FIXED POINT LEMMA

DANIEL R. GRAYSON

1. Introduction

Zermelo gave a beautiful proof in [6] that every set can be well ordered, and Kneser adapted it to give a direct proof of Zorn's lemma in [3]. Sources such as [4], [5], [2, p. 63], and most recently, [1], describe this proof, but it still doesn't seem to be generally known by mathematicians. In this note we adapt the proof, as suggested by lecture notes of Peter Loeb, to prove a theorem about fixed points of increasing functions on partially ordered sets.

2. The proof

A *partially* ordered set is a set X equipped with a relation $x \leq y$ satisfying $x \leq x$ and $x \leq y \leq z \Rightarrow x \leq z$ and $x \leq y \leq x \Leftrightarrow x = y$. (The last property is easily obtained by considering the quotient set for the equivalence relation $x \sim y \Leftrightarrow x \leq y \leq x$.) A *totally* ordered set is a partially ordered set where $x \leq y \vee y \leq x$. A *well* ordered set is a totally ordered set where every nonempty subset has a minimal element. A *closed* subset Y of a partially ordered set X is a subset satisfying $x \leq y \in Y \Rightarrow x \in Y$; we write $Y \leq X$, and if $Y \neq X$, too, then we write $Y < X$. If X is well ordered and $Y < X$, and we take x to be the smallest element of $X - Y$, then $Y = \{y \in X \mid y < x\}$.

Lemma 2.1. *Suppose X is a partially ordered set, and F is a collection of subsets which are well ordered by the ordering of X. Suppose also that for any $C, D \in F$, either $C \leq D$ or $D \leq C$. Let $E = \bigcup_{C \in F} C$. Then E is well ordered, and for each $C \in F$ we have $C \leq E$.*

Theorem 2.2. *An increasing function f on a partially ordered set X with upper bounds for its well ordered subsets has a fixed point.*

Proof. Suppose f has no fixed point. For f to be an increasing function on X means that $x \leq f(x)$ for all $x \in X$. For each well ordered subset $C \subseteq X$ define $g(C) = f(x)$ where x is some upper bound of C. (If X has least upper bounds for its well ordered subsets, then by using them one can avoid making any choices in the proof of this theorem.) Since $x < f(x)$ we see that $g(C) \notin C$. For each well ordered subset $C \subseteq X$ pick an upper bound $g(C) \notin C$. A well ordered subset $C \subseteq X$ such that $c = g(\{c' \in C \mid c' < c\})$ for every $c \in C$ will be called a g-set.

Intuitively, a g-set C, as far as it goes, is determined by g. For example, if C starts out with $\{c_0 < c_1 < c_2 < \dots\}$, then necessarily $c_0 = g(\{\})$, $c_1 = g(\{c_0\})$, $c_2 = g(\{c_0, c_1\})$, and so on. A pseudoproof of the theorem might go like this. We start with an empty collection of g-sets and add larger and larger g-sets to it. At each stage let W be the union of the g-sets encountered previously. We see that

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$W' = W \cup \{g(W)\}$ is a larger g-set, and we add it to our collection. Continue this procedure forever and let W be the union of the g-sets encountered along the way; it's again a g-set, and we can enlarge it once again, thereby encountering a g-set that isn't in our final collection and providing a contradiction. The problem with this pseudoproof is in interpreting the meaning of "forever", so now we turn to the real proof.

We claim that if C and D are g-sets, then either $C \leq D$ or $D \leq C$. To see this, let W be the union of the subsets $B \subseteq X$ satisfying $B \leq C$ and $B \leq D$. Since a union of closed subsets is closed, we see that $W \leq C$ and $W \leq D$, and W is the largest subset of X with this property. If $W = C$ or $W = D$ we are done, so assume $W < C$ and $W < D$, and pick elements $c \in C$ and $d \in D$ so that $W = \{c' \in C \mid c' < c\} = \{d' \in D \mid d' < d\}$. Since C and D are g-sets, we see that $c = g(W) = d$. Let $W' = W \cup \{g(W)\}$; it's a g-set larger than W with $W' \leq C$ and $W' \leq D$, contradicting the maximality of W.

Now let W be the union of all the g-sets. It's a g-set, too, and it's the largest g-set, but $W' = W \cup \{g(W)\}$ is a larger g-set, yielding a contradiction. □

University of Illinois at Urbana-Champaign
E-mail address: dan@math.uiuc.edu
URL: http://www.math.uiuc.edu/~dan