Introduction to Supermanifolds

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This talk is an introduction to supermanifolds — these are supposed to encode “odd fuzz” on regular manifolds. What you do is you enlarge the notion of “functions”: functions are usually sections of a trivial bundle, and we will instead enlarge that bundle.

1 Super Algebra

Definition: A super vector space is a \( \mathbb{Z}/2 \)-graded vector space (over \( \mathbb{R} \)) \( V = V_0 \oplus V_0 \), the even and odd parts. Morphisms are grading-preserving linear maps. There is a “parity reversing functor” \( \Pi \), for which \( (\Pi V)_0 = V_1 \) and \( (\Pi V)_1 = V_0 \).

Renato: These are just the even maps? Aaron: You can reverse parity, which is how you can talk about the other maps.

This category has a \( \otimes \) structure, which makes it equivalent as a monoidal category to the usual category of representations of \( \mathbb{Z}/2 \). But we choose different commutativity isomorphisms:

\[
c_{V,W} : V \otimes W \cong W \otimes V
\]

\[
v \otimes w \mapsto (-1)^{|w||v|} w \otimes v \quad \text{(assuming } v, w \text{ homogenous)}
\]

We can now talk about super algebras, commutative super algebras — use the commutativity isomorphism, so “commutative” means “skew commutative on odd things” —, super modules, super Lie algebras, etc. We can then talk about \( \text{Ssym}^*(V) \) and \( \text{S}\Lambda^n(V) \).

We now introduce the Berezinian. It will connect with integration — it’s a generalization of determinant, and when you change variables in an integral, you need to introduce a determinant.

Definition: A free module over a superalgebra \( \mathcal{A} \) is a module which is free as an ungraded module, but we do ask there to exist a homogenous basis.

Example: \( \mathcal{A}^{p|q} \) is freely generated by \( x_1, \ldots, x_p \) even and \( \theta_1, \ldots, \theta_q \) odd.

\[\square\]
A morphism \( T : \mathcal{A}^{p|q} \to \mathcal{A}^{r|s} \) will be given by a matrix
\[
\begin{pmatrix}
\text{even} & \text{odd} \\
\text{odd} & \text{even}
\end{pmatrix}
\]
Given such a linear transformation, its supertrace is
\[
\text{STr} \left( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) = \text{Tr}(A) - \text{Tr}(D)
\]
This is an even element of the underlying algebra \( \mathcal{A} \).

**Definition:** Let \( L \) be a free module (of finite type) over a commutative superalgebra \( \mathcal{A} \). Then we get a notion as above of \( \text{GL}(L) \). The Berezinian is a homomorphism \( \text{Ber} : \text{GL}(L) \to \mathcal{A}_{0}^{\times} \). We ask it to satisfy:

1. If \( \epsilon \) is even and \( \epsilon^2 = 0 \), then \( \text{Ber}(1 + \epsilon T) = 1 + \epsilon \text{STr}(T) \).
2. If \( T \) is diagonalizable \( T = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \), then \( \text{Ber}(T) = (\det A)(\det D)^{-1} \).
3. If 0 \to \( L' \to L \to L' \to 0 \) is a short exact sequence, and \( (T', T, T'') \) is an automorphism, then \( \text{Ber}(T) = \text{Ber}(T') \text{Ber}(T'') \).

**Harold:** Which is the definition? **Aaron:** You can do this for arbitrary \( A, B, C, D \), and it satisfies all of those, and we think those are enough to nail down the function.

Oh, I should have said: the canonical example of a supervector space is \( \Lambda^*(\mathbb{R}^q) \) — this is a commutative superalgebra, where the parity of the number of tensors is the grading.

## 2 Super Manifolds

The right way to do this is with locally ringed spaces. We will write \( (|M|, \mathcal{O}_M) \), where \( |M| \) is the topological space, and \( \mathcal{O}_M \) is the sheaf of functions. We say that such a locally ringed space is a supermanifold of dimension \( p|q \) if it is locally modeled on \( \mathbb{R}^{p|q} \). Here \( \mathbb{R}^{p|q} \) is the sheaf on \( |\mathbb{R}^{p|q}| = \mathbb{R}^p \) which assigns to an open set \( U \) the ring \( \mathcal{C}^\infty(U) \otimes \Lambda^*(\mathbb{R}^q) \). Sometimes you ask for Hausdorff, second countable, etc.

Given \( M = (|M|, \mathcal{O}_M) \), the reduced supermanifold is \( M^{\text{red}} = (|M|, \mathcal{O}_M/\text{Nil}) \), where Nil is the (sheaf of) ideal of nilpotent elements. **Renato:** So this is a manifold that we’re used to. **Aaron:** Yes, it’s an ordinary manifold. Or rather it’s a supermanifold of dimension \( p|0 \).

**Zach:** Is this really an ordinary manifold, or can it have twisting? **Aaron:** It is a locally ringed space modeled on \( \mathbb{R}^p \).

We always have \( M^{\text{red}} \hookrightarrow M \). This is the map that is the identity on topological spaces, and the quotient of rings.
Let \( \mathbb{R}^q \hookrightarrow E \rightarrow X \) be a vector bundle over an ordinary manifold. Then \( \Pi E \) is the “oddification” of the vector bundle. It is a supermanifold with \( |\Pi E| = X \) and \( O_{\Pi E} = \Gamma(\Lambda^*E^*) \), where \( E^* \) is the dual vector bundle, and so on. This has dimension \( p|q \) if \( \dim X = p \) as an ordinary manifold.

**Theorem (Batchelor ‘79):** We already described \( \Pi : \text{vector bundles} \rightarrow \text{supermanifolds} \). We define a map \( J \) in the other way, which sends \( M \) to the vector bundle over \( M^{\text{red}} \), with sections \( \Gamma(U, J(M)) = \text{Nil}(U)/\text{Nil}(U)^2 \).

The theorem is that \( J \circ \Pi \cong \text{id} \) is a natural isomorphism of functors. For each object \( M \), \( \Pi \circ J(M) \cong M \), but unnecessarily so — so we get a bijection on objects between \{vector bundles\} and \{supermanifolds\}.

In fact, if \( \dim M = p|q \) for \( p \geq 1 \) and \( q \geq 2 \), then there does not exist a contraction \( M \rightarrow M^{\text{red}} \) that is compatible with all automorphisms of \( M \). There are more supermanifold-morphisms than vector bundle morphisms.

**Theo:** Is it right to say that \( M^{\text{red}} \hookrightarrow M \) is a closed submanifold, and \( J(M) \) is the first-order neighborhood of \( M^{\text{red}} \), like the pull back of the tangent bundle? **Aaron:** That sounds right.

**Theo:** Is Batchelor’s theorem true in the analytic category? I see how to prove it in the smooth case, but my proof would require partitions of unity. **Aaron:** Nobody I’ve read said it wasn’t true, but I’m not sure.

### 3 Functors of points

I gave a local characterization, but local things can be hard to work with.

Given \( S, M \) supermanifolds, we have a bijection between \( \text{SUPERMAN}(S, M) \cong \text{SUPERALG}(O_M(M), O_S(S)) \). This clearly only holds in the smooth category. We call the elements of \( \text{SUPERMAN}(S, M) \) the \( S \)-points of \( M \).

We think of arbitrary functors \( \text{SUPERMAN}^{\text{op}} \rightarrow \text{Set} \) as “generalized” supermanifolds. It’s Yoneda’s lemma that the generalized supermanifold corresponding to a supermanifold is no loss of data.

**Example:** \( \text{SM}(B, C) \) is the generalized supermanifold that sends \( A \) to \( \text{SM}(A \times B, C) \). \( \diamond \)

I.e. \( (C^B)^A = C^{A \times B} \). \( 3^2 \) is the number of maps from the 2-element set to the 3-element set.

**Theorem:** \( \text{SM}(\mathbb{R}^{0|1}, M) = \Pi TM \).

We will now define the right hand side.

First, and aside on vector bundles:

**Definition:** A super vector bundle over a supermanifold is a locally free sheaf \( \mathcal{E} \) of \( O_M \) modules. This is the same as the usual non-super case.
**Example:** $TM$ the tangent bundle is the sheaf that on $U \in |M|$ gives $\text{Der}(\mathcal{O}_M(U))$. An \textit{even derivation} of a superalgebra is an even linear map $D$ satisfying $D(fg) = D(f)g + fD(g)$. An \textit{odd derivation} is an odd map satisfying $D(fg) = D(f)g + (-1)^{|f|}fD(g)$. This gives the supermodule of derivations. \hfill \Box

**Definition:** The total space of $\mathcal{E}$ has $S$-points $E(S) = \{(f, g) \text{ s.t. } f \in \text{SM}(S, M) \text{ and } g \in \Gamma(S, f^*\mathcal{E}^{\text{ev}})\}$. Then $\Pi : \text{SVect}_M \to \text{SVect}_M$ is the functor that reverses parities. It is $\Pi \mathcal{E} = \mathbb{R}^{0|1} \otimes \mathcal{E}$, where $\mathbb{R}^{0|1}$ is the trivial bundle with fiber $\mathbb{R}^{0|1}$.

**Renato:** The reduced manifold of $TM$ is $TM^{\text{red}}$. But you told me just a sheaf on $M^{\text{red}}$. **Theo:** He says that in general, a vector bundle over $M$ is a sheaf over $M^{\text{red}}$. Then he said: any vector bundle has a total space, which might have a much larger reduced space than $M$ has.

**Example:** The total space of $TM$ is $TM$, and is $2p|2q$-dimensional. \hfill \Box

**Proof (of theorem about $TM$):**

$$\text{SM}(S, \mathbb{S}\text{M}(\mathbb{R}^{0|1})) = \text{SM}(S \times \mathbb{R}^{0|1}, M)$$

defn of SM

$$\cong \text{SUPERALG}(\mathcal{O}_M(M), \mathcal{O}_S(S) \otimes \mathcal{O}_{\mathbb{R}^{0|1}})$$

proposition

$$\cong \text{SUPERALG}(\mathcal{O}_M(M), \mathcal{O}_S(S) \otimes \mathcal{O}_{\mathbb{R}^{0|1}})$$

defn of $M \times M'$

$$\cong \text{SUPERALG}(\mathcal{O}_M(M), \mathcal{O}_S(S) \otimes E(\theta))$$

$E(\theta) = \text{exterior algebra on } \theta$

Suppose $\phi : \mathcal{O}_M(M) \to \mathcal{O}_S(S) \otimes E(\theta)$. Write $\phi = f + \theta g$, where $f, g : \mathcal{O}_M(M) \to \mathcal{O}_S(S)$ are even,odd respectively linear maps. What is it to be an algebra map?

$$f(ab) + \theta g(ab) = \phi(ab) = \phi(a)\phi(b) = (f(a) + g(a))(f(b) + \theta g(b)) =$$

$$= f(a)f(b) + \theta(g(a)f(b) + (-1)^{|f(a)|\theta}f(a)g(b)) = f(a)f(b) + \theta(g(a)f(b) + (-1)^{|a|}f(a)g(b))$$

So $f$ is an algebra map, and $g$ is an odd derivation with respect to $f$.

But $\text{Der}_f^{\text{odd}}(U) = \Pi(TM)(U)$, so this is $\text{SM}(S, \Pi(TM))$. \hfill \Box

The reason I like $\mathbb{R}^{0|1}$ is that it (conjecturally) connects to homotopy theory. One goal in this class is to gain homotopical information from functors on the bordism category.

**Theorem:** $0|1$-$\text{EFT}^n(M) = \begin{cases} \Omega^{\text{even}}_{\text{closed}}(M), n \text{ even} \\ \Omega^{\text{odd}}_{\text{closed}}(M), n \text{ odd} \end{cases}$. So we can define $0|1$-$\text{EFT}^n[M]$ by imposing that $0|1$-$\text{EFT}^n$ is supposed to be a homotopy functor. You get even/odd de Rham cohomology of $M$, which is the same as that of $M^{\text{red}}$. So Euclidean Field Theories are a form of “cocycles” for de Rham cohomology.

**Theorem (Stolz-Teichner):** $1|1$-$\text{EFT}^n[M] \cong K^*(M)$.

This is very tantalizing. De Rham cohomology and K-theory are the 0th and 1st rungs on a ladder, called “chromatic homology”.

**Conjecture (Stolz-Teichner):** $2|1$-$\text{EFT}^n[M] \cong \text{TMF}^*(M)$.