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Quillen's work in algebraic K-theory

by

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Abstract

We survey the genesis and development of higher algebraic K-theory by Daniel Quillen.

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Introduction

This paper[1] is dedicated to the memory of Daniel Quillen. In it, we examine his brilliant discovery of higher algebraic K-theory, including its roots in and genesis from topological K-theory and ideas connected with the proof of the Adams conjecture, and his development of the field into a complete theory in just a few short years. We provide a few references to further developments, including motivic cohomology.

Quillen's main work on algebraic K-theory appears in the following papers: [65, 59, 62, 60, 61, 63, 55, 57, 64]. There are also the papers [34, 36], which are presentations of Quillen's results based on hand-written notes of his and on communications with him, with perhaps one simplification and several inaccuracies added by the author. Further details of the plus-construction, presented briefly in [57], appear in [7, pp. 84–88] and in [77]. Quillen's work on Adams operations in higher algebraic K-theory is exposed by Hiller in [45, sections 1-5]. Useful surveys of K-theory and related topics include [81, 46, 38, 39, 67] and any chapter in *Handbook of K-theory* [27].

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1. The Grothendieck group

Grothendieck introduced the abelian group $K(X)$, known as the *Grothendieck group*, where X is an algebraic variety, in order to formulate his generalization of

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the Riemann-Roch Theorem to higher dimensions[3, 21]. It is defined by generators and relations, where there is one generator $[E]$ for each vector bundle of finite rank E on X, and there is one relation $[E] = [E'] + [E'']$ for each short exact sequence $0 \to E' \to E \to E'' \to 0$. The tensor product operation $E' \otimes E''$ provides a multiplication operation on $K(X)$ that makes it into a commutative ring.

For an affine variety X with coordinate ring R we let $K(R)$ denote the group. In that case a vector bundle can be regarded as a finitely generated projective R-module, so all short exact sequences split. Thus the identity $[E' \oplus E''] = [E'] + [E'']$ implies the defining relation, so could also serve as a defining relation.

For $k \in \mathbb{N}$, the symmetric power operation $E \mapsto S^k E$ induces a natural function $\sigma^k : K(X) \to K(X)$, and the exterior power operation $E \mapsto \Lambda^k E$ induces a (closely related) natural function $\lambda^k : K(X) \to K(X)$; for $k > 1$ the operations are not compatible with direct sums, and thus the corresponding functions are not compatible with addition. The closely related and derivative *Adams operations* $\psi^k : K(X) \to K(X)$, for $k \in \mathbb{N}$, introduced in [4], are ring homomorphisms characterized by the identity $\psi^k[L] = [L^{\otimes k}]$, for any line bundle L.

Various cohomology theories provide a graded ring $H^*(X)$ for each smooth quasi-projective variety X and harbor Chern classes $c_k(E) \in H^k(X)$ for each vector bundle E with properties reminiscent those of σ^k and λ^k. Moreover, there is a Chern class function $c_k : K(X) \to H^k(X)$ and a related ring homomorphism $ch : K(X) \to H^*(X) \otimes \mathbb{Q}$, in terms of which the Grothendieck-Riemann-Roch theorem is phrased.

2. Topological K-theory

Atiyah and Hirzebruch [9] (see also [8, 6]), motivated by Grothendieck's work, considered a finite simplicial complex X and defined $K(X)$ to be the Grothendieck group of the category of topological vector bundles E on X. They used suspension to shift the degree by -1 and Bott periodicity to shift the degree by ± 2, thereby extending it to a graded ring and *generalized cohomology theory* $K^*(X)$, known as *topological K-theory*, with $K^0(X) = K(X)$ and with $K^*(X) \cong K^{*+2}(X)$. The *Atiyah-Hirzebruch spectral sequence* expresses its close relationship to the singular cohomology groups $H^k(X; \mathbb{Z})$.

3. K_1 and K_2

Motivated both by Grothendieck and by Atiyah-Hirzebruch, Bass, partly in collaboration with Alex Heller, sought the algebraic analogue of the topological K-groups $K^*(X)$, at least for an affine scheme with coordinate ring R. Thus they sought a graded ring $K_*(R)$ with $K_0(R) = K(R)$ (lowering the index to take into account the contravariance between rings and spaces). Bass [11] defined $K_1(R)$ to be (something isomorphic to) the abelianization of the infinite general linear group $GL_\infty(R)$, which consists of invertible matrices whose rows and columns are indexed by $\mathbb{N}$ and are equal to the identity matrix except in finitely many locations.

According to Bass: "The idea was that a bundle on a suspension is trivial on each cone, so the gluing on the 'equator' is defined by an automorphism of a trivial bundle, E, up to homotopy. Thus, topologically, K^{-1} is $Aut(E)/Aut(E)^o$, where $Aut(E)^o$ is the identity component of the topological group $Aut(E)$. Since unipotents are connected to the identity (by a straight line, in fact) they belong to the identity component. This indicated that the algebraic definition should at least include the elementary subgroup of $GL_n(R)$. Since that turned out (thanks to Whitehead) to be (stably) exactly the commutator subgroup, that led to the algebraic definition of K_1."

As Bass observed [11, § 20], the group $K_1(R)$ had been introduced (but not so named) by Whitehead [83, p. 4] in 1950 in the course of defining what is now known as the *Whitehead group*, which classifies simplicial homotopy equivalences between two homotopy equivalent simplicial complexes, modulo the simple homotopy equivalences.

The map $R^\times \to K_1(R)$ turns out to be an isomorphism when R is a field or a local ring.

Good evidence for the appropriateness of numbering the groups K_0 and K_1 as adjacent members of a series is provided by the *Localization Theorem* [15, p. 702] (see also [43, Theorem 10.5] and [11, p. 43]) for a Dedekind domain R, which provides an exact sequence

$$\oplus_{\mathfrak{p}} K_1(R/\mathfrak{p}) \to K_1(R) \to K_1(F) \to \oplus_{\mathfrak{p}} K_0(R/\mathfrak{p}) \to K_0(R) \to K_0(F) \to 0,$$

where $\mathfrak{p}$ runs over the maximal ideals of R, and F is the fraction field of R.

Further evidence is provided by various results about Laurent polynomial rings. For a regular noetherian ring R, there is an isomorphism

$$K_1(R[t,t^{-1}]) \cong K_0(R) \oplus K_1(R)$$

that mixes K_0 and K_1 (see [12, Theorem 2] and [43, Theorem 10.6]), contrasting with the more orderly behavior of the polynomial ring over R, for which there are isomorphisms $K_0(R) \cong K_0(R[t])$ (due to Grothendieck, see [69, Section 9]) and $K_1(R) \cong K_1(R[t])$ (see [12, Theorem 1]). For an arbitrary ring, there is Bass' *Fundamental Theorem* [15, Chapter XII, Theorem 7.4, p. 663], which provides the split exact sequences

$$0 \to K_1(R) \to K_1(R[t]) \to K_0(\mathrm{Nil}\, R) \to 0$$

and

$$0 \to K_1(R) \to K_1(R[t]) \oplus K_1(R[t^{-1}]) \to K_1(R[t,t^{-1}]) \to K_0(R) \to 0.$$

Here $\mathrm{Nil}\,R$ denotes the category of those $R[x]$-modules whose underlying R-module is finitely generated and projective, and upon which x acts nilpotently. There is also the related split exact sequence

$$0 \to K_1(R[t]) \to K_1(R[t,t^{-1}]) \to K_0(\mathrm{Nil}\,R) \to 0$$

of [25, Theorem 2] (which treats the twisted case).

See [17] for Bass' illuminating discussion of the history of K-theory up to this point.

Milnor introduced the abelian group $K_2(R)$ in [54], defining it as the kernel of a group homomorphism $St(R) \to GL_\infty(R)$, where $St(R)$, the *Steinberg group*, is defined by generators corresponding to the elementary row and column operations of linear algebra, together with a handful of obvious relations among them. Hence elements of $K_2(R)$ correspond to the non-obvious relations among the row operations.

Bass and Tate [54, Theorem 13.1] (see also [14]) extended the Localization Theorem by adding one more term on the left, namely $K_2(F)$, and Bass extended it by adding one more term, $K_2(R)$.

4. Quillen's first definition of the K-theory of a ring

Quillen's first definition of higher algebraic K-theory was announced in [57], which covered his work at the Institute for Advanced Study during the year 1969-70. Other contemporaneous and correct definitions of higher algebraic K-theory were presented by Swan [73], Gersten [28], Karoubi-Villamayor [47], and Volodin [76], but Quillen's was the most successful.

For a group G, there is a *classifying space* BG, whose only non-vanishing homotopy group is its fundamental group, with $\pi_1 BG \cong G$. Key to Quillen's proof [56] of the Adams Conjecture [5] (see also [24]) was his discovery that for each power q of a prime number p there is a homology equivalence $BGL_\infty(\mathbb{F}_q) \to F\psi^q$ (see [59, proof of Theorem 7], with, as Quillen says [59, p. 553], the key insight that $F\psi^q$ plays an important role here being provided by Dennis Sullivan. Here $\mathbb{F}_q$ is a finite field with q elements, and $F\psi^q$ is the homotopy fiber of $\psi^q - 1$ acting on the connective spectrum BU representing topological K-theory. The map is constructed using representation theory and *Brauer lifting* to pass from representations of $GL_n\mathbb{F}_q$ to representations of $GL_n\mathbb{C}$; on fundamental groups it amounts to taking the determinant of a matrix. Since $F\psi^q$ has a binary operation on it arising from direct sum of complex vector bundles, that makes it a monoid up to homotopy; such a space is called an *H-space*.

Quillen has attributed the following idea to Dennis Sullivan. One may add 2-cells to $BGL_\infty(\mathbb{F}_q)$, glued to it along their boundary circles, to kill the elements of the fundamental group corresponding to matrices of determinant 1; that also changes the second homology group, which can be restored to its former value by adding 3-cells. The resulting space has the same homology groups as $BGL_\infty(\mathbb{F}_q)$, and thus has the same homology as $F\psi^q$, by virtue of the homology equivalence of the previous paragraph. The space has the same fundamental group as $F\psi^q$, so one can check that it is homotopy equivalent to $F\psi^q$ (see [59, Theorem 7]). This way of adding cells to a space was probably motivated by well-known work of Kervaire on the construction of compact closed manifolds that are homology spheres [48]. He starts with a finitely presented group G with $H_1 G = H_2 G = 0$. Expressing G as a quotient F/R of a free group F, he finds a closed compact manifold N of dimension d (for any $d \geq 5$) with F as fundamental group. Using *spherical modifications*, an analogue of adding cells that ensures the result is still a manifold, Kervaire kills the subgroup R of $\pi_1 N = F$, obtaining a manifold N' whose fundamental group is G and with $H_1 N' = 0$. Then he kills the generators of the free abelian group $H_2 N'$, obtaining a manifold N'' whose fundamental group is G with the same homology as a sphere of dimension d. A rough parallel with Sullivan's idea is drawn by replacing F by $GL_\infty(\mathbb{F}_q)$, taking N to be $BGL_\infty(\mathbb{F}_q)$, taking R to be the subgroup $SL_\infty(\mathbb{F}_q)$, and observing that $H_1 R = 0$.

Replacing $\mathbb{F}_q$ by an arbitrary ring R, Quillen started with $BGL_\infty(R)$ and used Sullivan's idea to kill the classes in its fundamental group corresponding to *elementary* matrices, i.e., those matrices which differ from the identity matrix at a single off-diagonal spot. The result is a homology equivalence from $BGL_\infty(R)$ to a space he called $BGL_\infty(R)^+$; the process for constructing it is referred to as the *plus-construction*. The same cells work for every ring, rendering the operation functorial. (Curiously, it even suffices to use a single 2-cell and a single 3-cell.) Quillen used the direct sum of matrices, with rows and columns interleaved, to prove that $BGL_\infty(R)^+$ is an H-space; for that he had to show that the choice of interleaving function $\mathbb{N} \sqcup \mathbb{N} \cong \mathbb{N}$ does not matter. Evidently $\pi_1 BGL_\infty(R)^+ = H_1 BGL_\infty(R)^+ = H_1 BGL_\infty(R) = GL_\infty(R)^{ab} = K_1(R)$, and Quillen was able to produce an isomorphism $\pi_2 BGL_\infty(R)^+ \cong K_2(R)$ (see, for example, [65, Section 7, Corollary 5]). Quillen then defined the higher algebraic K-groups by setting $K_n(R) := \pi_n BGL_\infty(R)^+$ for $n > 0$. The groups are abelian: $K_1(R)$ by construction, and $K_n(R)$, for $n > 1$, because higher homotopy groups are abelian.

The homotopy equivalence between $F\psi^q$ and $BGL_\infty(\mathbb{F}_q)^+$, together with the definition of $F\psi^q$, yields a complete computation of the K-groups of $\mathbb{F}_q$, namely, that $K_{2i}\mathbb{F}_q = 0$ and $K_{2i-1}\mathbb{F}_q \cong \mathbb{Z}/(q^i - 1)\mathbb{Z}$, for $i > 0$ (see [59, Theorem 8]). There is an impressive corollary. Let $\bar{\mathbb{F}}$ denote the algebraic closure of $\mathbb{F}_p$; it is the filtered union of the finite fields $\mathbb{F}_q$. Then, for any prime number ℓ prime to p, the ℓ-adic completion of $BGL_\infty(\bar{\mathbb{F}})^+$ is homotopy equivalent to the ℓ-adic completion of BU. Thus the algebraic K-theory of $\bar{\mathbb{F}}$ is almost the same as topological K-theory, when viewed from the perspective of homotopy with finite or pro-finite coefficients. The generalization of that statement to any algebraically closed field of characteristic p was conjectured by Lichtenbaum and proved by Suslin in [72].

5. Two more definitions of the K-theory of a ring

Later, as reported in in [34], Quillen invented another way to construct a homology equivalence from $BGL_\infty(R)$ to an H-space $S^{-1}S$, which we may call the *localization construction*. It can replace the plus-construction as a definition of the groups $K_i(R)$. In [68], as a way of formulating some ideas of Quillen, Segal introduced Γ-objects, as a way to generalize the classifying space of a group; it is closely related to the localization construction and provides motivation for subsequent definitions of K-theory that incorporate exact sequences; it also provides an alternative definition of the groups $K_i(R)$. Both ways are better than the plus-construction, because the group $K_0(R)$ is not divorced from the higher homotopy groups, and natural deloopings (spectra) are available. (Because $K_0(R)$ participates in exact sequences involving K_1 where the boundary maps are nonzero, it would be unnatural to write the product $K_0(R) \times BGL_\infty(R)^+$ and expect it to serve a useful purpose, unless $K_0(R)$ happens to be generated by $[R^1]$.)

Suppose T is a monoid, and consider the construction of a universal homomorphism $T \to G$ to a group G. One way is to let G be the group with one generator $[t]$ for each element $t \in T$, modulo all relations of the form $[t' + t''] = [t'] + [t'']$. Another way, which works when T is commutative, is to let G be the set of equivalence classes of pairs $[t,t']$, where the equivalence relation is generated by the requirement that $[t + t'', t' + t''] = [t,t']$. Both ways can be realized topologically, as follows. Consider the space $X_1(T)$ with: one vertex v; an edge (t) starting and ending at v for each $t \in T$; a triangle bounded appropriately by (t'), (t''), and $(t' + t'')$ for each pair $t', t'' \in T$; and with higher dimensional simplices incorporated following the same pattern. It is a basic fact that $G \cong \pi_1 X_1(T)$. Consider also, for T commutative, the space $X_0(T)$ with: a vertex (t,t') for each pair; an edge connecting $(t + t'', t' + t'')$ to (t,t') for each triple; and with higher dimensional simplices incorporated following the same pattern. One sees that $G \cong \pi_0 X_0(T)$. In both cases, the advantage derived from incorporating higher dimensional simplices is that when T is a group G, the other homotopy groups of the two spaces vanish, and the space $X_1(G)$ is the classifying space BG.

Now consider replacing the monoid T by a category such as the category $\mathcal{P}(R)$ of finitely generated projective modules P over a ring R; a choice of direct sum operation $P \oplus P'$ makes $\mathcal{P}(R)$ into a commutative monoid, except that commutativity and associativity involve isomorphisms instead of equalities. Segal [68] takes the isomorphisms into account to produce a Γ*-category*, which we may call $\mathrm{Isom}\mathcal{P}(R)^{\oplus}$, consisting of the direct sum diagrams in $\mathcal{P}(R)$ and their isomorphisms; the main point is to avoid choosing a direct sum operation by incorporating all possible choices into the construction. From it is made a space $|\mathrm{Isom}\mathcal{P}(R)^{\oplus}|$, analogous to $X_1(T)$ above. Quillen's localization construction also takes the isomorphisms into account but makes a space $BS^{-1}S(\mathcal{P}(R))$ from $\mathcal{P}(R)$ analogous to $X_0(T)$ above. The space $BS^{-1}S(\mathcal{P}(R))$ turns out to be homotopy equivalent to the loop space of $|\mathrm{Isom}\mathcal{P}(R)^{\oplus}|$.

The desired homology equivalence $BGL_\infty(R) \to BS^{-1}S(\mathcal{P}(R))$ is constructed [34, p. 224] from the automorphisms of the free modules R^n. The idea of the proof is that, at least homologically, the localization construction can be considered as a direct limit construction, in the same way that a ring of fractions $S^{-1}R$, regarded as an R-module, can be viewed as a direct limit of copies of R. Thus one has the following pair of alternative definitions of *direct sum K-theory*: $K_i(R) := \pi_i BS^{-1}S(\mathcal{P}(R))$ and $K_i(R) := \pi_{i+1}|\mathrm{Isom}\mathcal{P}(R)^{\oplus}|$, for $i \geq 0$. An alternative approach to the same circle of ideas is presented in [65, Section 7].

6. K-theory of exact categories

Quillen's proof [61] of the extension, to higher K-groups, of the Localization Theorem for a Dedekind domain R, depends on the use of categories not of the form $\mathcal{P}(R)$, such as the category $\mathcal{M}(R)$ of finitely generated R-modules. In such a category, there are short exact sequences that do not split and hence do not arise from a direct sum diagram, so a new construction is required to produce the right *exact sequence K-theory* space from such a category.

The way to modify Segal's construction $|\mathrm{Isom}\mathcal{P}(R)^{\oplus}|$ to incorporate short exact sequences was perceived early, by Segal and by Waldhausen, and it is motivated directly by the definition of the Grothendieck group. For this purpose, suppose that $\mathcal{M}$ is an additive category equipped with a suitable notion of short exact sequence. The *S.-construction*, named thus for Segal by Waldhausen, yields a space $|S.\mathcal{M}|$, which is analogous to $X_1(T)$ above; it has one vertex v; it has an edge (M) starting and ending at v for each $M \in \mathcal{M}$; it has a triangle bounded appropriately by (M'), (M''), and (M) for each short exact sequence $0 \to M' \to M \to M'' \to 0$ of $\mathcal{M}$;

and it has higher dimensional simplices corresponding to filtrations. Its fundamental group is $K_0\mathcal{M}$, so $\pi_{i+1}|S.\mathcal{M}|$ would be an appropriate definition of $K_i(\mathcal{M})$.

Parenthetically we may mention that the appropriate way to incorporate short exact sequences into the definition of $BS^{-1}S(\mathcal{P}(R))$ to produce a space $|G\mathcal{M}|$ analogous to $X_0(T)$, with $K_i(\mathcal{M}) = \pi_i|G\mathcal{M}|$ for all i, was discovered in 1987 [32]. The first algebraic description of the K-groups of an exact category not involving homotopy groups was found in 2011 [41].

Quillen's letter to Segal [58] reveals an interesting bit of history. In it he says that he thought a lot[2] about Segal's S.-construction, but couldn't get very far, aside from using his *stable splitting theorem* [64, Section 5, Theorem 2'] to prove the equivalence of $|S.\mathcal{M}|$ with $|\mathrm{Isom}\mathcal{P}(R)^{\oplus}|$. Then, finally, in Spring, 1972, he freed himself "from the shackles of the simplicial way of thinking and found the category $Q\mathcal{M}$". Later work [37, 71] showed the simplicial way of thinking could accommodate proofs of many, but not all, of Quillen's basic theorems, and Waldhausen found that the S.-construction admits an additivity theorem and thereby produces a usable K-theory for certain non-additive categories arising from topology that he called *categories with cofibrations and weak equivalences* [78, Section 1.2].

We describe now the category $Q\mathcal{M}$, introduced in [61] and referred to above; it is known as the *Q-construction*. For this purpose, consider an *exact category* $\mathcal{M}$ as defined by Quillen [61]: it is an additive category together with a family of sequences $0 \to M' \to M \to M'' \to 0$ called *short exact sequences* that satisfies appropriate formal properties. An additive functor between exact categories is called *exact* if it sends exact sequences to exact sequences.

From each short exact sequence $0 \to M' \to M \to M'' \to 0$, by taking isomorphism classes, we get a *subobject* M' of M and a *quotient object* M'' of M, and we restrict our attention to these. A subobject of a quotient object of M is called a *subquotient object* of M; a quotient object of a subobject amounts to the same thing. By definition, an object of the category $Q\mathcal{M}$ is an object of $\mathcal{M}$, and an *arrow* $M_1 \to M$ of $Q\mathcal{M}$ is an isomorphism of M_1 with a subquotient object of M. One composes two arrows by viewing a subquotient object of a subquotient object of M as a subquotient object of M.

The next task is to make a space from the category $Q\mathcal{M}$ by interpreting the language of category theory in the language of topology. A space $B\mathcal{C}$, the *geometric realization* of $\mathcal{C}$, can be made from any category $\mathcal{C}$ as follows. The space has: a vertex (C) for each object $C \in \mathcal{C}$; an edge (f) connecting C' to C for each

[2] It probably took Quillen two years to come up with the Q-construction, from Spring, 1970, when he probably discussed the S.-construction with Segal at the Institute for Advanced Study, to Spring, 1972.

arrow $f : C' \to C$ in $\mathcal{C}$; a triangle (f,g) joining the three edges f, g, and gf for each composable pair of arrows $C'' \xrightarrow{f} C' \xrightarrow{g} C$ of $\mathcal{C}$; and higher dimensional simplices incorporated similarly. A functor $F : \mathcal{C} \to \mathcal{D}$ yields a continuous map $BF : B\mathcal{C} \to B\mathcal{D}$. A natural transformation $\eta : F \to F'$ between two functors yields a homotopy between BF and BF'. If $\mathcal{C}$ has an initial object C, then the space $B\mathcal{C}$ is contractible, with a contraction to C.

Quillen defined $K_i(\mathcal{M}) := \pi_{i+1} BQ\mathcal{M}$. Quillen has remarked that the shift of degrees by 1 here seems to be related to the advantage that the Q-construction has over the plus-construction for proving abstract foundational theorems. Alternatively, defining $K\mathcal{M} := \Omega BQ\mathcal{M}$ as the K-theory space of $\mathcal{M}$, where Ω denotes taking *the loop space* of a space, we may write $K_i(\mathcal{M}) := \pi_i K\mathcal{M}$.

Quillen then introduced two crucial combinatorial tools for proving homotopy-theoretic statements in K-theory: Theorem A [61, p. 93] provides a category-theoretic criterion for a functor to yield a homotopy equivalence on geometric realizations, together with its attendant isomorphisms on homotopy groups; Theorem B [61, p. 97] is a generalization of Theorem A that produces a homotopy fibration sequence and its attendant long exact sequence of homotopy groups. Quillen then proved the following amazing sequence of theorems, most of which were written down in January and February of 1973. Many of the proofs involve the clever introduction of ad hoc auxiliary categories and functors between them, to which Theorems A and B can be applied.

Theorem 1 [61, Theorem 1, p. 102] *$K_0(\mathcal{M})$ is isomorphic to the Grothendieck group of $\mathcal{M}$.*

Theorem 2 [58] *The space $BQ\mathcal{M}$ is homotopy equivalent to the space $|S.\mathcal{M}|$.*

The homotopy equivalence of the theorem sends the vertex v of $|S.\mathcal{M}|$ to the object 0 of $Q\mathcal{M}$, and, for $M \in \mathcal{M}$, sends the loop (M) in $|S.\mathcal{M}|$ to the loop that first traverses the arrow from 0 to M that expresses 0 as a subobject of M, and then traverses, in the reverse direction, the arrow from 0 to M that expresses 0 as a quotient object of M.

Theorem 3 (**Plus equals Q**) [34, p. 224] *For a ring R and for all $i \in \mathbb{Z}$, there is an isomorphism $K_i(R) \cong K_i(\mathcal{P}(R))$, implemented by a homotopy equivalence $\Omega BQ\mathcal{P}(R) \simeq BS^{-1}S(\mathcal{P}(R))$.*

Theorem 4 (**Additivity**) [61, Theorem 2, p. 105] *Letting $\mathcal{E}$ denote the exact category whose objects are the short exact sequences of $\mathcal{M}$, the map $BQ\mathcal{E} \to BQ\mathcal{M} \times BQ\mathcal{M}$ that forgets the middle object and the maps is a homotopy equivalence. Moreover, if $0 \to F' \to F \to F'' \to 0$ is an exact sequence of exact functors from an exact category $\mathcal{M}$ to an exact category $\mathcal{M}'$, then the maps*

$K_i\mathcal{M} \to K_i\mathcal{M}'$ induced by the functors satisfy the equation $F_* = F'_* + F''_*$.

Waldhausen provides a non-additive version of this basic theorem in [78, 1.3.2(4)], and a direct proof is given in [40].

Theorem 5 (Resolution) [61, Corollary 2, p. 110] *Let $\mathcal{P}_\infty(R)$ denote the exact category of R-modules that have a finite resolution by modules in $\mathcal{P}(R)$. Then the map $K(\mathcal{P}(R)) \to K(\mathcal{P}_\infty(R))$ is a homotopy equivalence. Thus if R is a regular noetherian ring, the map $K(\mathcal{P}(R)) \to K(\mathcal{M}(R))$ is a homotopy equivalence.*

In Quillen's paper, the theorem is stated more abstractly [61, Theorem 3, p. 108], and thus is widely applicable to situations where the objects in one exact category have finite resolutions by objects in a larger one.

Theorem 6 (Dévissage) [61, Theorem 4, p. 112] *Suppose that $\mathcal{M}$ is an abelian category and $\mathcal{N}$ is a full subcategory closed under taking subquotient objects and finite direct sums, such that any object of $\mathcal{M}$ has a finite filtration whose successive subquotients are in $\mathcal{N}$. Then the map $K(\mathcal{N}) \to K(\mathcal{M})$ is a homotopy equivalence. Thus if every object of $\mathcal{M}$ has finite length, there is an isomorphism $K_i\mathcal{M} \cong \bigoplus_D K_i(D)$ where D runs over the (division) endomorphism rings of representatives of the isomorphism classes of simple objects of $\mathcal{M}$.*

Theorem 7 (Localization for abelian categories) [61, Theorem 5, p. 113] *Suppose that $\mathcal{M}$ is an abelian category and $\mathcal{N}$ is a* Serre subcategory *of $\mathcal{M}$, i.e., it is a full subcategory closed under taking subquotient objects and extensions, so that the quotient abelian category $\mathcal{M}/\mathcal{N}$ exists. Then there is a long exact sequence $\cdots \to K_i(\mathcal{N}) \to K_i(\mathcal{M}) \to K_i(\mathcal{M}/\mathcal{N}) \to K_{i-1}(\mathcal{N}) \to \ldots$.*

Quillen once remarked that his first attempt to prove the theorem above involved focusing on the case where $\mathcal{M}$ is the category of finitely generated modules over a discrete valuation ring, and $\mathcal{N}$ consists of the torsion modules in $\mathcal{M}$. He made no progress until he switched to the general case, where he could observe that the opposite category $\mathcal{M}^{\mathrm{op}}$ is an abelian category and $\mathcal{N}^{\mathrm{op}}$ is a Serre subcategory of $\mathcal{M}^{\mathrm{op}}$. The symmetry allowed the construction of a short proof of the theorem.

Theorem 8 (Localization for Dedekind domains) [61, Corollary, p. 113] *Let R be a Dedekind domain with fraction field F. Then there is a long exact sequence $\cdots \to K_i(R) \to K_i(F) \to \bigoplus_{\mathfrak{p}} K_{i-1}(R/\mathfrak{p}) \to K_{i-1}(R) \to \ldots$, where the sum runs over the maximal ideals $\mathfrak{p}$ of R.*

The theorem is a corollary of Resolution, Dévissage, and Localization for abelian categories.

Theorem 9 (Localization for projective modules) [34, Theorem, p. 233] *Let R be a ring, let $S \subseteq R$ be a multiplicative set of central non-zero-divisors, let $S^{-1}R$*

be the ring of fractions, and let $\mathcal{H}$ be the category of finitely presented S-torsion R-modules of projective dimension 1*. Then there is a long exact sequence* $\cdots \to K_i(R) \to K_i(S^{-1}R) \to K_{i-1}(\mathcal{H}) \to K_i(R) \to \ldots.$

Theorem 10 (**Fundamental theorem for** K') [61, Theorem 8, p. 122] *If R is a noetherian ring, then, letting $K'_i(R)$ denote $K_i(\mathcal{M}(R))$, there are isomorphisms $K'_i(R[t]) \cong K'_i(R)$ and $K'_i(R[t,t^{-1}]) \cong K'_i(R) \oplus K'_{i-1}(R)$.*

Theorem 11 (**Fundamental theorem for regular rings**) [61, Corollary, p. 122] *If R is a regular noetherian ring, then there are isomorphisms $K_i(R[t]) \cong K_i(R)$ and $K_i(R[t,t^{-1}]) \cong K_i(R) \oplus K_{i-1}(R)$.*

Theorem 12 (**Fundamental theorem for rings**) [34, p. 237] *If R is a ring, then there is an exact sequence* $0 \to K_i(R) \to K_i(R[t]) \oplus K_i(R[t^{-1}]) \to K_i(R[t,t^{-1}]) \to K_{i-1}(R) \to 0.$

We consider only schemes that are noetherian and separated in the sequel. For such a scheme X, we let $\mathcal{P}(X)$ denote the exact category of locally free coherent sheaves on X, and we let $\mathcal{M}(X)$ denote the exact category of coherent sheaves on X. Then we define $K_i(X) := K_i(\mathcal{P}(X))$ and $K'_i(X) := K_i(\mathcal{M}(X))$.

Theorem 13 (**Resolution for schemes**) [61, p. 124] *If X is a regular scheme, then the map $K_i(X) \to K'_i(X)$ is an isomorphism.*

Theorem 14 (**Transfer map**) [61, p. 126] *If $f : X \to Y$ is a proper map of schemes, and either it is finite, or X has an ample line bundle, then there is a transfer map $f_* : K'_i(X) \to K'_i(Y)$.*

Theorem 15 (**Localization for schemes**) [61, Proposition 3.2, p. 127] *If Z is a closed subscheme of X and U is its complement, then there is a long exact sequence* $\cdots \to K'_i(Z) \to K'_i(X) \to K'_i(U) \to K'_{i-1}(Z) \to \ldots.$

Theorem 16 (**Homotopy property**) [61, Proposition 4.1, p. 28] *If the map $Y \to X$ is flat and its fibers are affine spaces, then the map $f^* : K'_i(X) \to K'_i(Y)$ is an isomorphism.*

This theorem generalizes the Fundamental Theorem for K', above.

Theorem 17 (**Projective bundle theorem**) [61, Proposition 4.3, p. 129] *Let $P \to X$ be the projective bundle associated to a vector bundle E of rank r on X. Assume X is quasi-compact. The powers of the tautological line bundle provide an isomorphism $K_i(P) \cong (K_i(X))^r$.*

The analogue of this theorem in topological K-theory follows from Bott periodicity and the decomposition of complex projective space of dimension $r-1$ into cells of dimension $0, 2, 4, \ldots, 2(r-1)$. The proof uses a novel fundamental functorial resolution of regular sheaves on projective space.

Theorem 18 (**K-theory of Severi-Brauer varieties**) [61, Theorem 4.1, p. 145] *Let X be a Severi-Brauer variety of dimension $r-1$ over a field F, and let A be the corresponding central simple algebra over F. Then there is a natural isomorphism $K_i(X) \cong \bigoplus_{n=0}^{r-1} K_i(A^{\otimes n})$.*

A *Severi-Brauer* variety is one that becomes isomorphic to projective space over an extension field of F, and a (finite-dimensional) *central simple* algebra over F is one that becomes isomorphic to a matrix algebra over an extension field of F, so this theorem is a generalization of the previous one. It was used in a crucial way in the paper [53] of Merkurjev-Suslin to prove a theorem relating $K_2(F)$ to the Brauer group of a field F, which can be interpreted as a purely algebraic result that shows how certain central simple algebras over F arise from *cyclic F-algebras*, which are given by explicit generators and relations. A generalization of the Merkurjev-Suslin theorem, the *Bloch-Kato conjecture*, served later as the central focus for research in motivic cohomology, as we mention in section 7.

Theorem 19 (**Filtration by support**) [61, Theorem 5.4, p. 131] *Suppose X has finite Krull dimension, and for $p \in \mathbb{N}$ let X_p denote the set of points of X of codimension p. Then there is a spectral sequence $E^1_{pq} = \bigoplus_{x \in X_p} K_{-p-q}(k(x)) \Rightarrow K'_{-p-q}(X)$.*

This theorem generalizes the localization theorem for Dedekind domains, above.

Theorem 20 (**Bloch's Formula**) [61, Theorem 5.19, p. 137] *If X is a regular scheme of finite type over a field, and $\mathcal{K}_i$ denotes the sheaf on X associated to the presheaf that sends an open set U to $K_i(U)$, then the cohomology group $H^i(X,\mathcal{K}_i)$ is isomorphic to the Chow group of algebraic cycles on X of codimension i modulo rational equivalence.*

An algebraic *cycle* on a scheme X is a formal finite linear combination, with integer coefficients, of irreducible subvarieties. The divisor of a rational function on a subvariety is declared to be *rationally equivalent* to 0.

The theorem above was novel to algebraic geometers, for formerly, the Chow group could be recovered only rationally from algebraic K-theory, in terms of the Grothendieck group $K_0(X)$, using the Grothendieck-Riemann-Roch theorem.

The proof involved proving *Gersten's Conjecture* for geometric rings, which states that the map $K_i(\mathcal{M}^p(R)) \to K_i(\mathcal{M}^{p-1}(R))$ is zero if R is a regular noetherian local ring. Here $\mathcal{M}^p(R)$ denotes the exact category of finitely generated R-modules whose support has codimension p or greater.

A slightly streamlined proof of the main lemma used in the proof was presented in [35]. A useful application of Quillen's techniques to motives of varieties is presented in [29]. One can turn Bloch's formula around and view it as a way to define intersection products for algebraic cycles in contexts where the moving lemma is not available, as in [30]. See also the survey [31].

Theorem 21 (**Lambda operations**) [45, 3.5 and 4.7] *If R is a commutative ring, then the exterior power operations on free R-modules induce operations $\lambda^k : K_n(R) \to K_n(R)$, which are homomorphisms for $n > 0$ and satisfy the identities of a (special) λ-ring. The corresponding Adams operations $\psi^k : K_n(R) \to K_n(R)$ become diagonalizable after tensoring with $\mathbb{Q}$, the eigenvalues have the form k^i with $i \geq 0$, and the eigenspaces depend only on i and not on $k > 1$. If p is a prime number and R is a ring of characteristic p, then the map on K-theory induced by the Frobenius endomorphism $x \mapsto x^p$ of R is equal to ψ^p.*

Taking into account the Atiyah-Hirzebruch spectral sequence for the topological analogue X of the scheme $\operatorname{Spec} R$, one sees that the Adams eigenspace in $K_n(R) \otimes \mathbb{Q}$ with exponent i serves as an algebraically-defined analogue of the topologically-defined rational cohomology group $H^{n-2i}(X;\mathbb{Q})$, with an added and novel dependence on the *weight* i, foreshadowing the later development of motivic cohomology.

Theorem 22 (**Finite generation**) [60, 36] *Let R be a finitely generated commutative ring of Krull dimension at most 1. Then the abelian group $K_i'(R)$ is finitely generated, for all i.*

Thus if R is the coordinate ring of a nonsingular algebraic curve over a finite field, or is the ring of integers in a number field, then it is regular noetherian, and $K_i(R)$ is finitely generated. The proof of this theorem depends on reduction theory for arithmetic groups and its analogue in finite characteristic. The case $i = 0$ amounts to finite generation of the ideal class group of R, and the case $i = 1$ amounts to finite generation of group $R^\times$ of units in R (the Dirichlet Unit Theorem) and the theorem of Bass-Milnor-Serre [13] that provides an isomorphism $K_1 R \cong R^\times$. Bass has asked [16, 9.1] whether the theorem remains true with no hypothesis on dimension; one may also ask the same question for motivic cohomology. Little progress has been made, aside perhaps from [19], which proves finite generation for K_0 of a regular arithmetic surface.

7. Later developments

Quillen's foundational work, summarized above, laid the foundation for a host of subsequent inspirational developments, many of which are exposed in detail in the *Handbook of K-theory* [27]. One could also peruse the many preprints at [1] or [2].

For example, there is the line of development that starts with Borel's computation [20] of the ranks of the K-groups of a ring of integers in an algebraic number field F. Borel's computation inspired Lichtenbaum's conjectures [49] that relate the ranks of the K-groups and the orders of the torsion subgroups of the K-groups to the behavior of the L-function of F at non-positive integers, generalizing the class number formula of analytic number theory, and Quillen showed [63, Section 9] how this ought to relate to étale cohomology. Soulé made concrete progress in [70], showing, for example, that $K_{22}\mathbb{Z}$ contains an element of order 691 and $K_{46}\mathbb{Z}$ contains an element of order 2294797. Then, motivated by Quillen's work on the eigenspaces of the Adams operations in Theorem 21 and by Bloch's work on regulators for K_2 of elliptic curves, Beilinson [18, 66] generalized the conjectures to arithmetic schemes of higher dimension, and the field blossomed during the 1980's and 1990's at the hands of Soulé, Bloch, Friedlander, Levine, Rost, Suslin, and Voevodsky (who won a Fields medal for his work), leading to the arrival of *motivic cohomology* and *motivic homotopy theory* as an ongoing enterprise [75, 22, 51], whose high point is the recent proof of the *Bloch-Kato conjecture*, spanning several papers, with [42] putting the final touches on the proof. Consequences for the K-groups of rings of integers in algebraic number fields are exposed in [80]. An important open problem is to prove the vanishing of motivic cohomology in negative degrees, as conjectured by Beilinson and Soulé.

There is also the line of development that starts with Waldhausen's variation on the $S.$-construction that expands its applicability so it accommodates not just exact categories, but also certain non-additive categories arising from topology, namely, the categories with cofibrations and weak equivalences mentioned above. His additivity theorem [78, 1.3.2(4)], fibration theorem [78, (1.6.4)] and approximation theorem [78, (1.6.7)] laid the basic foundation for his *algebraic K-theory of topological spaces*, which is broad enough to encompass the stable homotopy groups of the spheres as an example of algebraic K-groups, as well as to forge an important connection between geometric topology and the algebraic K-groups $K_i(\mathbb{Z})$, whose ranks were computed by Borel. As a result, Farrell and Hsiang [26] deduced the following result. Let $\mathrm{Diff}(S^n)$ denote the space of diffeomorphisms of the n-sphere, and assume $0 \le i < n/6 - 7$. Then the homotopy group $\pi_i \mathrm{Diff}(S^n)$ has rank 0 if i does not have the form $4k-1$, it has rank 1 if n is even and i has the form $4k-1$, and it has rank 2 if n is odd and i has the form $4k-1$. Surveys of that material include [50] and the more recent [82]. The statement and the careful and detailed proof of the stable parametrized h-cobordism theorem in [79] depend on the algebraic K-theory of topological spaces to relate the geometric topology of manifolds to algebraic K-theory in a succinct and powerful form. Waldhausen's basic theorems also laid the foundation for Thomason's extension of Quillen's K-theory that incorporates chain complexes naturally, leading to his localization theorem [74] for schemes.

A related line of development [33, 52, 23] considers a homomorphism $f : R \to S$ between *simplicial rings* or between *rings up to homotopy* and relates the relative K-theory $K(f)$ of the map, which fits into a fibration sequence $K(f) \to K(R) \to K(S)$, to the relative *cyclic homology* or to the relative *topological cyclic homology* of the map. Computability of the latter leads to consequences for the former, as in [44] and [10]. Here $K(R)$ is the K-theory space constructed by Waldhausen in [78, Section 2.3], which is not the same as the more-evident K-theory space constructed degree-wise, useful in other contexts.

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