ell_k \) are Taylor maps of a curved \( L_\infty \) algebra. We want to quantize this classical theory. So what does this mean? We want a collection of effective interactions \( \{I[L]\}_{L>0} \in \mathcal{O}(\mathcal{E})[[\hbar]] \)
satisfying
(1) the renormalization group equation;
(2) the quantum master equation;
(3) the asymptotic expansion in the \( L \to 0 \) limit of \( I[L] \) is \( I_{hCS} \).

9.1. The Renormalization Group Flow. The renormalization group flow gives a collection of maps
\[ RGE : \mathcal{O}(\mathcal{E})[[\hbar]] \to \mathcal{O}(\mathcal{E})[[\hbar]]. \]
We define this using Feynman diagrams.

So for each Feynman graph we can assign a weight \( W_\gamma(P(\epsilon, L), I) \) where \( P(\epsilon, L) \) is the propagator and \( I \in \mathcal{O}(\mathcal{E})[[\hbar]] \).

If \( I \) is cubic, then we can restrict our attention to trivalent graphs. These graphs will have internal vertices and tails. We put a propagator \( P \) on each internal edge, and an interaction \( I \) on each trivalent vertex. So what is this
propagator in our setting? We have \( P \in \text{Sym}^2(E) \). We want the heat kernel of the operator \( D = [Q, Q^{GF}] = [\overline{\partial}, \overline{\partial}^*] \). Call this kernel \( K_t \). Then

\[
P(\epsilon, L) = \int_{\epsilon}^{L} \partial K_t dt.
\]

In general we decompose

\[
I = \sum_{k \geq 0} I_{i,k} \epsilon^k h^i,
\]

then look at the various \( i \)-valent graphs.

So this propagator together with the interactions will allow us to give a weight to each graph \( \gamma \):

\[
W_{\gamma}(P(\epsilon, L), I) : E^{\otimes T(\gamma)} \to \mathbb{C}.
\]

These weights allow us to define the renormalization group flow

\[
W(I) := \sum_{\gamma_{\text{con}}} \frac{1}{|\text{Aut}(\gamma)|} W_{\gamma}(P(\epsilon, L), I) \epsilon^{g(\gamma)}
\]

where \( \gamma_{\text{con}} \) denote the connected graphs, and

\[
g(\gamma) = b_1(\gamma) + \sum_{v \in V(\gamma)} g_v
\]

where \( b_1 \) is the first Betti number of the graph and \( g_v \) is the “internal genus” of each vertex. This is called the renormalization group flow because it satisfies the renormalization group equation:

\[
I[L] = W(P(\epsilon, L), I[\epsilon])
\]

for all \( 0 < \epsilon < L \).

**Lemma 9.1.**

\[
W(P_1, W(P_2, L)) = W(P_1 + P_2, L),
\]

where \( P_i \in \text{Sym}^2(E) \).

Notice that (by simple properties of integrals):

\[
P(\epsilon, L_1) + P(L_1, L_2) = P(\epsilon, L_2).
\]

This is why we call this a renormalization group flow.

**9.2. Naive Quantization of Holomorphic Chern-Simons Theory.** We’d like to say

\[
I_{\text{naive}}[L] = \lim_{\epsilon \to 0} W(P(\epsilon, L), I_{hCS}).
\]

There are two problems with this:

1. This limit might not exist.
2. QME is not satisfied.
It will turn out that the first problem is “easy” in the sense that the limit does exist in this particular example, so we don’t need to choose a renormalization scheme. This follows (more or less) from all the symmetries we have in holomorphic Chern-Simons, together with a computation of a few integrals.

The second problem, however, is quite serious. As mentioned last time, there are topological obstructions to finding solutions to the quantum master equation.

9.3. Symmetries of Holomorphic Chern-Simons. We require quantization to be invariant under an action by $\mathbb{C}^\times$: for $(\alpha, \beta) \in \Omega^0 \otimes (\mathfrak{g}[1] \oplus \mathfrak{g}^*[1])$

$$t(\alpha, \beta) = (\alpha, t^{-1} \beta)$$

The above action gives us an action on $\mathcal{O}(\mathcal{E})[[\hbar]]$. Since $\mathbb{C}^\times$ acts on $\mathcal{E}(E)$ by weights $\leq 0$, it acts on $\mathcal{O}(\mathcal{E})[[\hbar]]$ by weights $\geq 0$. Call this action

$$\tilde{R}(t) : \mathcal{O}(\mathcal{E})[[\hbar]] \to \mathcal{O}(\mathcal{E})[[\hbar]], \quad t \in \mathbb{C}^\times.$$

We’re going to modify this action slightly so that the classical action defined by $I_{hCS}$ weight 0. So define

$$R(t) : \mathcal{O}(\mathcal{E})[[\hbar]] \to \mathcal{O}(\mathcal{E})[[\hbar]]$$

as $t^{-1}\tilde{R}(t)$.

**Lemma 9.2.** The RGF operator commutes with this action.

**Remark 9.3.** I think that the classical symmetries that descend to quantum symmetries are precisely those that commute with the RGE.

**Remark 9.4.** This action looks a little artificial (and I suspect is some supersymmetry in disguise, though this is a wild guess). Geometrically, it is coming from some action dilation action of $T^*X$ on the fibers.

This symmetry forces

$$I_{naive}[L] = I^{(0)}[L] + I^{(1)}[L] \hbar,$$

just by looking at weights and the fact that $I^{(0)}$ is linear in $\beta$. Also we see that $I^{(1)} \in \mathcal{O}(\Omega^0 \otimes \mathfrak{g}[1])$. We still have to check that the RGE is satisfied, but (because of the above form of $I$) we only need to look up to 1-loop terms to do this.

So we calculate

$$\lim_{\epsilon \to 0} W_\gamma(P(\epsilon, L), I_{hCS})$$

for $\gamma$ a 1-loop graph. We note that

$$\alpha \in \Omega^0 \otimes \mathfrak{g}[1] = C^\infty(E) \otimes (\mathbb{C}[\hbar] \otimes \mathfrak{g}[1])$$

and $W_\gamma$’s dependence on $\epsilon$ is only in the $C^\infty(E)$ part. We can further reduce our work to “trivalent wheels,” since these are the only things that could contribute a singular part to the limit.

After some calculus, we can show that there is a solution to the RGE because the above limit exists.
Remark 9.5. We don’t really have our hands on the quantization yet, because we don’t really know $I^{(1)}$ yet. This is good because otherwise we’d be in trouble when we try to solve the QME. This solution will depend on a choice of trivialization of $ch_2$, and so it would seem that there are as many solutions to the RGE as there are representatives of the class $ch_2$.

We define the length scale $L$ obstruction as

$$O[L] = h^{-1}(QI_{naive}[L] + \frac{1}{2}\{I_{naive}[L], I_{naive}[L]\}_L + h\Delta_L I_{naive}[L])$$

and clearly if $O[L] = 0$, the QME is satisfied.

Proposition 9.6.

$$\lim_{L \to 0} O[L]$$

exists.

If this obstruction actually vanishes, then we can choose $I^{(1)}$ so that both the QME and RGE are satisfied.

Remark 9.7. If a little confused in the computation about when we actually get $I^{(1)}$ and when we’re adding correction terms.

10. Finding the Witten Genus, Christian Blohmann

Recall the main statement of the paper:

Theorem 10.1. For any $L \in (0, \infty]$ there is an quasi-isomorphism

$$(\mathcal{O}(\mathcal{E}), \hat{Q}_L) \simeq (\Omega^{-\bullet}(B\mathfrak{g})[[h]], h\Delta + h\{\log \text{Wit}(X, E, \omega), -\}),$$

where recall that

$$\hat{Q}_L = Q_L + \{I_L, -\} + h\Delta.$$

Unfortunately, there are many calculations that go into this that we will be unable to complete, so instead we’ll just try to give a flavor for where this comes from.

Let’s begin by reminding ourselves of the ingredients in the above quasi-isomorphism:

1. $E = \mathbb{C}/\Lambda$ is an elliptic curve, with its holomorphic volume for $\omega$.
2. $\mathfrak{g}$ is a curved $L_\infty$ algebra associated to a complex manifold $X$.
3. $B\mathfrak{g}$ is a ringed space $(X, C^*(\mathfrak{g}))$ where

$$C^k(\mathfrak{g}) := \text{Hom}_{\Omega_X^k(\mathfrak{g} \otimes k, \Omega^k_X)^S k}.$$

4. Forms on $B\mathfrak{g}$ are defined as

$$\Omega^S_k(B\mathfrak{g}) = C^*(\mathfrak{g}, \Gamma^k(\mathfrak{g}^*[-1])) = C^*(\mathfrak{g}, (\text{Sym}^k \mathfrak{g}^*)[-k]).$$

Let’s keep track of what has changed from the AKSZ setup to where we are now in this $L_\infty$ gauge theory setup:

1. Input target manifold: $X \mapsto \mathfrak{g}_X$.
2. Target symplectic space: $T^*X \mapsto T^*B\mathfrak{g} = \mathfrak{g}[1] \oplus \mathfrak{g}^*[-1]$. 
(3) Space of fields: $\text{Map}(X, T^*X) \mapsto \Omega^0 \cdot E \otimes (g[1] \otimes g^*[-1]) =: \mathcal{E}$. This seems reasonable because we have linearized the space of maps, so we can consider it as some kind of tensor product.

(4) The functions on the space of fields: $O(E) = \prod_{n \geq 1} \text{Hom}_{\Omega^X}(\mathcal{E}^\otimes n, \Omega^X)^{S_n} = C^\bullet(\mathcal{E})$.

(5) Differential ("$Q$-structure"): $\hat{Q}_L = Q + \{I_{hCS}[L], -\} + h\Delta_L$, $Q = \mathcal{F} + \ell_1$, and the $\Delta_L, \{-,-\}$ are constructed in the BV story (from the odd symplectic structure).

Remark 10.2. The renormalization group flow induces a quasi-isomorphism $O(\mathcal{E})[[h]], \hat{Q}_L$ so it suffices to prove the theorem for $L = \infty$.

We haven’t been able to make sense out of the following:

$$C_{\text{red}}(C[dz] \otimes g[1]) = C_{\text{red}}(g, \text{Sym}^\bullet g^*) = \Omega^\bullet(Bg).$$

It’s unclear what red means, and how these things are actually isomorphic. Let’s take this as given, and we’ll explain things from here. We can think of $C[dz] \otimes g[1] \cong g \oplus g[1]$ via the decomposition, $C[dz] \cong C \oplus C \cdot dz$.

Remark 10.3. This has some flavor of deformation quantization, by thinking of this $\text{Sym}$ above as polyvector fields on $X$, in some sense. However, $\Omega^\bullet(Bg)$ is actually like formal holomorphic differential forms (not formal polyvector fields), though there is some kinda of $L_\infty$ morphism going on that looks a little like (the dual to) Kontsevich formality. I think this is called Tamarkin formality. Anyway, the precise analogy is not clear.

Now, $$\Omega^0 \cdot (E) = C^\infty(E) \otimes C[d\bar{z}].$$

Then $$\Omega^{-\bullet}(Bg) \subset O(H)$$

where $H$ are the "harmonic fields,"

$$H = C[dz] \otimes (g[1] \oplus g^*[-1]) \subset \mathcal{E}.$$ The odd symplectic pairing leads to $\Delta, \{-,-\}$ on the right-hand-side, where $\Delta = L_\pi$ for $\pi$ the Poisson tensor.

We need a more explicit description of the Witten genus for this calculation. So

$$\log W\text{it}(X, E, \omega) = \sum_{k>1} \frac{(2k - 1)!}{(2\pi i)^{2k}} E_{2k}(E, \omega) ch_{2k}(TBg)$$

where

$$E_{2k}(E, \omega) = \sum_{\lambda \in \Lambda - \{0\}} \lambda^{-2k}$$

is an Eisenstein series, and

$$ch_{2k}(TBg) = \frac{1}{(2k)!} Tr(\alpha^{2k})$$

is the Chern character.
10.1. Confusing Part of Proof. For the below we can say almost nothing, as in the current writing of the paper there is very limited justification.

\(H\) has an odd symplectic pairing which leads to \(\Delta_H\) and \([-,-]\) on \(O(H)\). Then \(O(E)\) has a BV operator, \(\Delta_\infty\).

Then Kevin says:

(1) \(O(E) \to O(H)\) is a BV-algebra morphism.

(2) \(\mathbb{C}[d\bar{z}] \hookrightarrow C^\infty(E) \otimes \mathbb{C}[d\bar{z}] \simeq \Omega^0 \cdot E\) is a quasi-isomorphism.

These two facts together give that

\[
(O(E)[[h]], Q + h\Delta + \{I[\infty], -\}) \simeq (O(H)[[h]], h\Delta_H + \{I[\infty]|_H, -\})
\]

is a quasi-isomorphism.

Then it is “straightforward” to check

\[
(O(H), \{I[\infty]|_H, -\}) \xrightarrow{\sim} (\Omega^{- \cdot}(T^*B\mathfrak{g}, 0)
\]

(though the 0 differential is our guess; Kevin doesn’t make it explicit). This map takes \(\Delta_H\) to \(L_\pi\), so we conclude that

\[
(O(H), h\Delta_H + \{I[0][\infty]|_H, -\} + h\{I[1][\infty]|_H, -\}) \simeq (\Omega^{- \cdot}(T^*B\mathfrak{g}), hL_\pi + h\{I[1][\infty]|_H, -\})
\]

(the \(T^*\) on the RHS is also our guess, this seems to be a typo).

10.2. (Less Confusing Part of) the Proof. You can check that the remaining part to show is that \(I^{(1)}\) at scale infinity is the (log of the) Witten genus.

**Proposition 10.4.** The 1-loop part of \(I^{(1)}[\infty]\) when restricted to \(\mathbb{C}[d\bar{z}] \otimes g[1] \subset E\) is equal to

\[
(stuff)Tr(\alpha^2) + \sum_{k \geq 2} \frac{1}{2k(4\pi^2)^{2k}} E_{2k}(E, \omega)Tr(\alpha^{2k}).
\]

Now we assume that the first term is 0 (i.e. that the obstruction to quantization vanishes). The reason it is identically zero and not just exact is that we are working with the harmonic fields, so that every cohomology class has a unique representative. Thus, if the obstruction to cohomologous to zero, it is identically zero.

Now we must compute

\[
I^{(1)}[\infty] = I^{(1)}_{\text{naive}} + J[\infty],
\]

and \(I^{(1)}_{\text{naive}} = I^{(1)}_{\text{wheels}} + I^{(1)}_{\text{others}}\). We will show that we only need to consider the wheel contribution.

\[
I^{(1)}_{\text{others}} = \sum_{\gamma} \frac{1}{|\text{Aut}(\gamma)|} W_\gamma(P(0, L), I_{hCS})
\]

where \(\gamma\) has one loop and no wheels.

**Lemma 10.5.** \(I_{\text{others}}[\infty]|_H = 0 = J[\infty]|_H\)
Idea of Proof.

The first step is to show that $I_{\text{tree}}|_{\mathcal{H}} = 0$.

So let $\gamma$ be a tree with $k + 1$ tails. We choose a root for our tree (giving us an operadic picture on the RHS):

$$W_\gamma : \mathcal{E}^{\otimes (k+1)} \to \mathbb{C} \implies W'_\gamma : \mathcal{E}^{\otimes k} \to \mathcal{E}$$

where

$$W_\gamma(\alpha_1, \ldots, \alpha_{k+1}) = \langle W'_\gamma(\alpha_1, \ldots, \alpha_k), \alpha_{k+1} \rangle.$$

Now we'll draw some trees. We put the order $m$ part of the holomorphic Chern-Simons action function at each $m$-valent vertex in the tree,

$$\ell_m : \Omega^{0,\bullet}(E) \otimes (g[1] \oplus g^*[-1])^n \to \Omega^{0,\bullet}(E) \otimes (g[1] \oplus g^*[-1]),$$

and the propagator on each edge. We note that $\ell_m$ maps harmonic forms to a harmonic form. Furthermore, $P(\epsilon, L) : \mathcal{E} \to \mathcal{E}$ will be zero on harmonic forms, since

$$\alpha \mapsto \mathcal{D}^s \int_0^L e^{-t(\mathcal{D}'\mathcal{D})} \alpha dt,$$

so $W_\gamma|_{\mathcal{H}} = 0$.

The second step is to consider a 1-loop non-wheel and show $I^{(1)}_{\text{other}}|_{\mathcal{H}} = 0$. Basically, there will be some tree somewhere, which will force the weight of the entire graph to be zero.

The third step is to show $J[\infty]|_{\mathcal{H}} = 0$, and this is also analogous to the above.

\[ \square \]

The following assumes this dubious equality connecting forms on $B\mathfrak{g}$ in negative degree to some Chevalley-Eilenberg complex.

**Lemma 10.6.** For all $k \geq 1$,

$$\sum_\gamma \frac{1}{|\text{Aut}(\gamma)|} W_\gamma(P(0, \infty), I_{\text{hCS}})|_{\mathcal{C}^{[\mathfrak{g}] \otimes \mathfrak{g}[1]}} = \frac{1}{2k(4\pi^2)^{2k}} E_{2k}(E, \omega) Tr(\alpha^{2k})$$

where $\gamma$ are 1-loop wheels with $2k$ vertices. The sum with $2k - 1$ vertices vanishes.

So now if we do this sum over all $2k$, we essentially get the log of the Witten genus.

Let's try to actually do part of this calculation. Assume $\gamma$ is a wheel with $2k$ vertices. First we need a nicer way to look at the propagator:

$$P(0, \infty) = \int_0^{\infty} \mathcal{D}^s K_t \otimes id_{\mathfrak{g}[1] \oplus \mathfrak{g}^*[-1]} = \mu^{-1} \frac{d}{dz} D^{-1} \otimes id$$

where $D$ is the Laplacian and $\mu = \pi i \int_E dz \wedge \bar{dz}$ is the symplectic volume of the elliptic curve.
Then
\[
\sum_{\gamma} \frac{1}{|\text{Aut}(\gamma)|} W_{\gamma}(P(0, \infty), I_{hCS})|_{\mathbb{C}[z] \otimes \mathbb{Q}[t]} = \frac{1}{2k} \text{Tr} \left( \left( \mu^{-1} \frac{d}{dz} D^{-1} \otimes \alpha \right)^{2k} \right) \\
= \frac{1}{2k} \text{Tr} \left( \mu^{-1} \frac{d}{dz} D^{-1} \right) \text{Tr}(\alpha^{2k})
\]
so now all we need to do is spot the Eisenstein series in the first factor of the last line.

Remark 10.7. It would be nice to have some indication of what’s going on here. This seems to be somewhat related to Kontsevich’s integrals where he gets Bernoulli numbers. This is somewhat more involved, but has a similar flavor.

Lemma 10.8. \( \text{Tr} \left( \mu^{-1} \frac{d}{dz} D^{-1} \right) = \frac{1}{(4\pi^2)^k} E_{2k} \)

Proof. Let \( a + ib, c + id \) be a basis of \( \Lambda \). Let
\[
det \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = -\frac{1}{2\pi} \mu
\]

Define a basis of \( \Lambda \)-invariant function on \( \mathbb{C} \),
\[
F_{n,m}(z, \overline{z}) = F_{n,m}(x, y) = \exp \left( \frac{2\pi i}{ad - bc} (nbx - nay + mdx - mcy) \right).
\]
Then (for \( (n, m) \neq (0, 0) \), which is a typo in the paper),
\[
\frac{d}{dz} D^{-1} F_{n,m} = -\frac{\mu(4\pi^2i)^{-1}}{nb + md - ina - imc} F_{n,m},
\]
and
\[
\text{Tr}_{C^\infty(E)} \left( \mu \frac{d}{dz} D^{-1} \right)^{2k} = \sum_{n,m \in \mathbb{Z}^2 \setminus \{0\}} \frac{(4\pi^2i)^{2k}}{(\text{stuff})} = \frac{1}{(4\pi^2i)^{2k}} \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{(i\lambda)^{2k}} = \frac{E_{2k}}{(4\pi^2i)^{2k}}.
\]

References

[Cos10b] ________, *A geometric construction of the witten genus II*.
[Cos10c] ________, *Renormalization and effective field theories*.