Proceedings of Symposia in Pure Mathematics
Volume 55 (1994), Part 1

Weight Filtrations in Algebraic K-Theory

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ABSTRACT. We survey briefly some of the K-theoretic background related to the theory of mixed motives and motivic cohomology.

1. Introduction

The recent search for a motivic cohomology theory for varieties, described elsewhere in this volume, has been largely guided by certain aspects of the higher algebraic K-theory developed by Quillen in 1972. It is the purpose of this article to explain the sense in which the previous statement is true, and to explain how it is thought that the motivic cohomology groups with rational coefficients arise from K-theory through the intervention of the Adams operations. We give a basic description of algebraic K-theory and explain how Quillen's idea **[42]** that the Atiyah-Hirzebruch spectral sequence of topology may have an algebraic analogue guides the search for motivic cohomology.

There are other useful survey articles about algebraic K-theory: **[50, 46, 23, 56, 40]**.

I thank A. Beilinson, W. Dwyer, E. Friedlander, S. Lichtenbaum, R. McCarthy, S. Mitchell, C. Soulé, and R. Thomason for frequent and useful consultations.

2. Constructing topological spaces

In this section we explain the considerations from combinatorial topology that give rise to the higher algebraic K-groups. The first principle is simple enough to state but hard to implement: when given an interesting group (such as the Grothendieck group of a ring) arising from some algebraic situation, try to realize it as a low-dimensional homotopy group (especially π_0 or π_1)

1991 *Mathematics Subject Classification.* Primary 19-02; Secondary 19E20.
Supported by NSF grant DMS 90-02715.
This paper is in final form and no version of it will be submitted for publication elsewhere.

of a space X constructed in some combinatorial way from the same algebraic inputs and study the homotopy type of the resulting space.

The groups which can easily be described as π_0 of some space are those that are presented as the set of equivalence classes for some equivalence relation on a set S. We may then take for the space T the graph that has S as its set of vertices, and as its edges some subset of $S \times S$ that generates the given equivalence relation.

The groups that can easily be described as π_1 of some space are those that are presented by generators and relations. One may then take for T the connected cell-complex constructed with one vertex, one edge for each generator of the group, and one 2-cell for each relation (appropriately glued into edges to impose the desired relations).

Neither of the two spaces T mentioned above has particularly interesting homotopy groups, due to the absence of higher-dimensional cells. One hopes that in the algebraic situation at hand there is some particularly evident and natural way of adding cells of higher dimension to T. The most natural and fruitful framework for adding higher-dimensional cells to such spaces hinges on the notion of geometric realization of a simplicial set, as invented by John Milnor in **[36]**. The cells used are simplices (triangles, tetrahedra, etc.), and a *simplicial set* is a sort of combinatorial object which amounts to a convenient way of labeling the faces of the simplices in preparation for gluing. For each integer $n \geq 0$ we give ourselves a set X_n and regard it as the set of labels for the simplices of dimension n to be used in the gluing construction. Then for each labeled simplex of dimension n we assign labels of the appropriate dimension to each of the faces.

The faces of a simplex of dimension n may be accounted for as follows. Let $\underline{\underline{n}}$ denote the ordered set $\{0 < 1 < 2 < \cdots < n\}$, and write the points of the standard n-dimensional simplex Δ^n as formal linear combinations $\sum_{i=0}^n a_i\langle i\rangle$, where $\langle i\rangle$ is simply a symbol and where the coefficients a_i are nonnegative real numbers with $\sum_{i=0}^n a_i = 1$. The faces of Δ^n are the affine-linear spans of subsets of $\{\langle 0\rangle, \dots, \langle n\rangle\}$. We can index the m-dimensional faces of Δ^n by the injective maps $s : \underline{\underline{m}} \to \underline{\underline{n}}$ that are *increasing* in the sense that $i \leq j \Rightarrow s(i) \leq s(j)$. We consider the unique affine-linear map $s_* : \Delta^m \to \Delta^n$ satisfying $s_*(\langle i\rangle) = \langle s(i)\rangle$, which embeds Δ^m as a face of Δ^n. If $x \in X_n$ is a label for an n-dimensional simplex, then the label we assign to the face given by the image of s_* should be an element of X_m which we will dub $s^*(x)$. We have to do this for each s and for each x. It turns out to be convenient to do this also for increasing maps s that are not necessarily injective.

The total system of compatibilities that this system of labels must satisfy is codified as follows. Let Ord denote the category of finite nonempty ordered sets $\underline{\underline{n}}$, where the arrows are the increasing maps. Then the collection of sets X_n together with the collection of maps $s^* : X_n \to X_m$ should constitute a contravariant functor from Ord to the category of sets. Such a functor X is called a *simplicial set*, and the corresponding space $|X|$ obtained by gluing simplices together is called the *geometric realization* of X.

A label $x \in X_n$ is called an *n-simplex* of X.

3. Nerves of categories

The primary example of a simplicial set arises from a category $\mathscr{C}$ in the following way. Interpret the ordered set $\underline{\underline{n}}$ as a category $0 \to 1 \to 2 \to \cdots \to n$, and let $\mathscr{C}_n = \mathscr{C}(\underline{\underline{n}})$ denote the set $\operatorname{Hom}(\underline{\underline{n}}, \mathscr{C})$ of functors from $\underline{\underline{n}}$ to $\mathscr{C}$. The resulting simplicial set, which we may also write as $\mathscr{C}$ without too much fear of confusion, is called the *nerve* of $\mathscr{C}$. The space $|\mathscr{C}|$ will have a vertex for each object of $\mathscr{C}$, an edge for each arrow of $\mathscr{C}$, and so on.

Geometric realization is a functor from the category of simplicial sets to the category of spaces, and taking the nerve is a functor from the category of small categories to the category of simplicial sets. There is a dictionary of corresponding notions in these three categories which are related by these two functors. Thus, the nerve of a functor $F : \mathscr{C} \to \mathscr{C}'$ is a natural transformation, i.e., a map of simplicial sets, and the geometric realization of that is a continuous map. The geometric realization of the category $0 \to 1$ can be identified with the unit interval $I = [0, 1]$, and investigating the extent to which compatibility with products holds shows that the nerve of a natural transformation $\mathscr{C} \times \underline{\underline{1}} \to \mathscr{C}'$ is a simplicial homotopy, and the geometric realization of that is a homotopy $|\mathscr{C}| \times I \to |\mathscr{C}'|$. In particular, if $\mathscr{C}$ has an initial object, then contraction along the cone of initial arrows amounts to a null-homotopy of $|\mathscr{C}|$, so that $|\mathscr{C}|$ is a contractible space.

4. Classifying spaces of groups

The simplest examples of categories for which it is useful to consider the geometric realization arise from groups. Let G be a group, and let $G[1]$ denote the category with one object $*$ and with G as its monoid of arrows. One finds that

$$\pi_i|G[1]| = \begin{cases} G & i = 1, \\ 0 & i \neq 1. \end{cases} \tag{4.1}$$

To prove this, one considers the category $\widetilde{G}$ whose set of objects is G and in which there is for each $g, h \in G$ a unique arrow $g \to h$ labeled hg^{-1}. The labels are there only for the purpose of describing the map $\widetilde{G} \to G[1]$ that sends an arrow of $\widetilde{G}$ labeled hg^{-1} to the arrow hg^{-1} of $G[1]$. One sees that G acts freely on $|\widetilde{G}|$ on the right and that the map $|\widetilde{G}| \to |G[1]|$ is the covering map corresponding to the quotient by this action. Moreover, the category $\widetilde{G}$ has an initial object, so $|\widetilde{G}|$ is contractible. Putting this information together yields the result.

Another name for the space $|G[1]|$ is BG, and it is called the *classifying space* of the group G.

An explicit calculation with simplicial chains shows that the singular homology group $H_n(BG, \mathbb{Z})$ is isomorphic to the group cohomology group $H^n(G, \mathbb{Z})$, as calculated using the bar resolution. In fact, the difference between the normalized bar resolution and the ordinary bar resolution amounts to the homeomorphism $|\widetilde{G}|/G \cong |G[1]|$.

5. Simplicial abelian groups

There is a third important class of examples of simplicial sets. Consider a simplicial abelian group A, which by definition is a contravariant functor from Ord to the category of abelian groups. If we forget the abelian group structure on each A_n, we are left with a simplicial set whose geometric realization $|A|$ we may consider. The striking fact here is that $\pi_i|A| = H_i NA$, where by NA we mean the normalized chain complex associated to A. The functor $A \mapsto NA$ is an equivalence of categories from the category of simplicial abelian groups to the category of homological chain complexes of abelian groups **[11, 28]**. (The appropriate definition of the inverse functor to N is easy to deduce from Yoneda's lemma.) This Dold-Kan equivalence allows us to embed the theory of homological algebra into homotopy theory.

We remark that if X is a simplicial set and we let $\mathbb{Z}[X]$ denote the simplicial abelian group whose group of n-simplices is the free abelian group $\mathbb{Z}[X_n]$ on the set X_n, then the homotopy groups of $|\mathbb{Z}[X]|$ are the homology groups of $|X|$. Thus $\mathbb{Z}[X]$ is a drastic form of abelianization for simplicial sets.

6. Eilenberg-Mac Lane spaces

Here is the fourth important class of examples of simplicial sets, obtained as a special case of the third. Let G be an abelian group, and let $n \geq 0$ be an integer. Then consider the homological chain complex that has G in dimension n and the group 0 in all the other dimensions. Let $G[n]$ denote the corresponding simplicial abelian group, obtained according to the Dold-Kan equivalence. We get a space $|G[n]|$ which has the abelian group $G = \pi_n|G[n]|$ as its only nonvanishing homotopy group; it is an *Eilenberg-Mac Lane space.*

Consider pointed spaces (CW-complexes) V and W, and let $[V, W]$ denote the set of homotopy classes of (base-point preserving) maps from V to W. There is a natural isomorphism $[V, |G[n]|] \cong H^n(V, G)$, which we will use later. To see the plausibility of the isomorphism, consider the case where V is a sphere and identify the two sides.

7. The lower K-groups

Grothendieck considered the group $K(R)$ generated by the isomorphism classes of finitely generated projective R-modules, modulo relations $[P] = [P'] + [P'']$ coming from short exact sequences $0 \to P' \to P \to P'' \to 0$. The group $\mathrm{Gl}(R)$ is the group of invertible matrices over R with a countable number of rows and columns, equal to the identity outside of some square. Bass defined $K_1(R)$ to be the abelianization $\mathrm{Gl}(R)^{\mathrm{ab}}$ of the infinite general linear group $\mathrm{Gl}(R)$ and renamed the Grothendieck group to $K_0(R)$ because of six-term exact sequences he was able to prove involving the Grothendieck group and his new group [2]. Milnor [37] found the correct definition for $K_2(R)$ and supported its correctness by extending the exact sequences of Bass.

8. The construction of the higher K-groups

We can now describe Quillen's first construction of algebraic K-theory for a ring R.

By adding a single two-cell and a single three-cell to the space $\mathrm{BGl}(R)$ Quillen was able to construct a space $\mathrm{BGl}(R)^+$ with the property that the map $\mathrm{BGl}(R) \to \mathrm{BGl}(R)^+$ induces an isomorphism on homology groups with integer coefficients and induces the map $\mathrm{Gl}(R) \twoheadrightarrow \mathrm{Gl}(R)^{\mathrm{ab}}$ on fundamental groups. (The construction provides a functor from rings to spaces, because the cells' attaching maps used in the case $R = \mathbb{Z}$ work for any ring R.) Quillen also proved that the space $\mathrm{BGl}(R)^+$ is an abelian group in the homotopy category of pointed spaces. This construction therefore serves as a modest form of abelianization for spaces such as this, whose commutator subgroup is perfect.

Quillen's first definition of the higher algebraic K-groups is given in terms of this plus-construction by setting $K_i(R) = \pi_i \mathrm{BGl}(R)^+$ for $i > 0$.

The Hurewicz mapping

$$K_i(R) \to H_i(\mathrm{BGl}(R)^+, \mathbb{Z}) \cong H_i(\mathrm{BGl}(R), \mathbb{Z}) \cong H_i(\mathrm{Gl}(R), \mathbb{Z}) \tag{8.1}$$

gives an initial stab at the relationship between the higher K-groups and the homology groups of the general linear group.

9. K-groups of exact categories

The plus-construction is a curious creature. It permits some explicit computations to be performed, by virtue of its close connection with the general linear group. For example, using it, Quillen proved that $K_1(R)$ agrees with the group of the same name defined by Bass and that $K_2(R)$ agrees with the group of the same name defined by Milnor in terms of the Steinberg group [37]. But it suffers from two drawbacks. First, one would prefer to have a space $K(R)$ which has all of the K-groups appearing as its homotopy groups, including $K_0(R)$. The naive construction of such a space, $K_0(R) \times \mathrm{BGl}(R)^+$, has such homotopy groups but, when regarded as an abelian group in the homotopy category, fails to mix π_0 with π_1 appropriately, unless it happens that $K_0(R) = \mathbb{Z}$. Second, the Grothendieck group is defined for almost any sort of category with a notion of exact sequence; the higher K-groups should be defined for such categories, too.

The definitions of K-theory that do not suffer from these two drawbacks deal with an *exact* category $\mathcal{M}$. An *exact* category $\mathcal{M}$ is an additive category equipped with a set of sequences $0 \to M' \to M \to M'' \to 0$ called *exact* sequences, which arises as a full subcategory $\mathcal{M} \subseteq \mathcal{A}$ closed under extensions in some (unspecified) abelian category $\mathcal{A}$, equipped with the collection of all short sequences of $\mathcal{M}$ that are exact in $\mathcal{A}$. As examples of exact categories, we mention the category $\mathcal{P}(R)$ of finitely generated projective left R-modules, the category $\mathcal{M}(R)$ of finitely generated left R-modules, the category $\mathcal{P}(Z)$ of locally free $\mathcal{O}_Z$-modules of finite type on a scheme Z, and the category $\mathcal{M}(Z)$ of quasi-coherent $\mathcal{O}_Z$-modules of finite type on a scheme Z. The corresponding higher K-groups are all of interest.

The first such definition of K-theory I intend to discuss is Quillen's Q-construction, [43]. The category $Q\mathcal{M}$ has the same objects as does $\mathcal{M}$, but an arrow $M' \to M$ of $Q\mathcal{M}$ is an isomorphism of M' with an admissible subobject of an admissible quotient object of M. Admissibility of a subobject refers to the requirement that the corresponding quotient object also lies in $\mathcal{M}$ or, more precisely, that the inclusion map for the subobject is part of a short exact sequence in $\mathcal{M}$. One may check eventually that the connected space $|Q\mathcal{M}|$ satisfies $\pi_1|Q\mathcal{M}| \cong K_0\mathcal{M}$, and one defines $K_i(\mathcal{M}) = \pi_{i+1}|Q\mathcal{M}|$ for all $i \geq 0$. It is then a theorem of Quillen [22] that $K_i(R) \cong K_i(\mathcal{P}(R))$. We define the K-theory space $K(\mathcal{M})$ to be the loop space $\Omega|Q\mathcal{M}|$ of $|Q\mathcal{M}|$, so that $K_i(\mathcal{M}) = \pi_i(K(\mathcal{M}))$.

If X is a scheme, we may define $K_i(X) = K_i(\mathcal{P}(X))$, where $\mathcal{P}(X)$ denotes the category of locally free $\mathcal{O}_X$-modules of finite type on X. Much of what is stated below for commutative rings R applies equally well to schemes X.

If R is a ring, we define $K'_i(R) := K_i(\mathcal{M}(R))$, where $\mathcal{M}(R)$ denotes the category of finitely generated R-modules. If X is a scheme, we define $K'_i(X) := K_i(\mathcal{M}(X))$, where $\mathcal{M}(X)$ denotes the category of quasi-coherent $\mathcal{O}_X$-modules locally of finite type.

A second definition of K-theory is provided by Waldhausen's S-construction [57]. The simplicial set $S\mathcal{M}$ is defined in such a way that its set of n-simplices consists of chains $0 = M_0 \rightarrowtail M_1 \rightarrowtail \cdots \rightarrowtail M_n$ of admissible monomorphisms of $\mathcal{M}$, together with objects of $\mathcal{M}$ representing all the quotient objects. One sees that there is exactly one vertex in the space $|S\mathcal{M}|$, one edge for each object M of $\mathcal{M}$, and one triangle for each short exact sequence $0 \to M' \to M \to M'' \to 0$ of $\mathcal{M}$. If we let $[M]$ denote the class in $\pi_1\mathcal{M}$ arising from the edge labeled by M, then the triangles are glued to the edges so that the relation $[M] = [M'] + [M'']$ is imposed. It follows that $\pi_1 S\mathcal{M} \cong K_0\mathcal{M}$—this fits in well with the earlier remark about expressing groups given by generators and relations as π_1 of a space. It is a theorem of Waldhausen that $|S\mathcal{M}|$ is homotopy equivalent to $|Q\mathcal{M}|$, so that $\pi_{i+1}|S\mathcal{M}| \cong K_i\mathcal{M}$ for all $i \geq 0$.

The S-construction or the Q-construction can be used to show that the space $K(\mathcal{M})$ is naturally an infinite loop space; the deloopings obtained thereby are increasingly connected, so the homotopy groups in negative dimension that arise thereby are all zero. This makes the space $K(\mathcal{M})$ into an Ω-*spectrum* (or an infinite loop space), which is essentially a space Z, together with compatible choices of deloopings $\Omega^{-n}Z$ for every $n > 0$. The sort of spectrum just obtained, in which the deloopings are increasingly connected, with no new homotopy groups arising in low degrees, is called *connective*.

Another equivalent definition for K-theory is presented in **[20]** by Gillet and Grayson. It hinges on the elementary fact that every element of $K_0(\mathcal{M})$ can be expressed as a difference $[P]-[Q]$, where P and Q are objects of $\mathcal{M}$ and $[P]$ denotes the class in $K_0(\mathcal{M})$. Thus $K_0(\mathcal{M})$ is a quotient of the set of pairs (P, Q) of objects of $\mathcal{M}$ by the equivalence relation where $(P, Q) \sim (P', Q')$ if and only if $[P]-[Q]=[P']-[Q']$. According to the earlier remark about groups whose elements are equivalence classes of a relation, we may try to express $K_0(\mathcal{M})$ as π_0 of some space. That space would have vertices that are pairs (P, Q) of objects, its edges would generate the equivalence relation, and its higher-dimensional simplices should be defined in a natural and simple way. It is an exercise to show that our equivalence relation is generated by the requirement that $(P, Q) \sim (P', Q')$ whenever there are admissible monomorphisms $P' \rightarrowtail P$ and $Q' \rightarrowtail Q$ whose cokernels are isomorphic. This suggests that an edge of our simplicial set should be a triple consisting of two such monomorphisms and an isomorphism of their cokernels. The simplicial set $G\mathcal{M}$ is defined by saying that an n-simplex is a pair of chains of admissible monomorphisms $P_0 \rightarrowtail P_1 \rightarrowtail \cdots \rightarrowtail P_n$ and $Q_0 \rightarrowtail Q_1 \rightarrowtail \cdots \rightarrowtail Q_n$, together with a commutative diagram of isomorphisms of quotients as illustrated here.

$$\begin{array}{ccccccc}
P_1/P_0 & \longrightarrow & P_2/P_0 & \longrightarrow & \cdots & \longrightarrow & P_n/P_0 \\
\cong\downarrow & & \cong\downarrow & & & & \cong\downarrow \\
Q_1/Q_0 & \longrightarrow & Q_2/Q_0 & \longrightarrow & \cdots & \longrightarrow & Q_n/Q_0
\end{array}$$

It is then a theorem that $K_i(\mathcal{M}) = \pi_i|G\mathcal{M}|$ for all $i \geq 0$ and that $|G\mathcal{M}|$ is a loop space of $|S\mathcal{M}|$. It turns out that having $K_0(\mathcal{M})$ appear as π_0 rather than as π_1 offers a technical advantage when constructing operations on K-groups that are homomorphisms on K_i only for $i > 0$, for then the operations may arise from maps of spaces **[24]**.

10. Some theorems of algebraic K-theory

Using the Q-construction, Quillen **[43]** was able to prove the following four foundational theorems (among others).

The first theorem is the additivity theorem. It states that if $0 \to F' \to F \to F'' \to 0$ is a short exact sequence of exact functors $\mathcal{M} \to \mathcal{M}'$, then the map $F_* : K_n(\mathcal{M}) \to K_n(\mathcal{M}')$ satisfies the formula $F_* = F'_* + F''_*$.

The second theorem is the Jordan-Hölder theorem for higher K-theory. It states that if $\mathcal{M}$ is an Artinian abelian category (i.e., every object has a composition series), then $K_n(\mathcal{M}) = \bigoplus_V K_n(\mathrm{End}(V))$, where the direct sum runs over the isomorphism classes of simple objects V of $\mathcal{M}$. An important application of this theorem is the following. Let R be a Noetherian ring, and let $\mathcal{M}^p(R)$ denote the category of finitely generated R-modules whose support has codimension $\geq p$. Then the quotient abelian category $\mathcal{M}^p/\mathcal{M}^{p+1}$ is Artinian, and the theorem implies that $K_n(\mathcal{M}^p/\mathcal{M}^{p+1}) = \bigoplus_x K_n(k(x))$, where x runs over points $x \in \mathrm{Spec}(R)$ of height p and $k(x)$ denotes the residue field at x. If we take $n = 0$ then we see that $K_0(\mathcal{M}^p/\mathcal{M}^{p+1}) = \bigoplus_x \mathbb{Z}$ is the group of algebraic cycles of codimension p. This is the way that algebraic cycles are related to algebraic K-theory.

The third theorem is the resolution theorem, which implies that if X is a regular Noetherian scheme, then $K_n(\mathcal{P}(X)) = K_n(\mathcal{M}(X))$. It hinges on the fact that any $M \in \mathcal{M}(X)$ has a resolution of finite length by objects of $\mathcal{P}(X)$. For X affine, that resolution is simply a projective resolution.

The fourth theorem is the localization theorem for abelian categories. It says that if $\mathcal{B}$ is a Serre subcategory of an abelian category $\mathcal{A}$ (which means that it is closed under taking subobjects, taking quotient objects, and taking extensions), then there is a fibration sequence $K(\mathcal{B}) \to K(\mathcal{A}) \to K(\mathcal{A}/\mathcal{B})$, where $\mathcal{A}/\mathcal{B}$ denotes the quotient abelian category. The main import of being a fibration sequence is that there results the following long exact sequence of K-groups:

$$\cdots \to K_n(\mathcal{B}) \to K_n(\mathcal{A}) \to K_n(\mathcal{A}/\mathcal{B}) \to K_{n-1}(B) \to \cdots.$$

This theorem can be applied notably in the case above where $\mathcal{A} = \mathcal{M}^p(R)$ and $\mathcal{B} = \mathcal{M}^{p+1}(R)$ or in the case where X is a Noetherian scheme, Y is a closed subscheme, $A = \mathcal{M}(X)$, $\mathcal{B} = \mathcal{M}(Y)$, and $\mathcal{A}/\mathcal{B} \cong \mathcal{M}(X - Y)$.

11. Some computations

We now present some explicit computations of some algebraic K-groups.

There is a map $R^\times = \mathrm{Gl}_1(R) \subseteq \mathrm{Gl}(R) \to K_1(R)$; let us use $\{u\}$ to denote the image of a unit u under this map, so that we can write the group law in $K_1(R)$ additively, $\{u\} + \{v\} = \{uv\}$. When R is commutative this map is split by the determinant map, so that $K_1(R)$ has $R^\times$ as a direct summand. It is known that when R is a field, a local ring, or a euclidean domain, then $K_1(R) \cong R^\times$.

Concerning the K-groups of a field F, we know that

$$\begin{aligned} &K_0F = \mathbb{Z}, \\ (11.1)\qquad &K_1F = F^\times, \\ &K_2F = (F^\times \otimes_{\mathbb{Z}} F^\times)/\langle a \otimes (1-a) \mid a \in F - \{0, 1\}\rangle. \end{aligned}$$

The standard notation for the image of $a \otimes b$ in K_2F is $\{a, b\}$, and the relation $\{a, 1-a\} = 0$ is called the Steinberg relation. From the Steinberg relation one can deduce that $\{a, b\} = -\{b, a\}$ **[37, p. 95]**.

For finite fields, the following K-groups are completely known [45]:

$$K_0\mathbb{F}_q = \mathbb{Z},$$
$$K_{2i+1}\mathbb{F}_q \cong \mathbb{Z}/(q^{i+1} - 1),$$
$$K_{2i+2}\mathbb{F}_q = 0.$$

In fact, what motivated Quillen's original definition of algebraic K-theory was the discovery of a space (the homotopy fixed-point set of the Adams operation ψ^q acting on topological K-theory) that has these homotopy groups, whose homology groups are isomorphic to the homology groups of the general linear group.

Suslin's important work on the K-groups of algebraically closed fields [54] shows that the torsion subgroup of $K_n\mathbb{C}$ is $\mathbb{Q}/\mathbb{Z}$ for n odd and is 0 for n even and nonzero. The quotient modulo the torsion is a uniquely divisible group. (For example, $K_1(\mathbb{C}) = \mathbb{C}^\times$, so the torsion subgroup is the group of roots of unity, thus isomorphic to $\mathbb{Q}/\mathbb{Z}$.) For a general exposition of these matters and others in the K-theory of fields, see [23].

As for the field $\mathbb{Q}$ of rational numbers, Tate proved [37, 11.6] that $K_2\mathbb{Q} \cong \mathbb{Z}/2 \oplus \bigoplus_{p \text{ prime}}(\mathbb{Z}/p)^\times$.

The higher K-groups of $\mathbb{Q}$ are not known exactly, but the ranks are. In fact, the ranks are known for any ring of algebraic integers; so consider now the case where F is a number field and $\mathcal{O}_F$ is the ring of integers in F. Write $F \otimes \mathbb{R} = \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$, so that r_1 is the number of real places of F, and r_2 is the number of complex places.

For $\mathcal{O}_F$ and, indeed, for any Dedekind domain, one knows that $K_0\mathcal{O}_F = \mathbb{Z} \oplus \operatorname{Pic}(\mathcal{O}_F)$, where $\operatorname{Pic}(\mathcal{O}_F)$ denotes the ideal class group of $\mathcal{O}_F$.

It is a theorem of Bass, Milnor, and Serre [3] that $K_1(\mathcal{O}_F) \cong \mathcal{O}_F^\times$. This is not a general fact about Dedekind domains.

Quillen [44] has shown that $K_n(\mathcal{O}_F)$ is a finitely generated abelian group for all $n \geq 0$. (In fact, it is conjectured by Bass that $K_n(R)$ is a finitely generated group whenever R is a finitely generated regular commutative ring.) We list the ranks as

$$\operatorname{rank} K_n\mathcal{O}_F = \begin{cases} 1 & \text{if } n = 0, \\ r_1 + r_2 - 1 & \text{if } n = 1, \\ 0 & \text{if } n = 2k \text{ and } k > 0, \\ r_1 + r_2 & \text{if } n = 4k+1 \text{ and } k > 0, \\ r_2 & \text{if } n = 4k+3 \text{ and } k \geq 0. \end{cases} \tag{11.2}$$

In this table, the case $n = 0$ amounts to the finiteness of the ideal class group; the case $n = 1$ is Dirichlet's unit theorem; Borel's theorem, [10], handles the case $n \geq 2$ through a detailed study of harmonic forms on symmetric spaces associated to arithmetic groups and depends on earlier work of Borel and Serre on compactifying these symmetric spaces. It is customary now to refer to a nontorsion element of $K_{2i-1}(\mathcal{O}_F)$ as a *Borel class*.

The interesting fact about these ranks is that in the odd cases, the rank of $K_{2i-1}(\mathcal{O}_F)$ is equal to the order of vanishing of the Dedekind zeta function $\zeta_F(s)$ at $s = 1 - i$. It was Lichtenbaum who spawned the current endeavor by predicting this coincidence, well before Borel computed the ranks of the K-groups.

Only the first four K-groups of $\mathbb{Z}$ are known. We know that $K_0(\mathbb{Z}) = \mathbb{Z}$ and $K_1(\mathbb{Z}) = \mathbb{Z}^\times = \mathbb{Z}/2$. Milnor showed [**37**, 10.1] that $K_2(\mathbb{Z}) \cong \mathbb{Z}/2$. Lee and Szczarba [**32**] proved that $K_3\mathbb{Z} \cong \mathbb{Z}/48$.

12. Products in K-theory

Henceforth we shall deal with rings R that are commutative. In that case the tensor product operation $P \otimes_R Q$ on finitely generated projective R-modules leads to an operation

$$K_mR \otimes K_nR \to K_{m+n}R \tag{12.1}$$

which endows $\bigoplus_{n=0}^{\infty} K_nR$ with the structure of a skew-commutative graded ring, by virtue of the essential commutativity and associativity of the tensor product operation. The unit element of the ring is $1 = [R] \in K_0(R)$. In the case $m = n = 1$ the product agrees with the Steinberg symbol (at least up to a possible sign), in the sense that $\{u\} \cdot \{v\} = \{u, v\}$.

One defines the Milnor ring of a field F to be the quotient of the tensor algebra of $F^\times$ by the ideal generated by the Steinberg relations. This ring is a graded ring, and we let $K_n^M F$ denote its degree n part. It follows from what we have said that there is a natural map $K_n^M F \to K_nF$ which is an isomorphism for $0 \le n \le 2$ and which is known to be a nonisomorphism in general for $n > 2$.

13. Nonlinear operations on K-theory

If $F : \mathcal{M} \to \mathcal{M}'$ is an exact functor, then there is a natural map $K(\mathcal{M}) \to K(\mathcal{M}')$ induced by F. The exterior power operation $\Lambda^k P$ on finitely generated projective R-modules P gives rise to operations $\lambda^k : K_0(R) \to K_0(R)$, as shown by Grothendieck [**26**], even though Λ^k is not an exact functor. These *lambda* operations are defined by letting $\lambda^k([P] - [Q])$ be the coefficient of t^k in $\lambda_t([P])/\lambda_t([Q])$, where $\lambda_t([P]) := \sum_{k=0}^{\infty}[\Lambda^k P]t^k$ in the power series ring $K_0(R)[[t]]$. One uses the identity $\lambda_t([P \oplus Q]) = \lambda_t([P])\lambda_t([Q])$ to show that λ_t (and hence λ^k) is well defined. When R is a field, then $K_0(R) = \mathbb{Z}$, and $\lambda^k(n) = \binom{n}{k}$; the usual definition of the binomial coefficient for $n < 0$ is the one we are led to using in the artifice above.

We point out, for later use, that when $P \cong L_1 \oplus \cdots \oplus L_n$, where each L_i

is a projective module of rank 1, then

$$\lambda_t([P]) = \prod_{i=1}^{n} \lambda_t([L_i]) = \prod_{i=1}^{n}(1 + t[L_i])$$
$$= \sum_{k=0}^{n} t^k \sigma_k([L_1], \dots, [L_n]),$$

where σ_k is the elementary symmetric polynomial of degree k. We deduce that $\lambda^k([\mathscr{P}]) = \sigma_k([L_1], \dots, [L_n])$. We will see that the operations λ^k are useful for the same reason the elementary symmetric polynomials are: we can write other symmetric functions in terms of them.

In [27] is presented Quillen's method for defining the lambda operations on the higher K-groups. The resulting functions $\lambda^k : K_n(R) \to K_n(R)$ are group homomorphisms except when $n = 0$ and $k \neq 1$. Perhaps it is startling at first glance that functions that are decidedly not additive on K_0 are closely related to functions on the higher K-groups which are, but there is no other possibility. Any sort of operation on higher homotopy groups of a space will presumably have to arise from a map of spaces and so cannot avoid being a homomorphism.

It is possible to repair the nonadditivity of the lambda operations on K_0, thereby increasing their utility, by means of a natural sort of abelianization procedure which produces new *Adams operations* $\psi^k : K_n(R) \to K_n(R)$ which are group homomorphisms, even for $n = 0$. On K_0, the salient feature of the Adams operations, aside from being homomorphisms, is that if L is a projective module of rank 1, then $\psi^k([L]) = [L^{\otimes k}]$. Consequently, if $P \cong L_1 \oplus \cdots \oplus L_n$ is a direct sum of projective modules of rank 1, then $\psi^k([P]) = [L_1^{\otimes k}] + \cdots + [L_n^{\otimes k}]$. The symmetric polynomial $x_1^k + \cdots + x_n^k$ can be expressed as a polynomial with integer coefficients in the elementary symmetric polynomials, so there is a formula for $[L_1^{\otimes k}] + \cdots + [L_n^{\otimes k}]$ in terms of the exterior powers of P. It does not really matter what this formula is, but it may be compactly recorded in terms of generating functions as follows:

$$\sum_{k=0}^{\infty} \psi^k(x)(-t)^k = \operatorname{rank} x - t\frac{d}{dt}\log \lambda_t(x).$$

This formula serves as the definition of $\psi^k(x)$ for any $x \in K_0(R)$.

A unit u of the ring R arises in $K_1(R)$ by virtue of being an automorphism of the free module $L = R$ of rank 1. As such it gives rise to the automorphism $u^k = u^{\otimes k}$ of $L = L^{\otimes k}$, so we see that $\psi^k(u) = u^k$, or, writing it additively, $\psi^k u = ku$.

Heuristically speaking, ψ^k raises the functions (i.e., the elements of R) entering into a construction of an element of K-theory, to the kth power. To the extent that constructions of elements of K-theory from several functions of R involve those functions in a multi-multiplicative way, the effect of ψ^k

on an element of K-theory can be used to count the number of functions entering into its construction.

Here we summarize some properties of the Adams operations.

(1) If $x \in K_n(R)$ and $y \in K_n(R)$ then $\psi^k(x+y) = \psi^k(x) + \psi^k(y)$.
(2) If x is the class $[L]$ of a line bundle (rank 1 projective R-module) L in $K_0(R)$, then $\psi^k(x) = [L^{\otimes k}]$.
(3) If x is the class in $K_0(R)$ of a free module, then $\psi^k(x) = x$.
(4) If $x \in K_p(R)$ and $y \in K_q(R)$ then $\psi^k(xy) = \psi^k(x)\psi^k(y) \in K_{p+q}(R)$.
(5) If $x \in R^\times$ then $\psi^k(\{x\}) = \{x^k\} = k\{x\}$.
(6) $\psi^k \circ \psi^\ell = \psi^{k\ell}$.
(7) If R is a regular Noetherian ring and M is a finitely generated R-module whose support has codimension $\geq p$, then $\psi^k([M]) = k^p[M]$ modulo torsion and classes of modules of codimension greater than p.

The last item above is related to the fact [21] that in K-theory with supports, if we have a regular sequence $t_1, \ldots, t_p$ in R, and if $M = R/(t_1, \ldots, t_p)$, then $\psi^k([M]) = [R/(t_1^k, \ldots, t_p^k)]$.

In the first part of [25] the reader may find a concrete description of the kth Adams operation on K_0 as the secondary Euler characteristic of the Koszul complex; the Koszul complex is introduced there by taking the mapping cone of the identity map on $P \in \mathscr{P}(R)$ and considering the kth symmetric power of that complex of length 1.

14. Weight filtrations in the K-groups

The Adams operations were used by Grothendieck to provide an answer (up to torsion) to the following question. Suppose that R is a regular Noetherian ring and that M is a finitely generated R-module with the codimension of its support at least p. We get a class $[M] \in K_0(\mathscr{M}(R))$. By the resolution theorem, we have an isomorphism $K_0(R) \cong K_0(\mathscr{M}(R))$, so we get a class $[M] \in K_0(R)$. Let $F^p_{\mathrm{top}}K_0(R)$ denote the subgroup of $K_0(R)$ generated by such classes; the collection of these subgroups is called the filtration by codimension of support. The question is, is there an algebraic construction of such a filtration which makes sense for any commutative ring R, not necessarily regular? The construction would have to be expressed in terms of projective modules alone, in the absence of the resolution theorem. In the next few paragraphs, we explore the construction of such a filtration.

Let us assume that R is a commutative domain. Then we have the *rank* homomorphism $K_0(R) \to \mathbb{Z}$ defined by $[P] \mapsto \operatorname{rank} P$. This map is an isomorphism when R has dimension 0, i.e., is a field. The kernel F^1_{alg} of the rank homomorphism consists of elements of the form $[P] - \operatorname{rank} P$, and we declare such elements to have *weight* ≥ 1.

The Picard group $\operatorname{Pic}(R)$ is the group of isomorphism classes $[L]$ of rank 1 projective R-modules L, with tensor product as the group operation. Given $P \in \mathscr{P}(R)$, we let $\det P$ denote the highest exterior power of P, and we let $[\det P]$ denote its isomorphism class in $\mathrm{Pic}(R)$. From the formula $\det(P \oplus P') \cong (\det P) \otimes (\det P')$ we find that there is a group homomorphism $K_0(R) \to \mathrm{Pic}(R)$. There is also a function $\mathrm{Pic}(R) \to K_0(R)$, defined by sending $[L] \in \mathrm{Pic}(R)$ to $[L] \in K_0(R)$, which is a group homomorphism from $\mathrm{Pic}(R)$ to the group of units $K_0(R)^\times$ of the ring $K_0(R)$. (We shall see shortly that $[L] - 1$ is a nilpotent element of the ring $K_0(R)$.)

Let us consider the simple case where R is a regular domain of dimension 1, i.e., is a Dedekind domain. In this case it is known that the homomorphism $K_0(R) \to \mathbb{Z} \oplus \mathrm{Pic}(R)$ given by $[P] \mapsto (\operatorname{rank} P, [\det P])$ is an isomorphism. The proof depends on the algebraic fact that any $P \in \mathscr{P}(R)$ of rank n is the direct sum of a free module R^{n-1} and a projective module L of rank 1, from which it follows that $L \cong \det P$, allowing us to recover $[P]$ from $\operatorname{rank} P$ and $\det P$ by the formula $[P] = [\det P] - 1 + \operatorname{rank} P$. (The nilpotence of $[L] - 1$ results from the isomorphism $L \oplus L \cong L^{\otimes 2} \oplus R$, which allows us to prove that $([L] - 1)^2 = 0$.)

For an arbitrary commutative domain R the kernel F^2_{alg} of the surjective homomorphism $K_0(R) \to \mathbb{Z} \oplus \mathrm{Pic}(R)$ is the subgroup generated by elements of the form $([\det P] - 1) - ([P] - \operatorname{rank} P)$, for the vanishing of such elements is all that is required for the proof from the previous paragraph. We declare such elements to bc of *weight* ≥ 2.

What ought we to use for the elements of weight ≥ 3? One problem confronting us is the lack of a function analogous to $\operatorname{rank} P$ and $\det P$ that would vanish on such elements. (Using the second Chern class associated with some cohomology theory seems unnatural and unprofitable.)

We try to get an idea by examining $([\det P] - 1) - ([P] - \operatorname{rank} P)$ when P is decomposable. For example, when $P \cong L_1 \oplus L_2$, where L_1 and L_2 have rank 1, we see that

$$\begin{aligned}([\det P] - 1) - ([P] - \operatorname{rank} P) &= [L_1 \otimes L_2] + 1 - [L_1 \oplus L_2] \\ &= ([L_1] - 1)([L_2] - 1),\end{aligned}$$

showing that our element of weight ≥ 2 is a product of two elements of weight ≥ 1. This suggests that the weight filtration we are constructing ought to be compatible with multiplication. That in turn fits in well with our original topological filtration, for if D_1 is a divisor whose defining ideal is isomorphic to L_1, then we have $[D_1] = 1 - [L_1]$, and if D_2 is a divisor whose ideal is isomorphic to L_2, and D_1 and D_2 intersect regularly, then $D_1 \cap D_2$ has codimension 2 and $[D_1 \cap D_2] = (1 - [L_1])(1 - [L_2])$.

A first attempt at a weight filtration might involve declaring a product such as

$$([L_1] - 1) \cdot \cdots \cdot ([L_k] - 1),$$

where each L_i is a rank 1 projective module, to have weight $\geq k$. This is not quite enough because it fails to assign weight ≥ 2 to the element

$([\det P] - 1) - ([P] - \operatorname{rank} P)$ when P is an arbitrary projective module, for P may not decompose as a direct sum of rank 1 modules; P may even fail to have a filtration whose successive quotients are rank 1 projectives.

Supposing $P \cong L_1 \oplus \cdots \oplus L_n$, where each L_i is a rank 1 projective, we let $x_i = [L_1] - 1$ and compute

$$\begin{aligned}
&([\det P] - 1) - ([P] - \operatorname{rank} P) \\
&\quad = ([L_1 \otimes \cdots \otimes L_n] - 1) - ([L_1 \oplus \cdots \oplus L_n] - n) \\
&\quad = (x_1 + 1) \cdot \cdots \cdot (x_n + 1) - (1 + (x_1 + \cdots + x_n)) \\
&\quad = \sum_{k=2}^{n} \sigma_k(x_1, \ldots, x_n).
\end{aligned}$$

One can show that each term in the latter sum has trivial rank and determinant, and that suggests singling them out. Setting $x = x_1 + \cdots + x_n = [P] - n$, we define $\gamma^k(x) := \sigma_k(x_1, \ldots, x_n)$. We justify the notation by showing that $\gamma^k(x)$ is a well-defined function of x as follows. Using the definition of the elementary symmetric polynomials, we compute

$$\begin{aligned}
\sum_{k=0}^{n} t^k \gamma^k(x) &= \sum_{k=0}^{n} t^k \sigma_k(x_1, \ldots, x_n) \\
&= \prod_{i=1}^{n} (1 + t x_i) \\
&= \prod_{i=1}^{n} (1 - t + t[L_i]) \\
&= (1 - t)^n \prod_{i=1}^{n} \left(1 + \frac{t}{1-t}[L_i]\right).
\end{aligned}$$

The latter product is $\lambda_t([P])$ with t replaced by $u := \frac{t}{1-t}$, and we use $\lambda_u([P])$ to denote it. We find that

$$\begin{aligned}
\sum_{k=0}^{n} t^k \gamma^k(x) &= (1 - t)^n \lambda_u([P]) = \left(1 + \frac{t}{1-t}\right)^{-n} \lambda_u([P]) \\
&= \lambda_u(1)^{-n} \lambda_u([P]) = \lambda_u([P] - n) = \lambda_u(x).
\end{aligned}$$

This formula shows that $\gamma^k(x)$ depends only on x and provides a definition of $\gamma^k(x)$ which is useable for any element $x \in K_0(R)$, without assuming that x has the form $\sum([L_i] - 1)$.

We observe that $\gamma^0(x) = 1$ and $\gamma^1(x) = x$. When $\operatorname{rank} x = 0$, from the equation $\gamma_t(x)\gamma_t(-x) = 1$ and the fact that both $\gamma_t(x)$ and $\gamma_t(-x)$ are polynomials with constant term 1, we see that the coefficients $\gamma^k(x)$ must be nilpotent for $k \geq 1$.

One can show that $\det(\gamma^k(x)) = 0$ when $k \geq 2$ and $\operatorname{rank} x = 0$, using the splitting principle. Thus $F^2_{\mathrm{alg}} K_0(R)$ is generated by such elements $\gamma^k(x)$.