A BRIEF INTRODUCTION TO ALGEBRAIC K-THEORY

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Abstract. We give a brief survey of higher algebraic K-theory and its connection to motivic cohomology. We start with limits and colimits, and then pass to the combinatorial construction of topological spaces by means of "systems of simplices", usefully mediated by simplicial sets. Definitions of K-theory are offered, and the main theorems are stated. A definition of motivic cohomology is offered and its major properties are listed. Many references to the literature are provided, especially for the development of motivic cohomology.

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Introduction

This chapter summarizes a six-lecture course on "Algebraic K-Theory" that I gave at the conference and summer school on "Cohomology of Groups and Algebraic K-Theory", held at the Center of Mathematical Sciences of Zhejiang University, in Hangzhou, China, July 2 to 12, 2007, with recent references to the literature added. I am grateful to the Center of Mathematical Sciences and to the Higher Education Press for the support that allowed me to visit China for the summer school. I am also grateful to Marco Varisco for taking notes of the lectures, in TEX, upon which this chapter is based.

Other useful surveys of (or textbooks about) algebraic K-theory include [3, 33, 24, 5, 43, 17, 1, 41, 31, 18, 39, 22, 11, 20, 26, 62].

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1. Limits, colimits, and equations

There are two things you can do with an equation (or a system of equations): you can ask for the solution set, or you can declare the equation to be true. Perhaps surprisingly, from the right point of view, these operations are *dual* to each other. (And what that means is that there are two points of view, symmetrically related to each other, from which the two concepts turn out to be identical.)

Example 1.1. *Suppose $F : V \to W$ is a linear map between vector spaces over a field, and you consider the equation $F(v) = 0$. The solution set is the* kernel *of the operator,* $\ker F = \{v \in V \mid F(v) = 0\}$*. On the other hand, declaring the equation to be true leads to the* cokernel *of the operator (where the equation is true), which is defined as the quotient space* $\operatorname{coker} F := W/F(V)$*.*

Example 1.2. *Suppose $F_1, \ldots, F_m$ are polynomials in the variables $x_1, \ldots, x_n$ over an algebraically closed field K. The solution set of the system $F(x) = 0$ (where the indices are inferred) is the algebraic set $\{\alpha \in K^n \mid F(\alpha) = 0\}$. On the other hand, declaring the equations to be true leads to the quotient ring $K[x]/(F(x))$, in which the equations are true. (The algebraic variety defined by the system incorporates both the solution set and the quotient ring.)*

The duality between these two operations is especially visible in an abstract category $\mathcal{C}$. The only operation available is the composition $f \circ g$ of arrows f and g, so the only equations available have the form $f \circ g = h$. Let I be a small "index" category, and consider a functor (or *diagram*) $F : \mathrm{I} \to \mathcal{C}$. The limit of F is the universal solution to the problem of finding an object $C \in \mathcal{C}$ and arrows $f_I : C \to F(I)$ for all $I \in \mathrm{I}$ that make the equation $f_J = F(\alpha) \circ f_I$ true, for every arrow $\alpha : I \to J$ in I. On the other hand, the colimit of F is the co-universal solution to the problem of finding an object $C \in \mathcal{C}$ and arrows $g_I : F(I) \to C$ for all $I \in \mathrm{I}$ that make the equation $g_J \circ F(\alpha) = g_I$ true, for every arrow $\alpha : I \to J$ in I.

In the category Top of topological spaces (and in the category Sets of sets) limits and colimits always exist. The limit is constructed as the subspace of the product $\prod_I F(I)$ where the equations $f_J = F(\alpha) \circ f_I$ are all true, whereas the colimit is constructed as the quotient space obtained from the disjoint union $\coprod_I F(I)$ by declaring all the equations $g_J \circ F(\alpha) = g_I$ to be true, mirroring the pairs of constructions in each of our first two examples.

Returning to example 1.1, one sees that $\ker F$ arises as

$$\lim \left(\begin{array}{ccc} & & V \\ & & \big\downarrow{\scriptstyle F} \\ 0 & \longrightarrow & W \end{array} \right),$$

and that $\operatorname{coker} F$ arises as

$$\operatorname{colim} \left(\begin{array}{ccc} & & W \\ & & {\scriptstyle F}\big\uparrow \\ 0 & \longleftarrow & V \end{array} \right).$$

The two diagrams are dual to each other (reverse the arrows and rename the objects).

The colimit of a diagram of spaces is said to be obtained by *gluing* the spaces of the diagram together.

2. Constructing spaces combinatorially

In algebraic K-theory we consider spaces that are large, but constructed from simple building blocks glued together in a simple way.

The building blocks are the *topological simplices* $\Delta^n_{\mathsf{Top}} := \{a \in \mathbb{R}^{n+1} \mid \forall i\ a_i \geq 0, \sum a_i = 1\} \in \mathsf{Top}$, for each $n \geq 0$. The maps between them that are usable in our diagrams are those that send vertices[1] to vertices, preserve the numerical ordering of the vertices, and are affine, in the sense that they preserve convex linear combinations[2]. Let Sim, the *category of simplices*, be the subcategory of Top with those objects and those maps.

The colimit (in Top) of a diagram in Sim will be called a *combinatorially constructed space*. All the important spaces of topology can be constructed in this way, even very complicated ones.

For example, a model for the topological n-sphere S^n_{Top} can be constructed combinatorially as

$$\operatorname{colim} \left(\begin{array}{ccc} \Delta^{n-1}_{\mathsf{Top}} & \overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}} & \Delta^n_{\mathsf{Top}} \\ \downarrow & & \\ \Delta^0_{\mathsf{Top}} & & \end{array} \right),$$

where the horizontal maps are the *face maps* (injective maps).

We can replace the category Sim by something entirely combinatorial, and thus slightly more convenient to use, as follows. Let Ord denote the category of finite ordered nonempty sets whose objects are of the form $\underline{\underline{n}} := \{0 < 1 < 2 < \cdots < n\}$, for $n \geq 0$; the arrows in the category are the order-preserving functions. The maps in Sim are determined by their effect on the vertices, so there is an equivalence $\mathsf{Ord} \cong \mathsf{Sim}$, so diagrams in Sim correspond to diagrams in Ord.

Let $\Delta_{\mathsf{Top}} \colon \mathsf{Ord} \to \mathsf{Top}$ denote the composite functor $\mathsf{Ord} \cong \mathsf{Sim} \hookrightarrow \mathsf{Top}$. To construct a space combinatorially, we may start with a diagram in Ord, compose it with Δ_{Top}, and take the colimit. The n-sphere arises in this way from the diagram

$$\begin{array}{ccc} \underline{\underline{n-1}} & \overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}} & \underline{\underline{n}} \\ \Big\downarrow & & \\ \underline{\underline{0}} & & \end{array} .$$

The horizontal maps used above are the injective order-preserving maps $\underline{\underline{n-1}} \hookrightarrow \underline{\underline{n}}$.

We abstract the situation above as follows. We say that a *system of simplices* in a category $\mathcal{C}$ is a functor $\Delta_{\mathcal{C}} \colon \mathsf{Ord} \to \mathcal{C}$. For each $n \geq 0$ we let $\Delta^n_{\mathcal{C}} := \Delta_{\mathcal{C}}(\underline{\underline{n}})$; we call it a *simplex* of $\mathcal{C}$.

When $\mathcal{C}$ has a system of simplices $\Delta_{\mathcal{C}}$, then for any object X of $\mathcal{C}$ we can construct a diagram of topological simplices from X by using the simplices of $\mathcal{C}$ to probe the

[1]The vertices of Δ^n_{Top} are the points with one nonzero coordinate.

[2]A *convex linear combination* of points p_i in an affine space is a sum $\sum a_i p_i$ where $a_i \geq 0$ for each i and $\sum a_i = 1$.

structure of X, as follows. The index category I_X of the diagram is the "comma category" where an object is a pair $(\underline{\underline{n}}, f : \Delta^n_{\mathcal{C}} \to X)$ and where an arrow from $(\underline{\underline{n}}, f : \Delta^n_{\mathcal{C}} \to X)$ to $(\underline{\underline{m}}, f : \Delta^m_{\mathcal{C}} \to X)$ is an arrow $\varphi : \underline{\underline{n}} \to \underline{\underline{m}}$ in Ord such that the triangle

$$\begin{array}{ccc} \Delta^n_{\mathcal{C}} & \longrightarrow & X \\ {\scriptstyle \Delta_{\mathcal{C}}(\varphi)}\big\downarrow & \nearrow & \\ \Delta^m_{\mathcal{C}} & & \end{array}$$

commutes. The diagram is the functor $F_X : \mathrm{I}_X \to \mathsf{Sim}$ that sends $(\underline{\underline{n}}, f : \Delta^n_{\mathcal{C}} \to X)$ to Δ^n_{Top}. We let $|X|$ denote the colimit of the diagram F_X; it is a topological space (and is a CW-complex); we call it the *geometric realization* of X formed with respect to the system of simplices $\Delta_{\mathcal{C}}$. The construction is natural, and we get a functor $\mathcal{C} \to \mathsf{Top}$ that sends X to $|X|$.

Exercise 2.1. *If $\Delta_{\mathcal{C}}$ is fully faithful and $X = \Delta^n_{\mathcal{C}}$, then I_X has a final object and $|X|$ is homeomorphic to Δ^n_{Top}.*

Example 2.2. *Let $\mathcal{C} = \mathsf{Top}$ equipped with the system of simplices Δ_{Top} defined above. For any space X the space $|X|$ is a* CW-approximation *to X. For example, the space $|X|$ is connected if and only if X is path-connected. In this case, Δ_{Top} is not fully faithful.*

Example 2.3. *Let $\mathcal{C}$ be the category PO of partially ordered sets, and let Δ_{PO} be the inclusion $\mathsf{Ord} \hookrightarrow \mathsf{PO}$, so that $\Delta^n_{\mathsf{PO}} = \underline{\underline{n}}$. In this case, Δ_{PO} is fully faithful. Example: the geometric realization of the partially ordered set $\{t \geq x \leq y \leq z\}$ can be visualized (using the exercise) as a 2-simplex xyz and a 1-simplex xt glued together at the vertex x.*

Exercise 2.4. *Interpreting $\underline{\underline{m}} \times \underline{\underline{n}}$ as a partially ordered set, show that the canonical map $|\underline{\underline{m}} \times \underline{\underline{n}}| \to |\underline{\underline{m}}| \times |\underline{\underline{n}}|$ is a homeomorphism.*

Example 2.5. *Let $\mathcal{C}$ be the category SS of all functors $X : \mathsf{Ord}^{\mathrm{op}} \to \mathsf{Sets}$, where $\mathsf{Ord}^{\mathrm{op}}$ denotes the opposite category of Ord. The arrows of the category are the natural transformations of functors. Let $\Delta_{\mathsf{SS}} : \mathsf{Ord} \to \mathsf{SS}$ be the functor that sends $\underline{\underline{n}}$ to the functor it represents; i.e., $\Delta^n_{\mathsf{SS}} = (\underline{\underline{m}} \mapsto \mathrm{Hom}_{\mathsf{Ord}}(\underline{\underline{m}}, \underline{\underline{n}}))$.*

An object X of SS is called a simplicial set. *An element of $X(\underline{\underline{n}})$ is called an n-simplex of X.*

Colimits and limits exist in SS. The geometric realization functor $\mathsf{SS} \to \mathsf{Top}$ commutes with colimits.

Using Yoneda's lemma, one may prove that Δ_{SS} is a fully faithful embedding, and that every simplicial set is canonically a colimit of representable ones. This allows us to regard SS as obtained from Ord by adding colimits in a formal way, a remark I first heard from Voevodsky; the same remark applies when Ord is replaced by any small category.

Remark 2.6. *For any $X \in \mathcal{C}$, the geometric realization $|X|$ can be realized in a natural way as the geometric realization of a simplicial set, as follows. We define the* nerve *$\mathsf{N}X \in \mathsf{SS}$ of X by $\mathsf{N}X(\underline{\underline{n}}) := \mathrm{Hom}_{\mathcal{C}}(\Delta^n_{\mathcal{C}}, X)$. One produces a natural homeomorphism $|\mathsf{N}X| \cong |X|$ by first using Yoneda's lemma to produce a natural equivalence $\mathrm{I}_{\mathsf{N}X} \cong \mathrm{I}_X$ of categories. Thus simplicial sets appear (via the nerve) as* *the universal intermediary between X and its geometric realization $|X|$, rendering simplicial sets ubiquitous wherever geometric realizations are used.*

Example 2.7. *Let* Ab *be the category of abelian groups. Let* SAb *be the category of all functors* $X : \mathsf{Ord}^{\mathrm{op}} \to \mathsf{Ab}$*. An object of the category is called a* simplicial abelian group*. Let* $\Delta_{\mathsf{SAb}} : \mathsf{Ord} \to \mathsf{SAb}$ *be the functor that sends* $\underline{\underline{n}}$ *to the simplicial abelian group that sends* $\underline{\underline{m}}$ *to the free abelian group on the set* $\mathrm{Hom}_{\mathsf{Ord}}(\underline{\underline{m}}, \underline{\underline{n}})$*. In this case* Δ_{SAb} *is not fully faithful. With respect to this system of simplices, the nerve* $\mathsf{N}X$ *is the simplicial set underlying* X*, obtained by forgetting the abelian group structure.*

Remark 2.8. *We may replace the topological simplices, as basic building blocks, by other simple things. For example, we may consider the functor* $\mathsf{Ord} \times \mathsf{Ord} \to \mathsf{Top}$ *that sends* $(\underline{\underline{m}}, \underline{\underline{n}})$ *to* $\Delta^m_{\mathsf{Top}} \times \Delta^n_{\mathsf{Top}}$*. That leads naturally to the notion of* bisimplicial set *and to* geometric realization of bisimplicial sets*. The spaces constructible in this way are, up to homeomorphism, the same ones as before.*

Example 2.9. *Let* CAT *be the category of small categories, and for each* $n \geq 0$ *let* Δ^n_{CAT} *be the category whose objects are* $0, 1, 2, \dots, n$*, and where there is an arrow* $i \to j$ *if an only if* $i \leq j$*, and when there is one, it's unique. The functor* Δ_{CAT} *is the one that arises by identifying the objects of* Δ^n_{CAT} *with the elements of* $\underline{\underline{n}}$*. In this case,* Δ_{CAT} *is fully faithful.*

The geometric realization of the category $0 \rightrightarrows 1$ *is homeomorphic to the circle, and is obtained by gluing together two* 1*-simplices along their vertices.*

If G *is a group, then let* $\underline{G}$ *denote the category with one object and with* G *as its set of arrows; the multiplication in* G *gives the composition in* $\underline{G}$*. The geometric realization* $|\underline{G}|$ *is a model for the classifying space* BG *of* G*.*

A natural transformation between two functors $F, G : \mathcal{C} \to \mathcal{D}$ *is conveniently encoded as a single functor* $h : \mathcal{C} \times \Delta^1_{\mathsf{CAT}} \to \mathcal{D}$*. Geometric realization yields a continuous map* $|\mathcal{C}| \times \Delta^1_{\mathsf{Top}} \to |\mathcal{D}|$ *and hence a homotopy between* $|F|$ *and* $|G|$*. This observation provides a convenient dictionary between the language of category theory and the language of homotopy theory.*

Example 2.10. *Let* $\mathcal{C}$ *be the category* CC *of positive chain complexes of abelian groups. Let* Δ^n_{CC} *be the cellular chain complex of* Δ^n_{Top}*, described explicitly as follows. In degree* k *it is the free abelian group* $(\Delta^n_{\mathsf{Top}})_k$ *whose basis consists of symbols* $\langle A \rangle$ *corresponding to the subsets* $A \subseteq \underline{\underline{n}}$ *of cardinality* $k+1$*. The boundary map* $d : (\Delta^n_{\mathsf{Top}})_k \to (\Delta^n_{\mathsf{Top}})_{k-1}$ *sends* $\langle A \rangle$ *to* $\sum_{i=0}^k (-1)^i \langle A_i \rangle$ *where* A_i *is obtained from* A *by removing its* i*-th element. Let* $\Delta_{\mathsf{CC}} : \mathsf{Ord} \to \mathsf{CC}$ *be the functor that assigns to a map* $\varphi : \underline{\underline{n}} \to \underline{\underline{m}}$ *the map* $\Delta^n_{\mathsf{CC}} \to \Delta^m_{\mathsf{CC}}$ *that sends* $\langle A \rangle$ *to* $\langle \varphi(A) \rangle$ *if* φ *is injective on* A*, and to* 0 *otherwise. In this case,* Δ_{CC} *is not fully faithful.*

If $X \in \mathsf{CC}$*, then* $|X|$ *is a space whose homotopy groups are the homology groups of* X*, i.e.,* $\pi_n|X| \cong H_n(X)$*, for all* $n \in \mathbb{Z}$*; see the works of Dold* [12] *and of Kan* [23]*.*

If $0 \to X \to Y \to Z \to 0$ *is a short exact sequence in* CC*, then the long exact sequence* $\cdots \to H_iX \to H_iY \to H_iZ \to H_{i-1}X \to \dots$ *of homology groups can be rewritten as a long exact sequence* $\cdots \to \pi_i|X| \to \pi_i|Y| \to \pi_i|Z| \to \pi_{i-1}|X| \to \dots$*. Much of algebraic* K*-theory is concerned with producing long exact sequences of homotopy groups from various sequences of spaces* $|X| \to |Y| \to |Z|$*. Such sequences of spaces are called fibration sequences, up to homotopy.*

The main thing to remember about this work of Dold and Kan is that it makes homological algebra into a part of homotopy theory, and thus it provides a source of motivation for results in homotopy theory.

Remark 2.11. *If $\mathcal{C}$ is a $\mathbb{Z}$-linear category, then for any $X \in \mathcal{C}$, the nerve can be regarded as a simplicial abelian group. Thus simplicial abelian groups appear (via the nerve) as the universal intermediary between X and its geometric realization $|X|$, rendering simplicial abelian groups ubiquitous wherever geometric realizations are used in a $\mathbb{Z}$-linear category.*

For $X \in \mathsf{SS}$ define $\mathbb{Z}X \in \mathsf{SAb}$ by replacing each set of X by the free abelian group it generates. The map $\pi_i|X| \to \pi_i|\mathbb{Z}X| \cong H_i|X|$ is the Hurewicz map. If $|X|$ is a sphere of dimension q, then we see that the connected component of $|\mathbb{Z}X|$ containing the basepoint is an Eilenberg-MacLane space of type $K(\mathbb{Z}, q)$, and thus it represents the cohomology functor $Y \mapsto H^i(Y, \mathbb{Z})$. Thus spheres, made "additive" and reduced, represent cohomology.

The category CC of example 2.10 is a $\mathbb{Z}$-linear category. The resulting nerve functor $\mathsf{N} : \mathsf{CC} \to \mathsf{SAb}$ is an equivalence of categories. Hence chain complexes are as useful as simplicial abelian groups to encapsulate the information retrieved by geometric realizations in a $\mathbb{Z}$-linear category, and that explains why homological algebra, conceived as the study of chain complexes, is so important. Alternatively, this shows that geometric realization in $\mathbb{Z}$-linear categories is the natural realm of homological algebra.

Example 2.12. *Let F be a field and let $\mathcal{C}$ be the category Sm_F of smooth varieties over F. Put $\Delta^n_{\mathsf{Sm}_F} = \operatorname{Spec}(F[T_0, \ldots, T_n]/(\sum T_i - 1))$; the equations are the same as for Δ^n_{Top}, but the inequalities are gone. Maps from these simplices to a smooth variety X often do not exist; for example, the Hurwitz formula tells that all such maps are constant for a curve X of positive genus, and thus $|X|$ turns out to be a discrete space equivalent to the set of rational points of X. In section 5 we see how to repair this.*

Example 2.13. *Let Y be a topological space. Let $\mathcal{C} = \mathsf{Top}$ equipped with the system of simplices defined by sending $\underline{\underline{n}}$ to $Y \times \Delta^n_{\mathsf{Top}}$. The realization of any other topological space X with respect to this system of simplices is a CW-approximation to the mapping space* $\operatorname{Map}(Y, X)$.

Example 2.14. *Suppose $\mathcal{C}$ is equipped with a system $\Delta_{\mathcal{C}}$ of simplices, suppose $Y \in \mathcal{C}$, and suppose $\mathcal{C}$ has products. As in example 2.13, the product $Y \times \Delta_{\mathcal{C}}$ is a new system of simplices. We may use $|Y, X|$ as notation for the geometric realization of $X \in \mathcal{C}$ with respect to it.*

3. Direct sum K-theory

Given a (small) additive category $\mathcal{M}$ the direct sum Grothendieck group $K_0^{\oplus}(\mathcal{M})$ is defined to be the abelian group with one generator $[M]$ for each object M, modulo the relations of the form $[M'] + [M''] = [M]$ whenever $M' \oplus M'' \cong M$.

For each $n \geq 0$, define a category $\operatorname{sub} \underline{\underline{n}}$ with objects (i, j) for each $0 \leq i \leq j \leq n$, and with a unique arrow $(i, j) \to (r, s)$ for each $i \leq r \leq s \leq j$. In this way we obtain a functor $\operatorname{sub} : \mathsf{Ord} \to \mathsf{CAT}$.

Given an additive category $\mathcal{M}$ with a (choice of) zero object $0 \in \mathcal{M}$, a functor $M\colon \operatorname{sub} \underline{\underline{n}} \to \mathcal{M}$ is called *additive* if:

(1) $M(i,i) = 0$ for all $i \in \underline{\underline{n}}$
(2) $M(i,k) \to M(i,j) \oplus M(j,k)$ is an isomorphism for all $i \le j \le k \in \underline{\underline{n}}$

An additive functor should be thought of as a direct sum diagram in $\mathcal{M}$. For example, an additive functor $\operatorname{sub}\underline{\underline{2}} \to \mathcal{M}$ boils down to three objects $P = M(0,1)$, $Q = M(1,2)$, and $R = M(0,2)$, of $\mathcal{M}$, together with an isomorphism $R \xrightarrow{\cong} P \oplus Q$.

Definition 3.1. *We let $S^{\oplus}\mathcal{M}$ denote the simplicial set that sends $\underline{\underline{n}}$ to the set of additive functors $M : \operatorname{sub}\underline{\underline{n}} \to \mathcal{M}$.*

Observe that $\pi_0|S^{\oplus}\mathcal{M}| = 0$ and $\pi_1|S^{\oplus}\mathcal{M}| \cong K_0^{\oplus}(\mathcal{M})$. We define the higher direct sum K-groups

$$K_n^{\oplus}\mathcal{M} := \pi_{n+1}|S^{\oplus}\mathcal{M}|.$$

Given an associative unital ring R we introduce the additive category $\mathcal{P}_R$ of finitely generated projective left R-modules. We define the higher K-groups of R by setting $K_nR := K_n^{\oplus}\mathcal{P}_R$.

The definition above is an example of the K-theory *gambit*:

> If you have a group K defined by generators and relations (*or* as a quotient set of a set) then make a space X with $\pi_1 X \cong K$ (*or* $\pi_0 X \cong K$, respectively) putting in higher dimensional simplices naturally. The space X is likely to have an interesting homotopy type, and its homotopy groups are worthy of study.

Here are the classical definitions of K_1 and K_2, in terms of group homology of the infinite general linear group $GL(R)$ of R, and its subgroup $E(R)$ generated by the elementary matrices:

$$K_1(R) = H_1(BGL(R), \mathbb{Z}), \quad K_2(R) = H_2(BE(R), \mathbb{Z}).$$

In order to relate these to our definitions we need first to find a way to construct a combinatorial model of the loop space $\Omega|S^{\oplus}\mathcal{P}_R|$.

Given two objects $\underline{\underline{m}}$ and $\underline{\underline{n}}$ of $\mathtt{Ord}$, let $\underline{\underline{m}} * \underline{\underline{n}}$ denote the ordered set in $\mathtt{Ord}$ (uniquely) isomorphic to the ordered set obtained by concatenating $\underline{\underline{m}}$ and $\underline{\underline{n}}$ so that the elements of $\underline{\underline{m}}$ are less than the elements of $\underline{\underline{n}}$, when viewed in $\underline{\underline{m}} * \underline{\underline{n}}$. In this way we obtain a functor $* : \mathtt{Ord} \times \mathtt{Ord} \to \mathtt{Ord}$, as well as natural injections $\underline{\underline{m}} \to \underline{\underline{m}} * \underline{\underline{n}}$ and $\underline{\underline{n}} \to \underline{\underline{m}} * \underline{\underline{n}}$.

Given a simplicial set Y with a base point y_0 we define a naive approximation ωY to the loop space $\Omega|Y|$ of Y as a simplicial set by setting

$$\omega Y(\underline{\underline{n}}) := \lim \left(\begin{array}{ccccccc} & & & & & & \{y_0\} \\ & & & & & & \downarrow \\ & & & & Y(\underline{\underline{0}} * \underline{\underline{n}}) & \longrightarrow & Y(\underline{\underline{0}}) \\ & & & & \downarrow & & \\ & & Y(\underline{\underline{0}} * \underline{\underline{n}}) & \longrightarrow & Y(\underline{\underline{n}}) & & \\ & & \downarrow & & & & \\ \{y_0\} & \longrightarrow & Y(\underline{\underline{0}}) & & & & \end{array} \right).$$

There is a natural map $|\omega Y| \to \Omega|Y|$, which can easily fail to be a homotopy equivalence.

The 0-simplices of $\omega S^{\oplus}\mathcal{M}$ are pairs (M, N) of objects of $\mathcal{M}$. A 1-simplex $(M', N') \to (M, N)$ consists of an object $P \in \mathcal{M}$ and a pair of isomorphisms $M \to M' \oplus P$, $N \to N' \oplus P$. Taking the isomorphisms to be identities yields 1-simplices $(M', N') \to (M' \oplus P, N' \oplus P)$. Alternatively, taking $P = 0$ allows such a 1-simplex $(\alpha, \beta) : (M', N') \to (M, N)$ to be constructed from a pair of isomorphisms $\alpha : M' \xrightarrow{\cong} M$, $\beta : N' \xrightarrow{\cong} N$. An elementary isomorphism $\pi_0|\omega S^{\oplus}\mathcal{M}| \cong K_0\mathcal{M}$ arises by associating the path-connected component containing the vertex (M, N) to the element $[M] - [N]$.

The following theorem is analogous to Quillen's original $+ = Q$ theorem [16], but since exact sequences are not involved, it is simpler, and is derived mainly from Waldhausen's discussion of his S-construction and of Segal's Γ-spaces in [59, 60].

Theorem 3.2 ([19, section 3])**.** *There is a homotopy equivalence* $|\omega S^{\oplus}\mathcal{M}| \xrightarrow{\sim} \Omega|S^{\oplus}\mathcal{M}|$.

Definition 3.3. *We introduce the K-theory space of an additive category $\mathcal{M}$ by setting $K^{\oplus}\mathcal{M} := |\omega S^{\oplus}\mathcal{M}|$, and the K-theory space of a ring R by setting $K^{\oplus}R := K^{\oplus}\mathcal{P}_R$.*

The theorem implies that $\pi_i K^{\oplus}\mathcal{M} \cong K_i\mathcal{M}$.

Quillen's original definition of K-theory for additive categories was the $S^{-1}S$-construction of [16]. It yields a category $S^{-1}S\mathcal{M}$ with $|S^{-1}S\mathcal{M}| \simeq |\omega S^{\oplus}\mathcal{M}|$. Indeed, $\omega S^{\oplus}\mathcal{M}$ can be viewed as a slight modification of $S^{-1}S\mathcal{M}$.

Let $\operatorname{Aut} M$ be the automorphism group of an object $M \in \mathcal{M}$. There is a map $\mathsf{N}\underline{\operatorname{Aut} M} \to \omega S^{\oplus}\mathcal{M}$ in SS that sends $\theta \in \operatorname{Aut} M$ to $(1_M, \theta)$. Thus we get a map $B \operatorname{Aut} M \to K^{\oplus}(\mathcal{M})$. Taking $\mathcal{M} = \mathcal{P}_R$ and $M = R^n$ we get a map $BGL_nR \to K^{\oplus}R$. The triangle

$$\begin{array}{ccc} BGL_n(R) & \longrightarrow & K^{\oplus}R \\ \downarrow & \nearrow & \\ BGL_{n+1}(R) & & \end{array}$$

commutes up to homotopy, and with some work this yields a homotopy class of maps $BGL(R) \to K^{\oplus}R$ whose image (since $BGL(R)$ is connected) is contained in the connected component of $K^{\oplus}R$ that contains the base point. We may call that connected component $BGL(R)^+$, although originally, Quillen constructed it a different way. Each of the path-connected components of $K^{\oplus}R$ is homotopy equivalent to $BGL(R)^+$, for we may use direct sum to translate from one component to another. *Warning:* avoid writing a homotopy equivalence $K^{\oplus}R \simeq K_0R \times BGL(R)^+$, for this equivalence is not canonical when $K_0R \ncong \mathbb{Z}$, and it's not possible to make the equivalence respect the H-space structures.

Theorem 3.4 (Quillen)**.** *The map $BGL(R) \to BGL(R)^+$ is* acyclic[3], *in particular*

$$H_*\big(BGL(R), \mathbb{Z}\big) \xrightarrow{\cong} H_*\left(BGL(R)^+, \mathbb{Z}\right)$$

is an isomorphism (even with local coefficients). On π_1 it is just abelianization.

[3] A continuous map $f : X \to Y$ is acyclic if for every local coefficient system $\mathcal{L}$ on Y and for all $n \in \mathbb{N}$, the map $H_n(X, f^*\mathcal{L}) \to H_n(Y, \mathcal{L})$ is an isomorphism.

Corollary 3.5. *$K_1(R)$ and $K_2(R)$ defined here agree with Bass' and Milnor's definitions.*

By the way, there is a way to interpret each set $S^{\oplus}\mathcal{M}(\underline{\underline{n}})$ as the objects of an additive category, allowing a new simplicial set $S^{\oplus}S^{\oplus}\mathcal{M}(\underline{\underline{n}})$ to be introduced. The resulting object $S^{\oplus}S^{\oplus}\mathcal{M}$ is a functor $\mathtt{Ord}^{\mathrm{op}} \times \mathtt{Ord}^{\mathrm{op}} \to \mathtt{Sets}$, which has a suitable geometric realization. It turns out that there is a homotopy equivalence $|S^{\oplus}\mathcal{M}| \simeq \Omega|S^{\oplus}S^{\oplus}\mathcal{M}|$. This process can be iterated, showing that the K-theory space $K^{\oplus}\mathcal{M}$ appears as the 0-th space in a connective (symmetric) Ω-spectrum (essentially because $\oplus$ is commutative).

Theorem 3.6 ([36])**.**

$$K_i(\mathbb{F}_q) \cong \begin{cases} \mathbb{Z} & i = 0, \\ \mathbb{F}_q^{\times} & i = 1, \\ 0 & i = 2n,\ n > 0, \\ \mathbb{Z}/(q^r - 1) & i = 2r-1,\ r \geq 1. \end{cases}$$

Idea of proof. Compare with $\mathbb{Z} \times BU = K^{\mathrm{top}}(\mathrm{pt})$. On $K(\mathbb{F}_q)$ the q-th Adams operation Ψ_q satisfies $\Psi_q = 1$. Use the *Brauer lifting* $BGL(\mathbb{F}_q)^+ \to K^{\mathrm{top}}(\mathrm{pt})$ to get a fibration sequence

$$BGL(\mathbb{F}_q)^+ \to K^{\mathrm{top}}(\mathrm{pt}) \xrightarrow{\Psi_q - 1} K^{\mathrm{top}}(\mathrm{pt})$$

and then use long exact sequences of homotopy groups. Compute the action of Ψ_q on $K_{2r}^{\mathrm{top}}(\mathrm{pt})$ by observing that $K_{2r}^{\mathrm{top}}(\mathrm{pt})$ is an infinite cyclic group generated by β^r, where $\beta \in K_2(pt)$ is the Bott element. Since $\Psi_q\beta = q\beta$, it follows from multiplicativity that $\Psi_q(\beta^r) = (q\beta)^r = q^r\beta^r$, i.e., Ψ_q is multiplication by q^r on $K_{2r}^{\mathrm{top}}(\mathrm{pt}) \cong \mathbb{Z}$. Hence $K_{2r}(\mathbb{F}_q) = 0$ and $K_{2r-1}(\mathbb{F}_q) \cong \mathbb{Z}/(q^r - 1)$ for all $r > 0$. □

4. Exact sequence K-theory

In a category such as $\mathcal{P}_R$, every short exact sequence is split, and hence direct sum diagrams provide all the information about exact sequences. Other categories, such as the category $\mathcal{M}_R$ of finitely generated left R-modules, have exact sequences that don't split, and $K_0\mathcal{M}_R$ is not necessarily isomorphic to $K_0^{\oplus}\mathcal{M}_R$. For such a category $\mathcal{M}$, the Grothendieck group $K_0(\mathcal{M})$ is defined to be the abelian group with one generator $[M]$ for each object M, modulo the relations of the form $[M']+[M''] = [M]$ whenever there is a short exact sequence $0 \to M' \to M \to M'' \to 0$.

The higher K-theory of such categories is handled as follows.

Definition 4.1 ([38])**.** *An* exact category *is an additive category $\mathcal{M}$, together with a collection of sequences in $\mathcal{M}$ of the form $0 \to M' \to M \to M'' \to 0$ (called* short exact sequences*), such that there is a fully faithful embedding of $\mathcal{M}$ in an abelian category $\mathcal{A}$ so that (1) any short exact sequence $0 \to M' \to A \to M'' \to 0$ of $\mathcal{A}$ with $M' \in \mathcal{M}$ and $M'' \in \mathcal{M}$ has $A \in \mathcal{M}$, too; and (2) the short exact sequences of $\mathcal{M}$ are exactly those sequences that are sent to exact ones in $\mathcal{A}$.*

Definition 4.2 ([38])**.** *An* admissible monomorphism *of an exact category $\mathcal{M}$ is a map $M' \rightarrowtail M$ that appears as the left hand map in a short exact sequence of $\mathcal{M}$. An* admissible epimorphism *of $\mathcal{M}$ is a map $M \twoheadrightarrow M''$ that appears as the right hand map in a short exact sequence of $\mathcal{M}$.*

In an exact category $0 \to M' \to M' \oplus M'' \to M'' \to 0$ is always a short exact sequence, so a short exact sequence is a generalization of a direct sum diagram. Generalizing the definition of $S^{\oplus}\mathcal{M}$ appropriately, one defines, as in [59], a simplicial set $S\mathcal{M}$ for any exact category $\mathcal{M}$ so that $K_0(\mathcal{M}) \cong \pi_1|S\mathcal{M}|$. The higher K-groups are defined as $K_i\mathcal{M} := \pi_{i+1}|S\mathcal{M}|$.

Theorem 4.3 ([14, 15])**.** *There is a natural homotopy equivalence* $|\omega S\mathcal{M}| \xrightarrow{\cong} \Omega|S\mathcal{M}|$*.*

We may then define $K\mathcal{M} := |\omega S\mathcal{M}|$, and it also appears as the first space in an Ω-spectrum.

Quillen's original definition of K-theory was the Q-construction [38]. It yields a category $Q\mathcal{M}$ with $|Q\mathcal{M}| \simeq |S\mathcal{M}|$. The objects of $Q\mathcal{M}$ are the objects of $\mathcal{M}$, and an arrow from M to N in $Q\mathcal{M}$ is an isomorphism of M with a subquotient object of N, where the subquotient objects under consideration are the ones that arise from admissible monomorphisms and admissible epimorphisms of $\mathcal{M}$. The proofs of some important theorems of Quillen in K-theory depend on the use of the Q-construction.

Theorem 4.4 (Quillen's resolution theorem [38])**.** *If $\mathcal{P} \subseteq \mathcal{M}$ is a full exact subcategory closed under extensions and closed under kernels and such that for every M in $\mathcal{M}$ there exists a short exact sequence $0 \to P' \to P \to M \to 0$ with P and P' in $\mathcal{P}$, then the map $K(\mathcal{P}) \to K(\mathcal{M})$ is a homotopy equivalence.*

Fact 4.5. *If X and Y are simplicial sets, then the projections induce a homeomorphism $|X \times Y| \xrightarrow{\cong} |X| \times |Y|$, provided the product is given the compactly generated topology, or if one of the factors is generated by a finite number of simplices. (Putting the compactly generated topology on a space does not change the homotopy groups.) Similarly for small categories.*

In particular, if $\mathcal{M}$ and $\mathcal{N}$ are exact categories then $K_*(\mathcal{M} \times \mathcal{N}) \cong K_*(\mathcal{M}) \times K_*(\mathcal{N})$. If R, S are rings then $K_*(R \times S) \cong K_*(R) \times K_*(S)$. This gives the first part of the following theorem.

Theorem 4.6 ([38])**.** *K-theory commutes with finite products and filtering colimits (of exact categories).*

If $F\colon \mathcal{A} \to \mathcal{C}$ is an exact functor between abelian categories then $\mathcal{B} = \ker F$ is abelian, too, and for all exact sequences $0 \to A' \to A \to A'' \to 0$ in $\mathcal{A}$ then $A \in \mathcal{B}$ if and only if $A', A'' \in \mathcal{B}$. A full subcategory $\mathcal{B}$ of an abelian category $\mathcal{A}$ satisfying this property is called a *Serre subcategory.*

Fact 4.7. *If $\mathcal{B} \subseteq \mathcal{A}$ is a Serre subcategory of an abelian category then there exists a* quotient *abelian category $\mathcal{A}/\mathcal{B}$ and an exact functor $G\colon \mathcal{A} \to \mathcal{A}/\mathcal{B}$ with* $\ker G = \mathcal{B}$ *such that for any exact functor $F\colon \mathcal{A} \to \mathcal{C}$ to any exact category $\mathcal{C}$ with $\mathcal{B} \subseteq \ker F$ there is an exact functor $\overline{F}\colon \mathcal{A}/\mathcal{B} \to \mathcal{C}$ (unique up to natural isomorphism) such that $\overline{F} \circ G \cong F$.*

Theorem 4.8 (Quillen's localization theorem for exact categories [38])**.** *If $\mathcal{B} \subseteq \mathcal{A}$ is a Serre subcategory of an abelian category then there is a natural long exact sequence in K-theory*

$$\cdots \to K_n(\mathcal{B}) \to K_n(\mathcal{A}) \to K_n(\mathcal{A}/\mathcal{B}) \to K_{n-1}(\mathcal{B}) \to \cdots .$$

We apply this to understand $K_*\mathbb{Z} \to K_*\mathbb{Q}$. First of all, the resolution theorem shows that $K_*\mathbb{Z} = K_*\mathcal{P}_\mathbb{Z} \xrightarrow{\cong} K_*\mathcal{M}_\mathbb{Z}$, and similarly for $\mathbb{Q}$. Now look at the exact functor $F\colon \mathcal{M}_\mathbb{Z} \to \mathcal{M}_\mathbb{Q}$ that sends M to $M \otimes \mathbb{Q}$. Then $\ker F$ is the abelian category $\mathcal{M}^1_\mathbb{Z}$ of torsion finitely generated $\mathbb{Z}$-modules, i.e., the category of finite abelian groups. One can check that the quotient Abelian category $\mathcal{M}_\mathbb{Z}/\mathcal{M}^1_\mathbb{Z}$ is equivalent to $\mathcal{M}_\mathbb{Q}$. Moreover,

$$\mathcal{M}^1_\mathbb{Z} \cong \operatorname*{colim}_{n\in\mathbb{N}} \mathcal{M}^1_\mathbb{Z}(n) \cong \operatorname*{colim}_{n\in\mathbb{N}} \mathcal{M}_{\mathbb{Z}/n\mathbb{Z}}$$

where $\mathcal{M}^1_\mathbb{Z}(n)$ is the full subcategory of $\mathcal{M}_\mathbb{Z}$ whose objects are the abelian groups annihilated by n.

Now what is $K_*\mathcal{M}_{\mathbb{Z}/n\mathbb{Z}}$? In terms of the prime factorization $n = p_1^{e_1}\cdots p_n^{e_n}$ the Chinese remainder theorem yields $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/p_1^{e_1}\mathbb{Z} \times \ldots \times \mathbb{Z}/p_n^{e_n}\mathbb{Z}$, hence $\mathcal{M}_{\mathbb{Z}/n\mathbb{Z}} \cong \mathcal{M}_{\mathbb{Z}/p_1^{e_1}\mathbb{Z}} \times \cdots \times \mathcal{M}_{\mathbb{Z}/p_n^{e_n}\mathbb{Z}}$, and $K_n\mathcal{M}_{\mathbb{Z}/n\mathbb{Z}} \cong K_n\mathcal{M}_{\mathbb{Z}/p_1^{e_1}\mathbb{Z}} \times \cdots \times K_n\mathcal{M}_{\mathbb{Z}/p_n^{e_n}\mathbb{Z}}$ so we need understand only $K_n\mathcal{M}_{\mathbb{Z}/p^e\mathbb{Z}}$.

Theorem 4.9 (Quillen's dévissage (untwisting) theorem [38])**.** *If $\mathcal{A}$ is an abelian category and $\mathcal{B}$ a full subcategory such that:*

1. *for all $B \in \mathcal{B}$ and $A \in \mathcal{A}$, if $A \rightarrowtail B$ then $A \in \mathcal{B}$;*
2. *for all $B \in \mathcal{B}$ and $A \in \mathcal{A}$, if $B \twoheadrightarrow A$ then $A \in \mathcal{B}$;*
3. *for all $B', B'' \in \mathcal{B}$ then $B' \oplus B'' \in \mathcal{B}$;*
4. *for all $A \in \mathcal{A}$ there is a filtration $A = A^0 \supseteq A^1 \supseteq \ldots \supseteq A^r = 0$ with $A^i/A^{i+1} \in \mathcal{B}$;*

then the map $K_*\mathcal{B} \xrightarrow{\cong} K_*\mathcal{A}$ is an isomorphism.

In particular $K_*\mathcal{M}_{\mathbb{Z}/p^e\mathbb{Z}} \xleftarrow{\cong} K_*\mathcal{M}_{\mathbb{Z}/p\mathbb{Z}} = K_*\mathcal{P}_{\mathbb{F}_p} = K_*\mathbb{F}_p$, and these groups are known, by Theorem 3.6. Then

$$\begin{aligned} K_*\mathcal{M}^1_\mathbb{Z} \cong \operatorname*{colim}_n K_*\mathcal{M}_{\mathbb{Z}/n\mathbb{Z}} &\cong \operatorname*{colim}_{p_1^{e_1},\ldots,p_n^{e_n}} K_*(\mathcal{M}_{\mathbb{Z}/p_1^{e_1}\mathbb{Z}} \times \ldots \times \mathcal{M}_{\mathbb{Z}/p_n^{e_n}\mathbb{Z}}) \\ &\cong \operatorname*{colim}_{p_1,\ldots,p_n} K_*\mathbb{F}_{p_1} \times \ldots \times K_*\mathbb{F}_{p_n} \cong \bigoplus_{\substack{p\in\mathbb{Z}\\ \text{prime}}} K_*\mathbb{F}_p \end{aligned}$$

All this is true more generally for the ring $\mathcal{O}_F$ of integers in a number field F and yields the following long exact sequence (where the sums run over all prime ideals $\mathfrak{p}$ of $\mathcal{O}_F$).

$$\begin{aligned} \cdots \to \bigoplus_{\mathfrak{p}} K_{2n}\mathcal{O}_F/\mathfrak{p} \to K_{2n}\mathcal{O}_F \to K_{2n}F \\ \to \bigoplus_{\mathfrak{p}} K_{2n-1}\mathcal{O}_F/\mathfrak{p} \to K_{2n-1}\mathcal{O}_F \to K_{2n-1}F \to \cdots \\ \to K_0\mathcal{O}_F \to K_0F \to 0 \end{aligned}$$

It is known that for any field F, the rank homomorphism $K_0F \to \mathbb{Z}$ is an isomorphism, and the determinant homomorphism $K_1F \to F^\times$ is an isomorphism. For any Dedekind domain, such as $\mathcal{O}_F$, the rank and the determinant maps yield an isomorphism $K_0\mathcal{O}_F \xrightarrow{\cong} \mathbb{Z} \oplus \operatorname{Pic}\mathcal{O}_F$, where $\operatorname{Pic}\mathcal{O}_F$ is the Picard group of isomorphism classes of invertible modules. The K-groups of the finite fields $\mathcal{O}_F/\mathfrak{p}$ are known, see Theorem 3.6. Bass, Milnor, and Serre [2] proved that $K_1\mathcal{O}_F \to K_1F$ is injective, and hence that $K_1\mathcal{O}_F \cong \mathcal{O}_F^\times$. Soulé [40] has generalized that to show that $K_n\mathcal{O}_F \to K_nF$ is injective for $n > 0$. Quillen has shown [37] that $K_n\mathcal{O}_F$ is a finitely generated abelian group for all n. The Dirichlet unit theorem shows that the rank of $K_1\mathcal{O}_F$ is r_1+r_2-1, where the isomorphism of rings $F\otimes\mathbb{R}\cong\mathbb{R}^{r_1}\times\mathbb{C}^{r_2}$ defines the numbers r_1 and r_2. Moreover, Borel [10] has computed the ranks of the groups $K_n\mathcal{O}_F$ for $n\geq 2$, showing that the ranks are $0, r_1+r_2, 0, r_2$ provided n is congruent to $0,1,2,3$ modulo 4, respectively.

Taking all that information into account, we see that the long exact sequence above splits into the folllowing simpler pieces, for $n>0$.

$$0\to K_{2n+1}\mathcal{O}_F \xrightarrow{\cong} K_{2n+1}F\to 0$$
$$0\to K_{2n}\mathcal{O}_F\to K_{2n}F\to\bigoplus_{\mathfrak{p}} K_{2n-1}\mathcal{O}_F/\mathfrak{p}\to 0$$
$$0\to\mathcal{O}_F^\times\to F^\times\to\bigoplus_{\mathfrak{p}}\mathbb{Z}\to\operatorname{Pic}\mathcal{O}_F\to 0$$

Specializing to the case $F=\mathbb{Q}$ one gets

$$0\to K_2\mathbb{Z}\to K_2\mathbb{Q}\to\bigoplus_p\mathbb{F}_p^\times\to 0$$

where the sum runs over all prime numbers p. We know that $\mathbb{F}_p^\times\cong\mathbb{Z}/(p-1)\mathbb{Z}$, and by Milnor [33] we know that $K_2\mathbb{Z}\cong\mathbb{Z}/2\mathbb{Z}$. It follows (after showing that the sequence splits) that

$$K_2\mathbb{Q}\cong\mathbb{Z}/2\mathbb{Z}\oplus\bigoplus_p\mathbb{Z}/(p-1)\mathbb{Z}.$$

Hence $K_2\mathbb{Q}$ is not a finitely generated abelian group, because there are infinitely many primes.

5. Weight filtrations and motivic cohomology

In this section we present the connection between K-theory and motivic cohomology. Based on ideas of Beilinson, motivic cohomology was developed initially by Bloch and Suslin, and then was further developed by Friedlander, Geisser, Levine, and Lichtenbaum. It was molded into its final form by the Fields Medal winning work of Voevodsky, who proved the Bloch-Kato conjecture in characteristic 2 and the related conjecture of Milnor. Work of Rost on norm varieties and their motives, complemented by work of Haesemeyer, Joukhovitski, Suslin, and Weibel, recently seems to have settled the Bloch-Kato conjecture for fields of higher characteristic.

Recall first étale cohomology, i.e., an algebraic construction of $H^i(Y,\mathbb{Z}/n\mathbb{Z})$, where Y is an algebraic variety. Let $\mathbb{G}_m=\operatorname{Spec}(\mathbb{C}[T,T^{-1}])$ be the multiplicative group. Its group of complex points $\mathbb{G}_m(\mathbb{C})\cong\mathbb{C}^\times$, with the analytic topology, is not simply connected; its fundamental group is $\mathbb{Z}$, and the universal covering space $\mathbb{C}\to\mathbb{C}^\times$ is given by the exponential map, which is not a polynomial map, so algebra alone cannot see that the fundamental group is infinite. However, algebra can see the finite quotients $\mathbb{Z}/n\mathbb{Z}$ of the fundamental group, because the n-fold covering map $\mathbb{C}^\times\to\mathbb{C}^\times$ sending z to z^n is a polynomial map. The construction of étale cohomology and the fact that it works for finite coefficients is based on that observation.

The goal of motivic cohomology is to provide a purely algebraic construction of something analogous to the singular cohomology groups $H^i(Y,\mathbb{Z})$, where Y is an algebraic variety. Following remarks 2.14 and 2.11, the idea is to work in an appropriate $\mathbb{Z}$-linear category, to look for "spheres" there, and to use them to represent cohomology.

We define a new $\mathbb{Z}$-linear category KSm_F with the same objects as Sm_F, but whose morphisms are "K_0-correspondences from X to Y". The morphisms from $X = \operatorname{Spec} A$ to $Y = \operatorname{Spec} B$ will be the elements of $K_0(\mathcal{P}(A,B))$, where $\mathcal{P}(A,B)$ is the exact category of A-B-bimodules M such that $M \in \mathcal{P}_A$. For non-affine varieties, a similar definition works.

Composition

$$K_0(\mathcal{P}(A,B)) \otimes K_0(\mathcal{P}(B,C)) \to K_0(\mathcal{P}(A,C))$$

is given by tensor product over B of bimodules.

We get a functor $\mathsf{Sm}_F \to \mathsf{KSm}_F$ and hence a system of simplices $\Delta_{\text{alg}} \colon \mathsf{Ord} \to \mathsf{KSm}_F$, better than that defined in example 2.12, because correspondences from the affine line to curves of high genus exist in abundance.

Now, by analogy with remark 2.11, we seek a suitable "algebraic sphere" S^q_{alg} in KSm_F. Motivated by the observation that $\mathbb{G}_m(\mathbb{C})$, with the analytic topology, is homotopy equivalent to the circle S^1_{Top}, we define $S^1_{\text{alg}} = \mathbb{G}_m = \mathbb{A}^1 - \{0\}$ and denote by $\widetilde{S}^1_{\text{alg}}$ its reduced version. That means that, later, after applying a functor with values in Ab to it, we will remove the direct summand arising as the image of the projection operator $\widetilde{S}^1_{\text{alg}} \to * \to \widetilde{S}^1_{\text{alg}}$. Alternatively, think of it as the pair $(\widetilde{S}^1_{\text{alg}}, *)$. Then define $\widetilde{S}^q_{\text{alg}} = \widetilde{S}^1_{\text{alg}} \wedge \ldots \wedge \widetilde{S}^1_{\text{alg}}$, where $\wedge$ denotes the product of varieties, mixed with the product of pairs.

As in 2.14, for a variety Y look at the system of simplices given by $Y \times \Delta_{\text{alg}}$ and the corresponding geometric realization $|Y, X|$. In topology, if we work in the $\mathbb{Z}$-linear category obtained from Top by replacing the Hom-sets by the free abelian groups they generate, $\pi_i |Y, \widetilde{S}^q_{\mathsf{Top}}|$ is $H^{q-i}(Y, \mathbb{Z})$ if $i \le q$ and is 0 if $i > q$. Thus we may expect (or hope) $\widetilde{S}^q_{\text{alg}}$ to represent cohomology in the algebraic context. We define motivic cohomology $H^i(Y, \mathbb{Z}(q))$ to be $\pi_{q-i}|Y, \widetilde{S}^q_{\text{alg}}|$, but only if $Y = \operatorname{Spec} A$ and A is a local ring, such as a field. (When Y is not local, one must use sheaf cohomology, too.) Unraveling the definitions, this is π_{q-i} of the simplicial set whose n-simplices are the elements of the group

$$\widetilde{K}_0(\mathcal{P}(A[T_0, \ldots, T_n]/\Sigma T_i - 1, F[U_1, U_1^{-1}, \ldots, U_q, U_q^{-1}])),$$

where $\widetilde{K}_0$ is obtained from K_0 as the quotient by the subgroup generated by the classes of all objects where one of the variables U_i acts trivially.

Theorem 5.1 ([19, 42, 20])**.** *Suppose $Y = \operatorname{Spec} A$ is smooth (local) over F. Then there is a filtration of the K-theory space of Y*

$$K(Y) = W^0 \leftarrow W^1 \leftarrow W^2 \leftarrow \ldots$$

and there are fibration sequences

$$W^{q+1} \to W^q \to \Omega^{-q}|Y, \widetilde{S}^q_{\text{alg}}|.$$

Therefore there is a motivic *spectral sequence*

$$E_2^{pq} = H^{p-q}(Y, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(Y)$$

analogous to the Atiyah-Hirzebruch spectral sequence in topology.

The spectral sequence of the theorem above was presented first by Bloch and Lichtenbaum in [9] in the case where Y is the spectrum of a field, and extended to the global case by Friedlander and Suslin in [13]. Levine's approach to the spectral sequence in [30] manages to extract it from formal properties of K-theory in a clean way.

Taking into account remark 2.11, we can interpret $|Y, \widetilde{S}^q_{\mathrm{alg}}|$ as an explicit chain complex of abelian groups. Taking into account its contravariant dependence on Y and sheafifying, we obtain a chain complex of sheaves, which is called simply $\mathbb{Z}(q)$, so that $H^i(Y, \mathbb{Z}(q))$ can be defined as its sheaf hypercohomology (for the Zariski topology on Y). This definition of motivic cohomology is not the one most often cited; we present it here because it is the one most closely related to algebraic K-theory.

Properties 5.2. *If ℓ is a prime number and char$F \nmid \ell$ and $Y \in \mathsf{Sm}_F$, then:*

(1) $\mathbb{Z}(0) \cong \mathbb{Z}$ *and* $\mathbb{Z}(1) \cong \mathbb{G}_m[-1]$*;*
(2) $\mathbb{Z}(q)$ *vanishes in degree greater than* q*;*
(3) *there are associative and commutative pairings* $\mathbb{Z}(q_1) \otimes \mathbb{Z}(q_2) \to \mathbb{Z}(q_1 + q_2)$*;*
(4) *after sheafification in the étale topology, there is an isomorphism* $\mu_\ell^{\otimes q} \cong \mathbb{Z}(q) \otimes \mathbb{Z}/\ell\mathbb{Z}$*, where* μ_ℓ *denotes the étale sheaf of* ℓ*-th roots of unity;*
(5) *the natural map* $K^M_q(F) \xrightarrow{\cong} H^q(\operatorname{Spec} F, \mathbb{Z}(q))$*, where* $K^M_q(F)$ *denotes the* q*-th Milnor* K*-group of* F*, is an isomorphism;*
(6) *there is a natural isomorphism* $H^{2q}(Y, \mathbb{Z}(q)) \cong CH^q(Y)$*, where* $CH^q(Y)$ *denotes the Chow group of algebraic cycles of codimension* q *on* Y *modulo rational equivalence;*
(7) localization: *if* Z *is a smooth subvariety of pure codimension* c *in* Y*, then there is a natural long exact sequence* $\cdots \to H^{i-1}(Y - Z, \mathbb{Z}(q)) \to H^{i-2c}(Z, \mathbb{Z}(q-c)) \to H^i(Y, \mathbb{Z}(q)) \to H^i(Y - Z, \mathbb{Z}(q)) \to \dots$*;*
(8) Beilinson-Lichtenbaum conjecture: *The natural map* $H^i(Y, \mathbb{Z}(q) \otimes \mathbb{Z}/\ell\mathbb{Z}) \to H^i_{\text{ét}}(Y, \mathbb{Z}(q) \otimes \mathbb{Z}/\ell\mathbb{Z})$ *is an isomorphism if* $i \leq q$*;*
(9) Bloch-Kato conjecture: *the natural map*

$$K^M_q(F) \otimes \mathbb{Z}/\ell\mathbb{Z} \xrightarrow{\cong} H^q_{\mathrm{cont}}(\mathrm{Gal}(\overline{F}/F), \mu_\ell^{\otimes q})$$

from Milnor K*-theory to Galois cohomology is an isomorphism.*
(10) Beilinson-Soulé vanishing conjecture: $H^i(Y, \mathbb{Z}(q)) = 0$ *for* $i \leq 0$*,* $q > 0$.

The first seven properties are known.

The Bloch-Kato conjecture (9) is the Beilinson-Lichtenbaum conjecture (8) in the case where $Y = \operatorname{Spec} F$ and $i = q$, interpreted using (4) and (5). It is known [46] that the Bloch-Kato conjecture (9) implies the Beilinson-Lichtenbaum conjecture (9), rendering them equivalent. For the latest word on the proof of the Bloch-Kato conjecture (9), which seems to be completely written down now, see [21].

The Beilinson-Lichtenbaum conjecture (8) and the motivic spectral sequence, together with computations of étale cohomology, lead to computations [61] of the K-groups of rings of integers in number fields, some of which depend on the (unsettled) Vandiver conjecture of number theory.

The Beilinson-Soulé vanishing conjecture (10) is completely out of reach at the moment.

For introductions to motivic cohomology see [28, 58, 32, 51]. See also [25, 4, 8, 7, 35, 6, 27, 45, 48, 47, 29, 34, 57, 49, 56, 50, 52, 54, 44, 55, 53, 63] for the development of the theory and the proofs of the properties above.

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