WELL ORDERING

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1. Introduction

Zermelo gave a beautiful proof in [1] that every set can be well ordered. We translate it here and provide a minor simplification at one point to make it more self-contained.

2. The proof

A *partially* ordered set is a set X equipped with a relation $x \leq y$ satisfying $x \leq x$ and $x \leq y \leq z \Rightarrow x \leq z$ and $x \leq y \leq x \Leftrightarrow x = y$. (The last property is easily obtained by considering the quotient set for the equivalence relation $x \sim y \Leftrightarrow x \leq y \leq x$.) A *totally* ordered set is a partially ordered set where $x \leq y \vee y \leq x$. A *well* ordered set is a totally ordered set where every nonempty subset has a minimal element. A *closed* subset Y of a partially ordered set X is a subset satisfying $x \leq y \in Y \Rightarrow x \in Y$; we write $Y \leq X$, and if $Y \neq X$, too, then we write $Y < X$. If X is well ordered and $Y < X$, and we take x to be the smallest element of $X - Y$, then $Y = \{y \in X \mid y < x\}$.

Lemma 2.1. *Suppose X is a set and $\mathcal{F}$ is a collection of subsets equipped with well orderings. Suppose also that for any $C, D \in \mathcal{F}$, either $C \leq D$ or $D \leq C$. Let $E = \bigcup_{C \in \mathcal{F}} C$. Then there is a unique ordering on E compatible with the ordering of each $C \in \mathcal{F}$; with that ordering E is well ordered, and for each $C \in \mathcal{F}$ we have $C \leq E$.*

Theorem 2.2 (Well-Ordering)**.** *Any set X can be well ordered.*

Proof. For each proper subset $C \subsetneqq X$ pick an element $g(C) \in X$ with $g(C) \notin C$. A subset $C \subseteq X$ equipped with a well ordering such that $c = g(\{c' \in C \mid c' < c\})$ for every $c \in C$ will be called a g-set.

Intuitively, a g-set C, as far as it goes, is determined by g. For example, if C starts out with $\{c_0 < c_1 < c_2 < \dots\}$, then necessarily $c_0 = g(\{\})$, $c_1 = g(\{c_0\})$, $c_2 = g(\{c_0, c_1\})$, and so on. The tricky part is seeing how to keep that going until all of X is exhausted.

We claim that if C and D are g-sets, then either $C \leq D$ or $D \leq C$. To see this, let W be the union of the subsets $B \subseteq X$ satisfying $B \leq C$ and $B \leq D$. Since a union of closed subsets is closed, we see that $W \leq C$ and $W \leq D$, and W is the largest subset of X with this property. If $W = C$ or $W = D$ the claim is established, so assume $W < C$ and $W < D$, and pick elements $c \in C$ and $d \in D$ so that $W = \{c' \in C \mid c' < c\} = \{d' \in D \mid d' < d\}$. Since C and D are g-sets, we see that $c = g(W) = d$. Let $W' = W \cup \{g(W)\}$, equipped with the ordering that

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declares $g(W)$ is larger than all the elements of W; it's a g-set larger than W with $W' \leq C$ and $W' \leq D$, contradicting the maximality of W.

Now let W be the union of all the g-sets, and equip it with the unique ordering compatible with the orderings on each of the g-sets. Using the lemma we see that it is a g-set, too, and it is the largest g-set. If $W \neq X$, then $W' := W \cup \{g(W)\}$, equipped with the ordering that declares $g(W)$ is larger than all the elements of W, is a larger g-set, yielding a contradiction. Hence $W = X$, and we have well ordered X. □

References

[1] Ernst Zermelo. Beweis, daß jede Menge wohlgeordnet werden kann. *Math. Ann.*, 59:514–516, 1904.

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