MORITA EQUIVALENCES OF CLIFFORD ALGEBRAS

For \( n \in \mathbb{N} \), let \( Cl_n \) denote the Clifford algebra with \( n \) generators \( e_i \) and \( e_i^2 = +1 \) for \( n > 0 \). Let \( Cl_{-n} \) denote the Clifford algebra with generators with \( n \) generators and \( f_i^2 = -1 \). Let \( Cl_n \) denote the corresponding complex Clifford algebra.

1. Complex Clifford algebras

We have an isomorphism of algebras \( Cl_2 \cong \text{End}(\mathbb{C}^{1|1}) \) this is given by

\[
(1) \quad e_1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 \mapsto \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.
\]

Then \( \mathbb{C}^{1|1} \) gives a Morita equivalence between \( Cl_2 \) and \( \mathbb{C} \).

2. Real Clifford algebras

For the real Clifford algebras, we have \( Cl_1 \otimes Cl_{-1} \cong \text{End}(\mathbb{R}^{1|1}) \) this is given by

\[
(2) \quad e_1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad f_1 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

We claim that \( Cl_3 \cong \mathbb{H} \otimes Cl_{-1} \) and \( Cl_{-3} \cong \mathbb{H} \otimes Cl_1 \) where \( \mathbb{H} \) is a purely even algebra. This isomorphism is given by

\[
e_1 \mapsto i \otimes f, \quad e_2 \mapsto j \otimes f, \quad e_3 \mapsto k \otimes f,
\]

and similarly for \( Cl_{-3} \).

We then have that

\[
Cl_4 \cong Cl_3 \otimes Cl_1 \cong \mathbb{H} \otimes Cl_{-1} \otimes Cl_1 \cong \mathbb{H} \otimes Cl_1 \otimes Cl_{-1} \cong Cl_{-3} \otimes Cl_{-1} \cong Cl_{-4},
\]

where \( \sigma \) is the braiding isomorphism in the category of super vector algebras. Now we have

\[
Cl_8 \cong Cl_4 \otimes Cl_4 \cong Cl_4 \otimes Cl_{-4} \cong (Cl_1 \otimes Cl_{-1})^\otimes 4 \cong \left( \text{End}(\mathbb{R}^{1|1}) \right)^\otimes 4 \cong \text{End}(\mathbb{R}^{8|8}).
\]

Hence, \( \mathbb{R}^{8|8} \) gives a Morita equivalence between \( Cl_8 \) and \( \mathbb{R} \).

3. The conjugation action on Morita equivalences

One can discover \( KO \)-theory as the fixed points under conjugation of complex \( K \)-theory: to any module over a complex Clifford algebra, we can extract a module over the real Clifford algebra. Implicit in this, we see that the 2-fold periodicity of the complex Clifford algebra become an 8-fold periodicity of the real Clifford algebras. We wish to make this explicit.

Any real periodicity can be complexified, so it suffices to look for Morita equivalences between even dimensional real Clifford algebras and \( \mathbb{R} \). As a sanity check, we note that the isomorphism \( Cl_2 \cong \text{End}(\mathbb{C}^{1|1}) \) does preserve the conjugation action on both sides, as can be seen by considering the image of \( e_2 \). Hence, the 2-periodicity of the complex Clifford algebras does not survive after taking fixed points (as one should expect).

However, we can identify this isomorphism as the complexification of the isomorphism \( Cl_1 \otimes Cl_{-1} \cong \text{End}(\mathbb{R}^{1|1}) \) by identifying \( if_1 \) with \( e_2 \). In particular, the fixed points of \( \text{End}(\mathbb{C}^{1|1}) \) under the conjugation action are isomorphic to \( Cl_1 \otimes Cl_{-1} \). Hence, for the fixed points of \( Cl(2n) \) to be identified with the fixed points of \( \text{End}(\mathbb{R}^{1|1})^{\otimes n} \), we require that \( Cl_n \otimes Cl_{-n} \cong Cl(2n) \). We claim that the first \( n \) for which this occurs is 4.

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