EXTERIOR POWER OPERATIONS ON HIGHER K-THEORY

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ABSTRACT. We construct operations on higher algebraic K-groups induced by operations such as exterior power on any suitable exact category, without appeal to the plus-construction of Quillen.

1. Results.

Let $\mathcal{M}$ be an exact category with a suitable notion of exterior power operations

$$M \mapsto \bigwedge^k M.$$

For example, we may take $\mathcal{M}$ to be the category $\mathcal{P}(X)$ of vector bundles on some scheme X. Or, we may fix a group G and a commutative ring R and take $\mathcal{M}$ to be the category $\mathcal{P}(R,G)$ of representations of G on projective finitely generated R-modules. One of the additional requirements is that we have a tensor product functor which is compatible with the exterior power functors in a certain sense; the tensor product functor is required to be bi-exact, and this prevents us from taking for $\mathcal{M}$ a category such as the category $\mathcal{M}(R)$ of finitely generated R-modules.

I provide a construction of maps

$$\lambda^k : K_*\mathcal{M} \to K_*\mathcal{M}$$

induced by the exterior power operations which is based on the construction of K-theory presented in [1].

One may also consider a sequence of categories $\mathcal{M}_n$ for $n \geq 0$ with appropriate operations $\bigwedge^k : \mathcal{M}_n \to \mathcal{M}_{nk}$. For example, we may fix a commutative ring R, and consider the sequence $\mathcal{P}(R, S_n)$ of exact categories, $n \geq 0$, where S_n is the symmetric group. In this case we provide an analogous construction of maps

$$\lambda^k : K_*\mathcal{M}_n \to K_*\mathcal{M}_{nk}.$$

The first K-groups and exterior power operations on them were invented by Grothendieck [3] for use in proving Riemann-Roch theorems. Construction of analogous operations on the higher K-groups of Quillen has been accomplished previously in certain cases: for the groups $K_*(R)$, with R a ring, by Quillen [4] and Kratzer [5]; for the groups $K_*(X)$ with X a regular noetherian scheme, by Soulé, using Brown-Gersten generalized sheaf cohomology; and for $K_0^Y(X)$ when Y is a closed subscheme of a noetherian scheme X, by Gillet-Soulé [2], in order to prove Serre's conjecture on vanishing of intersection multiplicities. I haven't yet managed to prove the special λ-ring properties for these operations, but I do give a proof that they agree with the ones defined by Quillen and Kratzer for the K-groups of a ring R.

It was Henri Gillet who explained to me how the construction of [1] ought to lead to a construction of these operations λ^k, and it was with this goal in mind that we wrote the paper [1].

The techniques of [1] have been recently generalized by Thomas Gunnarson, Roland Schwänzl, Rainer Vogt and Friedhelm Waldhausen in order to apply the techniques of this paper to the algebraic K-theory of rings up to homotopy and the algebraic K-theory of spaces.

I am grateful to Friedhelm Waldhausen and Ulf Rehmann for their hospitality at the University of Bielefeld, where the main part of this work was done.

1991 *Mathematics Subject Classification.* Primary 18F25; Secondary 19C30, 19D99.

Supported by NSF grants DMS 85-04692, 86-01980.

2. Introduction.

We give now a brief introduction to the problem of defining exterior power operations on K-groups. Consider the Grothendieck group $K_0(\mathcal{M})$ first. Given an exact sequence

$$0 \to U \to W \to X \to 0$$

one has the usual formula

$$[\textstyle\bigwedge^k W] = \sum_{i=0}^{k} [\bigwedge^i U] [\bigwedge^{k-i} X]. \tag{2.1}$$

One introduces the variable t and defines a power series

$$\lambda_t(W) = \sum_{i=0}^{\infty} [\textstyle\bigwedge^i W] t^i \in K_0\mathcal{M}\,[[t]]\,.$$

The formula (2.1) can be collected for $k = 1, \dots, \infty$ into the single formula $\lambda_t(W) = \lambda_t(U)\lambda_t(X)$, which in turn allows one to construct a group homomorphism

$$K_0\mathcal{M} \to (1 + t \cdot K_0\mathcal{M}\,[[t]])^{\times}\,.$$

Taking the coefficient of t^k defines a function $\lambda^k : K_0\mathcal{M} \to K_0\mathcal{M}$. For a typical element $[V] - [W]$ of $K_0\mathcal{M}$ the element $\lambda^k([V] - [W])$ is the coefficient of t^k in the power series quotient $\lambda_t(V)/\lambda_t(W)$; the first few of these are:

$$\begin{aligned}
\lambda^1(V - W) &= V - W \\
\lambda^2(V - W) &= \lambda^2 V - V \cdot W - \lambda^2 W + W \cdot W \\
\lambda^3(V - W) &= \lambda^3 V - \lambda^2 V \cdot W - V \cdot \lambda^2 W \\
&\quad - \lambda^3 W + V \cdot W \cdot W + \lambda^2 W \cdot W \\
&\quad + W \cdot \lambda^2 W - W \cdot W \cdot W
\end{aligned} \tag{2.2}$$

An important observation of Gillet is that if one writes down a similar formula for $\lambda^k(V - W)$, then the number of terms is 2^k, and there is a natural way to position the terms at the vertices of a k-dimensional cube, in such a way that the signs associated to the terms at either end of any edge are opposite. To see this, one can compute that

$$[\lambda^k(V - W)] = \sum_{a+b_1+\cdots+b_u=k} (-1)^u [\textstyle\bigwedge^a V] [\bigwedge^{b_1} W] \cdots [\bigwedge^{b_u} W]. \tag{2.3}$$

Here the sum runs over all ordered partitions $a + b_1 + \cdots + b_u = k$ with $a \geq 0$ and $b_1, \dots, b_u \geq 1$. To generate these partitions, consider all words of length k in the letters L and R. (It is even more revealing to use the symbols $\wedge$ and $\otimes$ in place of the letters L and R, respectively.) For each such word, we insert after each of its letters a V or a W, the rule being that to the left of the first R we insert only V's and to the right of there, we insert only W's. Next, delete all occurences of the letter L from the word. We let a be the number of initial V's and let $b_1, \dots, b_u$ be the lengths of the subsequent runs of W's. Such runs are initiated by the letter R, so the number u of them is the number of R's in the original word, and its parity gives the sign in the formula.

A second observation is that for $k > 1$ the operation λ^k is definitely not a group homomorphism, for else the formulas (2.1) and (2.2) would have just two terms.

Now recall that the K-groups are defined in terms of Quillen's Q-construction by $K_i\mathcal{M} = \pi_{i+1}|Q\mathcal{M}|$, and consider the question of whether one could define maps λ^k on the higher K-groups by first constructing a map $\lambda^k : |Q\mathcal{M}| \to |Q\mathcal{M}|$ of topological spaces. Any such map of spaces would certainly induce a homomorphism on $\pi_1|Q\mathcal{M}| = K_0\mathcal{M}$, and we know that λ^k is not a homomorphism on K_0. Thus, this approach is doomed to failure.

That is why all previous constructions of lambda operations are based on the plus-construction of Quillen, where one has $K_iR = \pi_i BG\ell(R)^+$ for $i \geq 1$. Even if we modify the construction so that K_0R appears as π_0 of the space, we don't get into trouble; there is no reason that a map of spaces is required to induce a homomorphism on π_0, even if π_0 happens to be a group and the space happens to be an H-space. Nevertheless, approaching λ-operations via the plus-construction is somewhat unsatisfactory because it divorces the Grothendieck group from the other K-groups, and works only for K-groups of rings, or for K-groups expressible in terms of K-groups of rings. In particular it does not work for the K-groups of an arbitrary scheme.

In [1] Gillet and I provide an alternate definition for the K-groups of any exact category, $K_i(\mathcal{M}) = \pi_i G\mathcal{M}$, which has the advantage that the Grothendieck group appears as π_0 and yet is not divorced from the higher K-groups. It is this definition which allows us to construct the lambda operations on the K-groups.

3. Review of the G-construction.

A simplicial set $G\mathcal{M}$ is defined in [1] for any exact category $\mathcal{M}$, in such a way that $K_i(\mathcal{M}) = \pi_i G\mathcal{M}$. It may be described loosely as follows. The vertices of $G\mathcal{M}$ are the pairs (M, N) of objects of $\mathcal{M}$, and the edges

$$(M, N) \xrightarrow{(\alpha,\beta,\theta)} (M', N')$$

are the triples (α, β, θ) where

$$\alpha : M' \rightarrowtail M$$
$$\beta : N' \rightarrowtail N$$

are admissible monomorphisms, and

$$\theta : \operatorname{ckr} \alpha \xrightarrow{\cong} \operatorname{ckr} \beta$$

is an isomorphism of the cokernels. The higher dimensional simplices are defined analogously, a k-simplex being two admissible filtrations

$$M_0 \subseteq M_1 \subseteq \cdots \subseteq M_k$$
$$N_0 \subseteq N_1 \subseteq \cdots \subseteq N_k$$

together with compatible isomorphisms $\theta_{j,i} : M_j/M_i \xrightarrow{\cong} N_j/N_i$. It is an easy exercise to show, from this description, that $\pi_0 G\mathcal{M} = K_0\mathcal{M}$.

In actuality, the precise definition of $G\mathcal{M}$ is formulated in a way that avoids introducing quotient constructions, as in Waldhausen's work. We describe this now.

Let Δ denote the category of finite nonempty totally ordered sets. We let $[n]$ denote the ordered set $\{0 < 1 < \cdots < n\}$ for $n \geq 0$.

We introduce two new symbols L and R which will serve as elements of partially ordered sets about to be constructed. They denote the words "Left" and "Right". Given $A \in \Delta$ we construct a partially ordered set $\gamma(A)$ whose underlying set is the disjoint union $\{L, R\} \cup A$. We order it in such a way that A is a partially ordered subset of $\gamma(A)$, but we also decree that $L < a$ and $R < a$ for each $a \in A$. The elements L and R are not comparable.

For $j \leq i$ in a partially ordered set we let i/j denote the arrow from j to i, regarding $\gamma(A)$ as a category in the usual way. We let $\Gamma(A)$ denote the category of those arrows i/j in $\gamma(A)$ which have $i \in A$. (Recall that an arrow in an arrow category is simply a commutative square.) Notice that if $A \to B$ is an arrow of Δ there is a natural functor $\Gamma(A) \to \Gamma(B)$. We call a functor $M : \Gamma(A) \to \mathcal{M}$ *exact* if the sequence

$$0 \to M(j/k) \to M(i/k) \to M(i/j) \to 0$$

is exact for each $k \leq j \leq i$ in $\gamma(A)$ with $i, j \in A$, and if also $M(i/i) = 0$ for each $i \in A$. (Here 0 denotes a previously chosen zero object of $\mathcal{M}$.)

In [1] we used the category of all arrows in $\gamma(A)$ for $\Gamma(A)$. Here we are omitting the two identity arrows R/R and L/L from consideration, but in light of the axiom $M(i/i) = 0$ (which was also in effect in [1]), it amounts to the same thing, as far as the exact functors from $\Gamma(A)$ to $\mathcal{M}$ are concerned.

We define the value of the functor

$$G\mathcal{M} : \Delta^{\mathrm{op}} \to \{\text{sets}\}$$

on the object A to be the set $G\mathcal{M}(A) = \mathrm{Exact}(\Gamma(A), \mathcal{M})$ of *exact* functors $\Gamma(A) \to \mathcal{M}$.

We have defined $G\mathcal{M}$ on the objects of Δ; on arrows one uses the evident composition.

If one applies this more precise definition to the sets $A = [0]$ and $A = [1]$ one recovers something equivalent to the loose definition given above, except that the choice of object representing the cokernels has been added to the data.

The G-construction can be iterated k times, to provide a k-fold multisimplicial set we denote $G^k\mathcal{M} = G \cdots G\mathcal{M}$. It is the functor

$$G^k\mathcal{M} : (\Delta \times \cdots \times \Delta)^{\mathrm{op}} \to \{\text{sets}\}$$

defined on objects to be set

$$(G^k\mathcal{M})(A_1, \ldots, A_k) = \mathrm{Exact}(\Gamma(A_1) \times \cdots \times \Gamma(A_k), \mathcal{M})$$

of *multiexact* functors (exact in each variable separately). Notice that the vertices of $G^k\mathcal{M}$, i.e. the elements of the set

$$(G^k\mathcal{M})_{0,\ldots,0} = (G^k\mathcal{M})([0], \ldots, [0]),$$

are the same thing as 2^k-tuples of objects of $\mathcal{M}$. There is a homotopy equivalence $|G\mathcal{M}| \cong |G^k\mathcal{M}|$ in [1] which has the effect, on $K_0\mathcal{M}$, of taking the alternating sum of the objects of the 2^k-tuple.

All this tells us that we ought to expect λ^k to appear as a map from $G\mathcal{M}$ to $G^k\mathcal{M}$; this can't be exactly right, though, because $G\mathcal{M}$ is a simplicial set and $G^k\mathcal{M}$ is a k-fold multisimplicial set. The next section presents the solution to this problem.

4. Edgewise subdivision.

It turns out that to define λ^k we need to introduce a simplicial subdivision especially adapted to the situation. To see the necessity for this, consider a one-step filtration

$$U \subseteq W$$

in $\mathcal{M}$. The most natural filtration on $\bigwedge^k W$ to consider is the k-step filtration which proves the formula (2.1), namely

$$\begin{aligned} \bigwedge^k U = & U \wedge U \wedge \cdots \wedge U \wedge U \subseteq \\ & U \wedge U \wedge \cdots \wedge U \wedge W \subseteq \\ & U \wedge U \wedge \cdots \wedge W \wedge W \subseteq \\ & \cdots \subseteq \\ & U \wedge W \wedge \cdots \wedge W \wedge W \subseteq \\ & W \wedge W \wedge \cdots \wedge W \wedge W = \bigwedge^k W \end{aligned} \tag{4.1}$$

An edge in $G\mathcal{M}$ is a bit more than a one-step filtration; it is actually a pair of one-step filtrations with an isomorphism of the cokernels. Nevertheless, since $\bigwedge^k$ converts one-step filtrations into k-step filtrations, one may expect the map λ^k to map a single edge onto a chain of k edges. To turn such a map into a simplicial map, it is necessary to subdivide each edge of the *domain* of the map into a chain of k edges. It turns out that once this is done, our map can be described in a purely combinatorial fashion.

This "k-fold edge-wise subdivision", which we are about to describe, has also been discovered independently by Marcel Bökstedt and Thomas Goodwillie. I discovered it for myself in 1985, but the definition was completely forced by the desire to make exterior powers work.

Consider the concatenation functor

$$\begin{aligned} \Delta^k &\to \Delta \\ (A_1, A_2, \ldots, A_k) &\mapsto A_1 A_2 \cdots A_k \end{aligned}$$

The concatenation $A_1 A_2 \cdots A_k$ is the disjoint union of the ordered sets $A_1, \ldots, A_k$, ordered in such a way that each A_i is an ordered subset, together with the additional declaration that $a < b$ for $a \in A_i$ and $b \in A_j$ with $i < j$.

If X is a simplicial set, then it is a functor $X : \Delta^{\mathrm{op}} \to \{\text{sets}\}$, and composing it with concatenation yields a functor

$$\operatorname{Sub}_k X : (\Delta^k)^{\mathrm{op}} \to \{\text{sets}\},$$

i.e. a k-fold multisimplicial set, which we call the k-fold edgewise subdivision.

Given an object A of Δ we let $\Delta(A)$ denote the standard simplex on A, namely the set of functions (not necessarily order preserving) from A to the unit interval $[0,1]$ whose values sum to 1.

If Y is a k-fold multisimplicial set, we let $|Y|$ denote the k-fold geometric realization. This is different (but homeomorphic to) the geometric realization of the diagonal, as its cells are k-fold cartesian products of standard simplices, and is the evident quotient of

$$\coprod Y(A_1, \ldots, A_k) \times \Delta(A_1) \times \cdots \times \Delta(A_k).$$

There is a map

$$\psi : \Delta(A_1) \times \cdots \times \Delta(A_k) \to \Delta(A_1 A_2 \cdots A_k)$$

defined for any $a \in A_1 \cdots A_k$ by $\Psi(f_1, \ldots, f_k)(a) = f_i(a)/k$ where i is chosen so that $a \in A_i$. Letting X be a simplicial set, we may assemble the maps

$$1 \times \psi : X(A_1 \cdots A_k) \times \Delta(A_1) \times \cdots \times \Delta(A_k) \to X(A_1 \cdots A_k) \times \Delta(A_1 A_2 \cdots A_k)$$

to get a continuous map

$$\Psi : |\operatorname{Sub}_k X| \to |X|.$$

The map Ψ is a homeomorphism; we check this assertion by defining an inverse map Φ to Ψ. Given a point P of $|X|$ we represent it by some $(x, f) \in X(A) \times \Delta(A)$ for some $A \in \Delta$ and consider for each $a \in A$ the interval

$$I_a = \left[\sum_{b<a} f(b), \sum_{b \le a} f(b) \right] \subseteq [0,1].$$

Pick subsets $A_i \subseteq A$ for each i with $1 \le i \le k$ so that $A_1 \le A_2 \le \cdots \le A_k$ (in the sense that $a \le b$ whenever $a \in A_i$ and $b \in A_{i+1}$), and so that $\left[\frac{i-1}{k}, \frac{i}{k}\right] \subseteq \bigcup_{a \in A_i} I_a$. Let $\phi : A_1 \ldots A_k \to A$ be the surjective map arising from the various inclusions. Define functions $f_i : A_i \to [0,1]$ according to the formula

$$f_i(a) = k \cdot \operatorname{length}\left(I_a \cap \left[\frac{i-1}{k}, \frac{i}{k} \right] \right).$$

We remark that $\sum_{a \in A_i} f_i(a) = k \cdot \operatorname{length}\left[\frac{i-1}{k}, \frac{i}{k}\right] = 1$, so $f_i \in \Delta(A_i)$. Finally we let $\Phi(P) = (\phi^* x, f_1, \ldots, f_k)$. One may check that $\Phi : |X| \to |\operatorname{Sub}_k X|$ is a well-defined map, continuous, and an inverse to Ψ.

Thus Ψ and Φ are homeomorphisms, and this justifies calling $\operatorname{Sub}_k X$ a subdivision of X.

One may easily see that this subdivision does to edges what we advertised earlier. Let's use the notation $|g|$ to denote the cell in $|X|$ arising from a simplex g of a simplicial set X. If g is an 1-simplex (in $X([1])$) then the image under Ψ of the edge $|g|$ is the union of the edges $|g_i|$, where $g_i \in (Sub_k X)([0], \ldots, [0], [1], [0], \ldots, [0])$ is the evident degeneracy of g. (Here the [1] occurs in the i-th spot.) Moreover, the starting point of $|g_i|$ coincides with the ending point of $|g_{i+1}|$.

(4.2)
$$\bullet \xrightarrow{|g_k|} \bullet \xrightarrow{|g_{k-1}|} \bullet \cdots \bullet \xrightarrow{|g_2|} \bullet \xrightarrow{|g_1|} \bullet$$

The illustration here should be compared with the filtration (4.1).

5. The functor Ξ.

For $A \in \Delta$ we introduce a category (actually a partially ordered set) $\Gamma^k(A)$. We will have $\Gamma^1(A) = \Gamma(A)$. We take for objects of $\Gamma^k(A)$ those collections

$$\alpha = (i_1/\ell_1, *_2, i_2/\ell_2, *_3, \dots, *_k, i_k/\ell_k)$$

where for each r we have

(A1) $i_r \in \gamma(A)$, $\ell_r \in \gamma(A)$, and $*_r \in \{\wedge, \otimes\}$,
(A2) $\ell_r \le i_r$ and $i_r \in A$,
(A3) if $*_r = \wedge$ and $r > 1$ then $\ell_{r-1} = \ell_r$ and $i_{r-1} \le i_r$.

In particular, we see that each i_r/ℓ_r is an object of $\Gamma(A)$.

Given a pair α, α' of objects of $\Gamma^k(A)$, there will be an arrow from $\alpha \to \alpha'$ (and then there will be only one) if for each r we have

(B1) $i_r \le i'_r$,
(B2) $\ell_r \le \ell'_r$, and
(B3) if $*_r = \wedge$ and $*'_r = \otimes$ then $i_{r-1} \le \ell'_r$.

Finally, we will call a sequence

$$\alpha' \to \alpha \to \alpha''$$

exact if there exist integers $r \le s$ such that

(C1) for any p with $p < r$ or $s < p$ we have $i'_p = i_p = i''_p$, $\ell'_p = \ell_p = \ell''_p$, and $*'_p = *_p = *''_p$;
(C2) for any p satisfying $r < p \le s$ we have $*'_p = *_p = *''_p = \wedge$, and $i'_p = i_p = i''_p$;
(C3) $\ell_r = \ell'_r \le i'_r = \ell''_r \le i''_r = i_r$, $*'_r = *_r$, and $*''_r = \otimes$.

We introduce an elementary functor

$$\Xi : \Gamma(A_1) \times \cdots \times \Gamma(A_k) \to \Gamma^k(A_1 \cdots A_k)$$

defined by the formula

$$\Xi(i_1/j_1, \dots, i_k/j_k) = (i_1/\ell_1, *_2, \dots, *_k, i_r/\ell_r)$$

where we define $\ell_1 = j_1$, and then inductively for $r > 1$ we declare:

(D1) if $j_r = L$ then $*_r = \wedge$ and $\ell_r = \ell_{r-1}$
(D2) if $j_r \ne L$ then $*_r = \otimes$ and $\ell_r = j_r$

One can check that Ξ is a multi-exact functor (exact in each variable separately), and that Ξ is natural in each of the variables A_i.

6. Proofs for the previous section.

We check first that $\Gamma^k(A)$ is a category. Identity arrows exist. Given arrows $\alpha \to \alpha' \to \alpha''$ we see that there is an arrow $\alpha \to \alpha''$ as follows.

(B1) $i_r \le i'_r \le i''_r$.
(B2) $\ell_r \le \ell'_r \le \ell''_r$.
(B3) Assuming $*_r = \wedge$ and $*''_r = \otimes$ we see that $i_{r-1} \le i'_{r-1} \le \ell''_r$ if $*'_r = \wedge$, and that $i_{r-1} \le \ell'_r \le \ell''_r$ if $*'_r = \otimes$.

We check that $\Xi(i_1/j_1, \dots, i_k/j_k)$ is an object of $\Gamma^k(A_1 \cdots A_k)$. Notice that $\ell_r \in \{L, R\} \cup A_1 \cup \cdots \cup A_r$.

(A1) Clear.
(A3) $i_{r-1} \in A_{r-1}$ and $i_r \in A_r$, so $i_{r-1} < i_r$ in $A_1 \cdots A_k$.
(A2) If $r = 1$ or $j_r \ne L$ then $\ell_r = j_r \le i_r$. Otherwise $\ell_r = \ell_{r-1} \le i_{r-1} \le i_r$.

We check that Ξ is a functor. Suppose we are given an arrow

$$(i_1/j_1, \dots, i_k/j_k) \to (i'_1/j'_1, \dots, i'_k/j'_k).$$

We have the inequalities $i_r \le i'_r$, $j_r \le j'_r$, $j_r \le i_r$, and $j'_r \le i'_r$ to work with, in checking that there is an arrow

$$(i_1/\ell_1, *_2, \dots, *_k, i_r/\ell_r) \to (i'_1/\ell'_1, *'_2, \dots, *'_k, i'_r/\ell'_r)$$

(B1) Clear.

(B2) For $r = 1$ it is clear. For $r > 1$ we prove it by induction, and there are three cases. In the case where $L < j_r$ we have $\ell_r = j_r \leq j'_r = \ell'_r$. In the case where $L = j_r = j'_r$ we have $\ell_r = \ell_{r-1} \leq \ell'_{r-1} = \ell'_r$. In the case where $L = j_r < j'_r$ we have $\ell'_r = j'_r \in A_r$ but $\ell_r = \ell_{r-1} \in \{L, R\} \cup A_1 \cup \cdots \cup A_{r-1}$, so $\ell_r < \ell'_r$.

(B3) Assume $*_r = \wedge$ and $*'_r = \otimes$. Then $L = j_r < j'_r$, so $j'_r \in A_r$, and we see that $\ell'_r = j'_r \geq i_{r-1}$, because $i_{r-1} \in \{L, R\} \cup A_1 \cup \cdots \cup A_{r-1}$.

We check that Ξ is exact in the r-th variable by applying it to a sequence

$$(i'_r/j'_r) \to (i_r/j_r) \to (i''_r/j''_r)$$

where $j_r = j'_r \leq i'_r = j''_r \leq i''_r = i_r$. For $p \neq r$ we write $i'_p = i_p = i''_p$ and $j'_p = j_p = j''_p$. To prove that the resulting sequence

$$(i'_1/\ell'_1, *'_2, \ldots, *'_k, i'_r/\ell'_r) \to (i_1/\ell_1, *_2, \ldots, *_k, i_r/\ell_r) \to (i''_1/\ell''_1, *''_2, \ldots, *''_k, i''_r/\ell''_r)$$

is exact we take s in the range $r \leq s \leq k$ as large as possible so that $j_{r+1} = j_{r+2} = \cdots = j_s = L$. The properties (C1), (C2), and most of (C3) are clear. To check that $i'_r = \ell''_r$ we observe that $j''_r = i'_r \in A_r$, so $j''_r \neq L$, and thus (following (D2)) we have $*''_r = \otimes$ and $\ell''_r = j''_r = i'_r$.

7. The construction.

Suppose we have a sequence of exact categories $\mathcal{M}_n$ defined for $n \geq 0$. We assume we are given bi-exact functors

$$\otimes : \mathcal{M}_n \times \mathcal{M}_p \to \mathcal{M}_{n+p}.$$

Let $F_k(\mathcal{M})$ denote the category of chains $V_1 \rightarrowtail \cdots \rightarrowtail V_k$ of admissible monomorphisms in the exact category $\mathcal{M}$, whose arrows are the evident commutative diagrams. We assume we are given exterior power functors

$$F_k(\mathcal{M}_n) \to \mathcal{M}_{nk}$$
$$V_1 \rightarrowtail \cdots \rightarrowtail V_k \mapsto V_1 \wedge \cdots \wedge V_k.$$

We let $\bigwedge^k V$ denote $V \wedge \cdots \wedge V$, where the identity map is used for the monomorphism.

We assume the operations $\otimes$ and $\wedge$ satisfy the following compatibility conditions:

(E1) Given $V \rightarrowtail \cdots \rightarrowtail W \rightarrowtail X \rightarrowtail \cdots \rightarrowtail Y$ there is a natural map

$$V \wedge \cdots \wedge W \otimes X \wedge \cdots \wedge Y \to V \wedge \cdots \wedge W \wedge X \wedge \cdots \wedge Y.$$

These maps are associative in the obvious sense.

(E2) Given $V \rightarrowtail \cdots \rightarrowtail W \rightarrowtail X \rightarrowtail \cdots \rightarrowtail Y$ there is a natural map

$$V \wedge \cdots \wedge W \wedge X \wedge \cdots \wedge Y \to V \wedge \cdots \wedge W \otimes \frac{X}{W} \wedge \cdots \wedge \frac{Y}{W}.$$

These maps are associative in the obvious sense. The abuse of notation occurring here through the use of quotients is to be repaired by letting the condition apply to any choice of representatives for the quotient objects $X/W, \ldots, Y/W$.

(E3) Given $U \rightarrowtail \cdots \rightarrowtail V \rightarrowtail W \rightarrowtail \cdots \rightarrowtail X \rightarrowtail Y \rightarrowtail \cdots \rightarrowtail Z$, the following diagram commutes.

$$\begin{array}{ccc} U \wedge \cdots \wedge V \wedge W \wedge \cdots \wedge X \otimes Y \wedge \cdots \wedge Z & \longrightarrow & U \wedge \cdots \wedge V \wedge W \wedge \cdots \wedge X \wedge Y \wedge \cdots \wedge Z \\ \downarrow & & \downarrow \\ U \wedge \cdots \wedge V \otimes \frac{W}{V} \wedge \cdots \wedge \frac{X}{V} \otimes \frac{Y}{V} \wedge \cdots \wedge \frac{Z}{V} & \longrightarrow & U \wedge \cdots \wedge V \otimes \frac{W}{V} \wedge \cdots \wedge \frac{X}{V} \wedge \frac{Y}{V} \wedge \cdots \wedge \frac{Z}{V} \end{array}$$

(E4) Given $U \rightarrowtail \cdots \rightarrowtail V \rightarrowtail W \rightarrowtail \cdots \rightarrowtail X \rightarrowtail Y \rightarrowtail \cdots \rightarrowtail Z$, the following diagram commutes.

$$\begin{array}{ccc} U \wedge \cdots \wedge V \otimes W \wedge \cdots \wedge X \wedge Y \wedge \cdots \wedge Z & \longrightarrow & U \wedge \cdots \wedge V \wedge W \wedge \cdots \wedge X \wedge Y \wedge \cdots \wedge Z \\ \downarrow & & \downarrow \\ U \wedge \cdots \wedge V \otimes W \wedge \cdots \wedge X \otimes \frac{Y}{X} \wedge \cdots \wedge \frac{Z}{X} & \longrightarrow & U \wedge \cdots \wedge V \wedge W \wedge \cdots \wedge X \otimes \frac{Y}{X} \wedge \cdots \wedge \frac{Z}{X} \end{array}$$

(E5) Given $U \rightarrowtail \cdots \rightarrowtail V \rightarrowtail W' \rightarrowtail W \rightarrowtail X \rightarrowtail \cdots \rightarrowtail Y$ the sequence

$$0 \to U \wedge \cdots \wedge V \wedge W' \wedge X \wedge \cdots \wedge Y \to U \wedge \cdots \wedge V \wedge W \wedge X \wedge \cdots \wedge Y \to U \wedge \cdots \wedge V \otimes \frac{W}{W'} \wedge \frac{X}{W'} \wedge \cdots \wedge \frac{Y}{W'} \to 0$$

is an exact sequence.

We define a function (for each $A \in \Delta$)

$$\bigwedge^k : \mathrm{Exact}(\Gamma(A), \mathcal{M}_n) \to \mathrm{Exact}(\Gamma^k(A), \mathcal{M}_{nk})$$

as follows. Given $M \in \mathrm{Exact}(\Gamma(A), \mathcal{M}_n)$ and given $(i_1/\ell_1, *_2, \dots, *_k, i_k/\ell_k) \in \Gamma^k(A)$ we define the functor $\bigwedge^k M$ on objects with the formula

$$(\bigwedge^k M)(i_1/\ell_1, *_2, \dots, *_k, i_k/\ell_k) = M(i_1/\ell_1) *_2 \cdots *_k M(i_k/\ell_k).$$

Property (A3) ensures that $\ell_{r-1} = \ell_r$ whenever $*_r = \wedge$, and thus the use of multiwedges is permissible (i.e. we are actually applying the multiwedge to a filtration). Properties (E1)-(E4) ensure that the obvious approach to defining $\bigwedge^k M$ on arrows yields a well-defined functor, and property (E5), together with exactness of M and $\otimes$, ensures that $\bigwedge^k M$ is exact. We see that $\bigwedge^k$ is natural in the variable A.

If we replace A by a concatenation $A_1 \cdots A_k$ and follow $\bigwedge^k$ by composition with Ξ we get a natural function

$$\lambda^k : \mathrm{Exact}(\Gamma(A_1 \cdots A_k), \mathcal{M}_n) \to \mathrm{Exact}(\Gamma(A_1) \times \cdots \times \Gamma(A_k), \mathcal{M}_{nk})$$

which is nothing more than a simplicial map

$$\lambda^k : \mathrm{Sub}_k\, G\mathcal{M}_n \to G^k\mathcal{M}_{nk}.$$

whose geometric realization gives up the desired map

$$\lambda^k : K_*(\mathcal{M}_n) \to K_*(\mathcal{M}_{nk})$$

8. Agreement on K_0.

To compute what happens on $K_0\mathcal{M}$ we consider a vertex (V, W) of $G\mathcal{M}_n$, i.e. a functor $N : \Gamma([0]) \to \mathcal{M}$ defined by $N(0/L) = V$ and $N(0/R) = W$. In the subdivision $\mathrm{Sub}_k(G\mathcal{M}_n)$ this corresponds to the $(0, \dots, 0)$-simplex M defined by $M(0_r/L) = V$, $M(0_r/R) = W$, and $M(0_s/0_r) = 0$, for $1 \le r \le s \le k$. Here 0_r denotes the 0 of the r-th $[0]$ in the concatenation $[0] \cdots [0]$. (The precise statement is that the subdivision homeomorphism Ψ of section 4 sends M to a certain k-fold degeneracy of N.)

To compute the $(0, \dots, 0)$-simplex $\lambda^k(V, W) = \lambda^k M$ of $G^k\mathcal{M}_{nk}$, we take any $j_1, \dots, j_k \in \{L, R\}$ and examine

$$(\lambda^k M)(0/j_1, \dots, 0/j_k) = M(0_1/\ell_1) *_2 \cdots *_r M(0_r/\ell_r)$$

Here ℓ_i and $*_i$ are defined as in (D1,2) above, namely: if $j_r = L$ then $*_r = \wedge$ and $\ell_r = \ell_{r-1}$, but if $j_r = R$ then $*_r = \otimes$ and $\ell_r = R$. It is apparent that if, for any r, we have $j_r = R$, then for all $s \ge r$ we have $\ell_s = R$. Thus the sequence $\ell_1, \dots, \ell_r$ consists of some L's followed by some R's. If we let u denote the number of j's which are equal to R, then we see that $(\lambda^k M)(0/j_1, \dots, 0/j_k)$ has the form

$$\bigwedge^a V \otimes \bigwedge^{b_1} W \otimes \cdots \otimes \bigwedge^{b_u} W.$$

As the indices $j_1, \dots, j_k$ take all 2^k possible combinations of values from the set $\{L, R\}$, we obtain all possible ordered partitions $a + b_1 + \cdots + b_u = k$ with $a \ge 0$ and $b_1, \dots, b_u \ge 1$; there are 2^k of these.

Taking the alternating sum in $K_0\mathcal{M}_{nk}$ yields $\lambda^k([V] - [W])$, according to the formula (2.3). This proves agreement on K_0 for our operations with the those discussed in section 2.

9. Agreement with the operations on K_*R.

Let R be a commutative ring, and let $\mathcal{M} = \mathcal{P}(R)$. Quillen [4] and Kratzer [5] have defined λ-operations $\lambda^k : K_*(R) \to K_*(R)$. In this section we check that our construction agrees with theirs.

If $\mathcal{C}$ is a category and n is a positive integer, we introduce a k-fold category $\mathrm{Multi}_n\,\mathcal{C}$ by setting

$$(\mathrm{Multi}_n\,\mathcal{C})(A_1, \ldots, A_n) = \mathcal{C}(A_1 \times \cdots \times A_n).$$

We remark that $\mathrm{Multi}_2\,\mathcal{C}$ was called $\mathrm{bi}(\mathcal{C})$ in [6, p. 168]. It can be checked that $|\mathcal{C}|$ and $|\mathrm{Multi}_n\,\mathcal{C}|$ are homotopy equivalent, although we will not actually use this fact.

If $\mathcal{C}$ is a category we let $\mathrm{Iso}(\mathcal{C})$ denote the subcategory of $\mathcal{C}$ whose arrows are the isomorphisms of $\mathcal{C}$.

Consider the sub-k-fold-simplicial-set $IG^k\mathcal{M}$ of $G^k\mathcal{M}$ whose simplices are those where the admissible monomorphisms in the filtrations are isomorphisms, or equivalently, are those multiexact functors $M \in (G^k\mathcal{M})(A_1, \ldots, A_k)$ such that $M(i_1/j_1, \ldots, i_k/j_k) = 0$ whenever $j_r \notin \{L, R\}$ for some r. By forgetting the superfluous zero values of the functors M we see that

$$\begin{aligned}(IG^k\mathcal{M})(A_1, ..., A_k) &= \mathrm{Hom}((A_1 \times \{L,R\}) \times \cdots \times (A_k \times \{L,R\}), \mathrm{Iso}\,\mathcal{M}) \\ &= \mathrm{Hom}(A_1 \times \cdots \times A_k, \mathrm{Iso}\,\mathcal{M}^{\{L,R\}^k}) \\ &= (\mathrm{Multi}_k\,\mathrm{Iso}\,\mathcal{M}^{\{L,R\}^k})(A_1, \ldots, A_k)\end{aligned}$$

and thus $IG^k\mathcal{M}$ is nothing more that a product of 2^k copies of $\mathrm{Multi}_k\,\mathrm{Iso}\,\mathcal{M}$. A path in $IG^k\mathcal{M}$ can be written as an alternating sum of 2^k paths, each of which has all but one of its 2^k components being the constant path at the basepoint 0. Each term of the sum can then be thought of as a path in $\mathrm{Multi}_k\,\mathrm{Iso}\,\mathcal{M}$. Now the edges of $|\mathrm{Multi}_k\,\mathrm{Iso}\,\mathcal{M}|$ are of k different types, depending on their "direction" : an edge in the i-th direction is a $0, \ldots, 0, 1, 0, \ldots, 0$-simplex, where the 1 occurs in the i-th spot. But commutative squares like

$$\begin{array}{ccc} \bullet & \xrightarrow{1} & \bullet \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle f} \\ \bullet & \xrightarrow[f]{} & \bullet \end{array}$$

show that, for a given arrow f of $\mathcal{C}$, the edges of $|\mathrm{Multi}_k\,\mathrm{Iso}\,\mathcal{M}|$ formed from f in the k directions are all homotopic. We make use of this remark implicitly in what follows.

We remark that our map λ^k maps $IG\mathcal{M}$ to $IG^k\mathcal{M}$.

Consider an arbitrary group H and a virtual representation ρ of H on finitely generated projective R-modules. From ρ we may obtain a homotopy class of maps $|\rho| : |H| \to |G\mathcal{M}|$. One way to do this is to represent ρ as a difference $[V] - [W]$ of representations and use the map $H \to IG\mathcal{M}$ which sends h to the pair $(h_{|V}, h_{|W})$. We can do something similar with an alternating sum of 2^k representations and $|G^k\mathcal{M}|$, because any map $H \to IG^k\mathcal{M}$ is an alternating sum of maps for the H-space structure on $|G^k\mathcal{M}|$, as can be seen by picking out the 2^k components of the map.

Making use of the λ-operations in $K_0(\mathcal{P}(R,H))$ we obtain another virtual representation $\lambda^k\rho$. An examination of the construction in [4] and [5] tells us it is enough to check that the diagram

$$\begin{array}{ccc} |H| & \xrightarrow{|\rho|} & |G\mathcal{M}| \\ \| & & \downarrow{\scriptstyle |\lambda^k|} \\ |H| & \xrightarrow[|\lambda^k\rho|]{} & |G\mathcal{M}| \end{array}$$

commutes up to homotopy.

Let (V, W) represent ρ as before. In the previous section we checked that we get the right result on the underlying R-modules, which means that $|\lambda^k| \circ |\rho|$ and $|\lambda^k\rho|$ and can be written in a parallel fashion as an alternating sum of 2^k terms, one of which we single out by selecting $(j_1, \ldots, j_k) \in \{L, R\}^k$. Now it suffices to examine a single element h of H, since a homotopy class of maps from a space to the geometric realization of a groupoid is determined by where it sends the base point and the loops. The edge $\rho(h)$ in $G\mathcal{M}$ is the exact functor $M : \Gamma([1]) \to \mathcal{M}$ determined by

$$\begin{aligned} M(0/L) &= V \\ M(1/L) &= V \\ M(0/L \to 1/L) &= (V \xrightarrow[h]{\cong} V) \\ M(0/R) &= W \\ M(1/R) &= W \\ M(0/R \to 1/R) &= (W \xrightarrow[h]{\cong} W) \end{aligned}$$

Our computation implicitly involves applying the subdivision homeomorphism Φ of section 4, for we have written $|G\mathcal{M}|$ in place of $|\operatorname{Sub}_k G\mathcal{M}|$ in the upper right corner. After referring to the diagram (4.2) and the discussion surrounding it, one eventually sees that the homotopy of loops we desire results from the fact that the composite of the maps

$$\begin{aligned} M(0/j_1) *_2 M(0/j_2) *_3 \ldots *_{k-1} M(0/j_{k-1}) *_k M(0/j_k) & \\ &\to M(0/j_1) *_2 M(0/j_2) *_3 \cdots *_{k-1} M(0/j_{k-1}) *_k M(1/j_k) \\ &\to \ldots \\ &\to M(0/j_1) *_2 M(1/j_2) *_3 \cdots *_{k-1} M(1/j_{k-1}) *_k M(1/j_k) \\ &\to M(1/j_1) *_2 M(1/j_2) *_3 \cdots *_{k-1} M(1/j_{k-1}) *_k M(1/j_k), \end{aligned}$$

is the map induced on $M(0/j_1) *_2 M(0/j_2) *_3 \cdots *_{k-1} M(0/j_{k-1}) *_k M(0/j_k)$ by h.

10. Examples.

For our first example we take a scheme X and let $\mathcal{M}_n = \mathcal{P}(X)$ (for each $n \geq 0$). For $V \otimes W$ we take the usual tensor product, and given admissible monomorphisms $V_1 \subseteq \cdots \subseteq V_k$ we take $V_1 \wedge \cdots \wedge V_k$ to be the image of $V_1 \otimes \cdots \otimes V_k$ in $\wedge^k V_k$. The latter could also be defined locally as the quotient of $V_1 \otimes \cdots \otimes V_k$ by the submodule generated by all elements of the form $v_1 \otimes \cdots \otimes v_k$ with $v_i = v_{i+1}$ for some i, and this point of view helps check the properties required.

For our second example we take a commutative ring R and let $\mathcal{M}_n = \mathcal{P}(R, G)$ (for each $n \geq 0$). For $\otimes$ and $\wedge$ we take the operations over R (as described in the first example) and let the group G act diagonally.

For our third example we take a commutative ring R and let $\mathcal{M}_n = \mathcal{P}(R, S_n)$ as in the introduction. We define our two operations as follows. Given $V \in \mathcal{M}_n$ and $W \in \mathcal{M}_p$ we let $V \otimes W$ be the induced representation $\operatorname{Ind}_{S_n \times S_p}^{S_{n+p}} V \otimes_R W$. Given $V_1 \subseteq \cdots \subseteq V_k$ in $\mathcal{M}_n$ we let

$$V_1 \wedge \cdots \wedge V_k = \operatorname{Ind}_{S_n \int S_k}^{S_{nk}} \sum_{f \in S_k} V_{f(1)} \otimes_R \cdots \otimes_R V_{f(k)}$$

where the sum is understood to be taken as a submodule of $V_k \otimes_R \cdots \otimes_R V_k$.

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