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ADAMS OPERATIONS ON HIGHER K-THEORY

Daniel R. Grayson

University of Illinois at Urbana-Champaign

Abstract. We construct Adams operations on higher algebraic K-groups induced by operations such as symmetric powers on any suitable exact category, by constructing an explicit map of spaces, combinatorially defined. The map uses the S-construction of Waldhausen, and deloops (once) earlier constructions of the map.

1. Introduction.

Let $\mathcal{P}$ be an exact category with a suitable notion of tensor product $M \otimes N$, symmetric power $S^k M$, and exterior power $\Lambda^k M$. For example, we may take $\mathcal{P}$ to be the category $\mathcal{P}(X)$ of vector bundles on some scheme X. Or we may take $\mathcal{P}$ to be the category $\mathcal{P}(R)$ of finitely generated projective R-modules, where R is a commutative ring. Or we may fix a group Γ and a commutative ring R and take $\mathcal{P}$ to be the category $\mathcal{P}(R, \Gamma)$ of representations of Γ on projective finitely generated R-modules. We impose certain exactness requirements on these functors, so that in particular the tensor product is required to be bi-exact, and this prevents us from taking for $\mathcal{P}$ a category such as the category $\mathcal{M}(R)$ of finitely generated R-modules.

In a previous paper [5] I showed how to use the exterior power operations on modules to construct the *lambda* operations λ^k on the higher K-groups as a map of spaces $\lambda^k : |G\mathcal{P}| \to |G^k\mathcal{P}|$ in a combinatorial fashion. Here the simplicial set $G\mathcal{P}$, due to Gillet and me [3], provides an alternate definition for the K-groups of any exact category, $K_i\mathcal{P} = \pi_i G\mathcal{P}$, which has the advantage that the Grothendieck group appears as π_0 and is not divorced from the higher K-groups. The Q-construction of Quillen and the S-construction of Waldhausen are the original definitions for the K-groups of $\mathcal{P}$, but involve a shift in degree, so that $K_i(\mathcal{P}) = \pi_{i+1}|S.\mathcal{P}| = \pi_{i+1}|Q\mathcal{P}|$; since the lambda operations are not additive on K_0, but any function on π_1 arising from a map would be a homomorphism, neither of these two spaces could be used to define lambda operations combinatorially.

The Adams operation ψ^k is derived from the lambda operation λ^k by a natural procedure which makes ψ^k additive on K_0. Thus there is no apparent obstruction to the presence of a combinatorial description for ψ^k that involves $|S.\mathcal{P}|$ or $|Q\mathcal{P}|$. The purpose of this paper is to present such a combinatorial construction of the Adams operation as a map of spaces $\psi^k : |S.\mathcal{P}| \to |S.\tilde{G}^{(k)}\mathcal{P}|$. The map works by considering symmetric powers of acyclic complexes of length one, and by introducing a sort of symmetric *product* of the members of a filtration of acyclic complexes. The map is a delooping of the Adams operation map

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derivable from the lambda operation maps. I don't know whether further delooping is possible without inverting some integers, and I suspect that this one-fold delooping is new, even on the level of $\mathbb{Z} \times BU$. (One may refer to [12] for methods that can be used to transfer these results to topological K-theory.)

The construction $\tilde{G}^{(k)}$ appearing in the target of the map is a $(k-1)$-dimensional cube of exact categories, each of which involves acyclic complexes of length k as well as total complexes of multi-dimensional complexes that are acyclic in two directions. It is arranged so that the target of the map is yet another space whose homotopy groups are the K-groups, and, in fact, there is a natural, combinatorially defined, homotopy equivalence $|S.\mathcal{P}| \to |S.\tilde{G}^{(k)}\mathcal{P}|$.

In [13] Schechtmann gives a construction of operations analogous to the one I present here, but it yields a homotopy class of maps rather than a single explicit map; at the expense of tensoring with the rational numbers, he shows that the Adams operations are infinite loop maps, whereas we deloop only once in this paper. Alexander Nenashev will write a paper in which he constructs lambda operations based on techniques in [5], but using long exact sequences instead of cubes, as suggested in [3]. For other discussions of lambda-operations and Adams operations on algebraic K-theory, the reader may wish to refer to [7], [8], [9], [4], and [10].

I thank Henri Gillet for useful discussions and the idea of using the secondary Euler characteristic. I thank David Benson for the definition of the symmetric power of a complex that I use; the one I was originally using was based on the theory of non-additive derived functors of Dold and Puppe, [2]. I thank Pierre Deligne and Jens Franke, who explained to me that it ought to be possible to realize the eigenspaces of the Adams operations on the rational K-groups as the rational homotopy groups of spaces; perhaps the construction of this paper is a step in that direction, and thus might help analyze the relationship between K-theory and motivic cohomology.

2. Symmetric powers of complexes and symmetric products of filtered complexes.

We will write about finitely generated projective R-modules for convenience of exposition below, but it will be apparent that any of the constructions we use will work equally well for locally free sheaves of finite type (vector bundles) on a scheme X, or for representations of a group G in finitely generated projective R modules. All tensor products will be over R.

If R is a commutative ring and M is an R-module, then the k-th symmetric power $S^k M$ of M is defined to be the quotient of $M^{\otimes k}$ by the relations

$$x_1 \otimes \cdots \otimes x_i \otimes x_{i+1} \otimes \cdots \otimes x_k \sim x_1 \otimes \cdots \otimes x_{i+1} \otimes x_i \otimes \cdots \otimes x_k.$$

Similarly, the k-th exterior power $\Lambda^k M$ of M is defined to be the quotient of $M^{\otimes k}$ by the relations

$$x_1 \otimes \cdots \otimes x_i \otimes x_{i+1} \otimes \cdots \otimes x_k \sim -x_1 \otimes \cdots \otimes x_{i+1} \otimes x_i \otimes \cdots \otimes x_k.$$

and

$$x_1 \otimes \cdots \otimes x_i \otimes x_{i+1} \otimes \cdots \otimes x_k \sim 0$$

if $x_i = x_{i+1}$. The first of these two relations follows easily from the second.

Now let M be a $\mathbb{Z}$-graded R-module, (or even a $\mathbb{Z}/2\mathbb{Z}$-graded R-module). If $x \in M_p$, then we say that x is a homogeneous element of M and that $\deg x = p$. We may mix the relations for symmetric and exterior powers mentioned above, and define the k-th symmetric power $S^k M$ of M to be the quotient of $M^{\otimes k}$ by the relations among homogeneous elements x_i of M,

$$x_1 \otimes \cdots \otimes x_i \otimes x_{i+1} \otimes \cdots \otimes x_k \sim (-1)^{\deg x_i \cdot \deg x_{i+1}} x_1 \otimes \cdots \otimes x_{i+1} \otimes x_i \otimes \cdots \otimes x_k,$$

and

$$x_1 \otimes \cdots \otimes x_i \otimes x_{i+1} \otimes \cdots \otimes x_k \sim 0$$

whenever $x_i = x_{i+1}$ and $\deg x_i$ is odd. We let $x_1 \cdot x_2 \cdot \ldots \cdot x_k$ denote the image in $S^k M$ of $x_1 \otimes \cdots \otimes x_k$.

If M is concentrated in even degrees, then $S^k M$ is the k-th symmetric power of the underlying module, and if M is concentrated in odd degrees, then $S^k M$ is the k-th exterior power.

The module $S^k M$ is itself a graded module, with

$$\deg(x_1 \cdot \ldots \cdot x_k) = \deg x_1 + \cdots + \deg x_k$$

If the graded module M is free, (which we take to mean that each component M_p is free), then we may take a basis $\{e_j\}$ for it that consists of homogeneous elements. We say that a tensor product $e_{i_1} \otimes \cdots \otimes e_{i_k}$ or its image $e_{i_1} \cdot \ldots \cdot e_{i_k}$ in $S^k M$ is a *monomial*. We may write $S^k M$ as the quotient of $M^{\otimes k}$ by the following *monomial relations*:

$$e_{j_1} \otimes \cdots \otimes e_{j_i} \otimes e_{j_{i+1}} \otimes \cdots \otimes e_{j_k} \sim (-1)^{\deg e_{j_i} \cdot \deg e_{j_{i+1}}} e_{j_1} \otimes \cdots \otimes e_{j_{i+1}} \otimes e_{j_i} \otimes \cdots \otimes e_{j_k},$$

and

$$e_{j_1} \otimes \cdots \otimes e_{j_i} \otimes e_{j_{i+1}} \otimes \cdots \otimes e_{j_k} \sim 0$$

whenever $e_{j_i} = e_{j_{i+1}}$ and $\deg e_{j_i}$ is odd. Repeated application of the first of these types of relations to a monomial will accumulate a sign which is the sign of the permutation affecting the factors of odd degree; if we are ever led thereby to a relation of form $e_{j_1} \cdot \ldots \cdot e_{j_k} \sim -e_{j_1} \cdot \ldots \cdot e_{j_k}$, then we must have a repeated factor of odd degree, so that $e_{j_1} \cdot \ldots \cdot e_{j_k} \sim 0$ is a consequence of the second relation. These remarks make it clear that sorting the factors in a monomial modulo the two relations is a well-defined operation, so that $S^k M$ is a free R-module, with a basis consisting of those monomials $e_{j_1} \cdot \ldots \cdot e_{j_k}$ such that $j_1 \leq \cdots \leq j_k$, and $j_i = j_{i+1}$ only if $\deg e_{j_i}$ is even.

Now suppose that M is a chain complex of R-modules, so it is a $\mathbb{Z}$-graded module with a differential d of degree -1. We define a differential d on $M^{\otimes k}$ by means of the usual Leibniz rule

$$d(x_1 \otimes \cdots \otimes x_k) = \sum_{i=1}^{k} (-1)^{\deg x_1 + \cdots + \deg x_{i-1}} x_1 \otimes \cdots \otimes dx_i \otimes \cdots \otimes x_k$$

and observe that this respects the relations defining the quotient $S^k M$, thereby defining a differential on $S^k M$ and making it into a chain complex.

An important special case arises when M is the mapping cone $CN = C1_N$ of the identity map on a finitely generated projective R-module N, so that M is an acyclic chain complex of length 1, with a copy of N in degrees 0 and 1. In this case one sees that $S^k CN$ is the usual Koszul complex of N, in which $(S^k M)_p = S^{k-p} M_0 \otimes \Lambda^p M_1 = S^{k-p} N \otimes \Lambda^p N$. It is known [1, p. 528] that the Koszul complex $S^k CN$ is acyclic when $k > 0$, and is the ring R concentrated in degree 0 when $k = 0$; a simple proof can be given based on the multilinearity property below (2.1), by induction on k and the rank of N.

We remark that if M is an acyclic free complex concentrated in degrees 1 and 2, then $S^k M$ is not in general acyclic. For example, with $k = 2$, one gets a complex $S^2 N \to N \otimes N \to \Lambda^2 N$ which fails to be exact in the middle because of elements like $x \otimes x$ which are not hit.

We proceed now to the next generalization. We will overload the subscript notation a bit, and use subscripts to denote both the members of a filtration and the components of a graded module. Let M be a filtered complex with k steps, so that we have complexes $M_1 \subseteq \cdots \subseteq M_k = M$. If we need it, we will refer to the degree p component of the complex M_i as M_{ip}. We define the *symmetric product* $M_1 \cdot \ldots \cdot M_k$ of M to be the image of $M_1 \otimes \cdots \otimes M_k$ in $S^k M_k$.

We will always assume that M is an *admissible* filtered complex of finitely generated projective R-modules, so that every module M_{ip} in it is a finitely generated projective module, and so that each inclusion $M_{i-1,p} \subseteq M_{ip}$ is *admissible* in the sense that its cokernel is projective. We say that M is *free* if every M_{ip} is free, and every quotient $M_{ip}/M_{i-1,p}$ is free. A *basis* for a free admissible filtered complex M will be a collection of bases for each M_{ip} that are upward compatible, and thus induce bases on the quotients $M_{ip}/M_{i-1,p}$. We remark that an admissible filtered complex M is locally free.

The symmetric product of an admissible filtered complex M can also be defined by generators and relations (and this might be a preferable definition when M is not admissible, or does not consist of projective modules). It is the quotient of $M_1 \otimes \cdots \otimes M_k$ by those relations used before where the i-th factor in the tensor is required to lie in M_i. To be precise, the relations among tensor products of homogeneous elements x_i of M are

$$x_1 \otimes \cdots \otimes x_i \otimes x_{i+1} \otimes \cdots \otimes x_k \sim (-1)^{\deg x_i \cdot \deg x_{i+1}} x_1 \otimes \cdots \otimes x_{i+1} \otimes x_i \otimes \cdots \otimes x_k$$

whenever $x_j \in M_j$ for all j, and moreover $x_{i+1} \in M_i$, and

$$x_1 \otimes \cdots \otimes x_i \otimes x_{i+1} \otimes \cdots \otimes x_k \sim 0$$

whenever $x_j \in M_j$ for all j, $x_i = x_{i+1}$ and $\deg x_i$ is odd. To prove this assertion, we may localize sufficiently to ensure that M is free, and then we may pick a basis $\{e_j\}$ for M and order it in such a way that the basis elements for M_1 come first, and then come some more elements to complete a basis for M_2, and so on. The relations mentioned suffice to sort the factors of any monomial drawn from $M_1 \otimes \cdots \otimes M_k$, and allow us to write down an explicit basis for the quotient, consisting of those monomials $e_{j_1} \otimes \cdots \otimes e_{j_k}$ where $e_{j_i} \in M_i$ for each i, $j_1 \leq \cdots \leq j_k$, and $j_i = j_{i+1}$ only if $\deg e_{j_i}$ is even. Since these monomials are a subset of the monomials that serve as basis for $S^k M_k$, and are the same monomials that span the image of $M_1 \otimes \cdots \otimes M_k$ in $S^k M_k$, we have proved our assertion.

The main fact about symmetric products of admissible filtered complexes governs what happens when one of the terms in the filtration is perturbed slightly, and is a property we will call *multilinearity*. Suppose M is an admissible filtered complex, and suppose M'_{j+1} is an alternative for the step M_{j+1} in the filtration M, which we take to mean that

$$M_1 \subseteq \cdots \subseteq M_j \subseteq M'_{j+1} \subseteq M_{j+1} \subseteq \cdots \subseteq M_k$$

is an admissible filtrations of complexes. By localizing sufficiently to make everything free, one sees that $M_1 \cdot \ldots \cdot M'_{j+1} \cdot \ldots \cdot M_k$ is an admissible subcomplex of $M_1 \cdot \ldots \cdot M_{j+1} \cdot \ldots \cdot M_k$. The multilinearity property identifies the quotient via a certain natural isomorphism:

$$\frac{M_1 \cdot \ldots \cdot M_{j+1} \cdot \ldots \cdot M_k}{M_1 \cdot \ldots \cdot M'_{j+1} \cdot \ldots \cdot M_k} \cong \big(M_1 \cdot \ldots \cdot M_j\big) \otimes \big(\frac{M_{j+1}}{M'_{j+1}} \cdot \ldots \cdot \frac{M_k}{M'_{j+1}}\big). \tag{2.1}$$

Indeed, both sides of this isomorphism are quotients of $M_1 \otimes \cdots \otimes M_k$ by various explicit relations, and all one has to do is to check that the two sets of relations are equivalent; this can be done. Another way is to localize sufficiently so that all the everything is free, pick an ordered basis $\{e_j\}$ for M compatible with the filtration as we did above, and observe that the same set of monomials gives a basis for both sides.

Here is an important corollary of the multilinearity of symmetric products. Suppose M is an *acyclic* admissible filtered complex of length 1, which we take to mean that (in addition to begin admissible) each step M_i in the filtration is an acyclic complex of length 1. I claim that the symmetric product $M_1 \cdot \ldots \cdot M_k$ is an acyclic complex. The proof goes by induction on k; making use of multilinearity and the fact that a tensor product of two acyclic complexes is acyclic allows us to modify $M_2, \ldots, M_k$ successively so that they all equal M_1, reducing us to the previously mentioned result about Koszul complexes being acyclic.

Here is an example of the symmetric product. In the case where $k = 2$ and $M = CN$ is the mapping cone of a admissible filtered module $N_1 \subseteq N_2$ we find that $M_1 \cdot M_2$ is the acyclic complex

$$0 \to N_1 \wedge N_2 \to (N_1 \otimes N_2) + (N_2 \otimes N_1) \to N_1 \cdot N_2 \to 0$$

which sits as an admissible subcomplex of the Koszul complex of N_2:

$$0 \to \Lambda^2 N_2 \to N_2 \otimes N_2 \to S^2 N_2 \to 0.$$

Here we use $N_1 \wedge N_2$ to denote the image of $N_1 \otimes N_2$ in $\Lambda^2 N_2$, and $N_1 \cdot N_2$ to denote the image of $N_1 \otimes N_2$ in $S^2 N_2$.

We have seen that the symmetric product of an admissible filtered acyclic complex of length one is a natural generalization of the Koszul complex. There is another conceivable generalization of the Koszul complex that also turns out to be acyclic, but which we do not need in the sequel; uninterested readers may skip to the beginning of the next section now. For an admissible filtration $N_1 \subseteq \cdots \subseteq N_k$ of finitely generated projective modules it looks like

$$0 \to \Lambda^k N_1 \to \cdots \to N_1 \cdot \ldots \cdot N_{k-p} \otimes \Lambda^p N_{k-p+1} \to \cdots \to N_1 \cdot \ldots \cdot N_k \to 0.$$

It can be constructed from the symmetric product $CN_1 \cdot \ldots \cdot CN_k$ by an interesting pruning procedure, which I describe now.

Suppose that a complex M of length k has a filtration $0 = M_{-1} \subseteq \cdots \subseteq M_k = M$ with the property that each quotient M_p/M_{p-1} is a complex of length p whose homology vanishes except in degree p. A new complex $\tilde{M}$, also of length k, can be defined by setting $\tilde{M}_p = H_p(M_p/M_{p-1})$. A straightforward diagram chase defines the differentials in $\tilde{M}$, shows that $\tilde{M}$ is a complex, constructs a map $\tilde{M} \to M$, and shows that the map $\tilde{M} \to M$ is a quasi-isomorphism. (This is related to the way that the skeletal filtration of a cell-complex leads to the complex of cellular chains from the complex of singular chains.) Instead of doing the diagram chase, one could regard the spectral sequence associated to M, and take $\tilde{M}$ to be the nonvanishing row of the E^1 term. We say that $\tilde{M}$ is obtained from M by *pruning*.

We may prune the symmetric product $W = CN_1 \cdot \ldots \cdot CN_k$ by means of the filtration whose p-th step is $W_p = N_1 \cdot \ldots \cdot N_{k-p} \cdot CN_{k-p+1} \cdot \ldots \cdot CN_k$. Here we regard each module N_i as a complex by concentrating it in degree 0; in this way it is a subcomplex of CN_i. By multilinearity (2.1) the quotient W_p/W_{p-1} is

$$N_1 \cdot \ldots \cdot N_{k-p} \otimes \frac{CN_{k-p+1}}{N_{k-p+1}} \cdot \ldots \cdot \frac{CN_k}{N_{k-p+1}}.$$

We may modify the latter complex so that $N_{k-p+2}, \ldots, N_k$ are successively replaced by N_{k-p+1}, without changing the quasi-isomorphism class, by using the multilinearity property with the acyclicity of complexes of the form

$$\frac{CN_{\ell+1}}{CN_\ell} \cdot \ldots \cdot \frac{CN_k}{CN_\ell}.$$

The result, after the modifications, is

$$N_1 \cdot \ldots \cdot N_{k-p} \otimes \frac{CN_{k-p+1}}{N_{k-p+1}} \cdot \ldots \cdot \frac{CN_{k-p+1}}{N_{k-p+1}} = N_1 \cdot \ldots \cdot N_{k-p} \otimes \Lambda^p N_{k-p+1}[-p].$$

We conclude that W_p/W_{p-1} has homology only in degree p, and that pruning W leads to the complex announced above.

3. The Adams operation as the secondary Euler characteristic of the Koszul complex.

Use the symbol $[N]$ to denote the class of a finitely generated projective R-module N in the Grothendieck group K_0R, or the class of a vector bundle N on a scheme X in the Grothendieck group K_0X. All complexes below will be bounded chain complexes. Let M be a complex of finitely generated projective R-modules with differential $d_p : M_{p+1} \to M_p$, and recall the Euler characteristic $\chi(M) = \sum_p (-1)^p [M_p]$. If M is acyclic, then $\chi(M) = 0$, and the secondary Euler characteristic may be defined as $\chi'(M) = \sum_p (-1)^{p+1} p[M_p]$ or as $\chi'(M) = \sum_p (-1)^p [\operatorname{im} d_p]$. If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of acyclic complexes, then $\chi'(M) = \chi'(M') + \chi'(M'')$.

We say that a bicomplex is *doubly acyclic* if each row and each column are acyclic. The tensor product of two acyclic complexes of projective modules (regarded as a bicomplex) is doubly acyclic. If M is a doubly acyclic bicomplex, and $\operatorname{Tot} M$ is its total complex, then $\chi'(\operatorname{Tot} M) = 0$; one proves this by considering the filtration on $\operatorname{Tot} M$ arising from the canonical filtration with respect to the columns and using the additivity of χ' to show that $\chi'(\operatorname{Tot} M)$ is the alternating sum of χ' of the columns of M, which is then zero because the columns of M fit into a long exact sequence. Even more is true: if $d = d' + d''$ is the differential on $\operatorname{Tot} M$, where d' and d'' are the horizontal and vertical differentials on M, then the projective modules $\operatorname{im} d_p$ may be assembled into an acyclic complex by using the maps induced by d' (or by d'') as differential. The proof (for the ring case) goes by filtering M in both directions in such a way that the successive quotients are doubly acyclic bicomplexes of size 1 by 1, in which case the statement can be checked easily.

Let ψ^k denote the k-th Adams operation on $K_0 R$ or $K_0 X$. I claim that for any N as above the following formula holds.

$$\psi^k[N] = \chi'(S^k CN) \tag{3.1}$$

We prove this by verifying, for the right hand side of the equation, the two properties that (according to the splitting principle) characterize ψ^k. Firstly, when $\operatorname{rank} N = 1$ the Koszul complex $S^k CN$ is just $C(N^{\otimes k})$, so $\chi'(S^k CN) = N^{\otimes k}$. Secondly, if $0 \to N' \to N \to N'' \to 0$ is a short exact sequence, then we can verify the additivity $\chi'(S^k CN) = \chi'(S^k CN') + \chi'(S^k CN'')$ of the right hand side by making use of multilinearity (2.1). From the filtration

$$\begin{aligned} S^k CN' &= CN' \cdot \ldots \cdot CN' \cdot CN' \\ &\subseteq CN' \cdot \ldots \cdot CN' \cdot CN \\ &\subseteq CN' \cdot \ldots \cdot CN \cdot CN \subseteq \ldots \\ &\subseteq CN \cdot \ldots \cdot CN \cdot CN = S^k CN \end{aligned}$$

we deduce that

$$\begin{aligned} \chi'(S^k CN) &= \chi'(S^k CN'') + \chi'(S^k CN') + \sum_{i=1}^{k-1} \chi'(S^i CN'' \otimes S^{k-i} CN') \\ &= \chi'(S^k CN'') + \chi'(S^k CN'). \end{aligned}$$

The cross-terms drop out because the secondary Euler characteristic of a tensor product of acyclic complexes is zero.

As an example, we may compute $\psi^2[N]$. In this case, the complex $S^2 CN$ is $0 \to \Lambda^2 N \to N \otimes N \to S^2 N \to 0$, and $\chi'(S^2 CN) = [S^2 N] - [\Lambda^2 N]$.

If we let $L_p^k N$ denote the image of d_p in the Koszul complex $S^k CN$. The functor $L_p^k N$ is the Schur functor corresponding to the Young diagram $(k-p, 1, \ldots, 1)$ of hook type. We see that $\psi^k[N] = \sum (-1)^p [L_p N]$.

We remark that formula (3.1) is like the nonstandard definition of the differential of a C^∞-map $f : M \to N$ of manifolds. If we think of M and N as being embedded manifolds containing the origin, the differential of f at the origin can be written as $(df)_0(v) =$ *standard part of* $\big(\frac{1}{\epsilon} f(\epsilon v)\big)$, where v is a vector tangent to M at 0, and ϵ is an infinitesimal number. Comparing with (3.1) we see that multiplication of v by ϵ is analogous to forming the mapping cone of the identity map on N. This suggests that we regard acyclic complexes as being infinitesimal in size when compared to arbitrary complexes, and that we regard the category of complexes as being an enlargement of the category of modules. We also see that the final step of dividing by ϵ and taking the standard part is analogous to taking the secondary Euler characteristic of an acyclic complex. The fact that terms in the expansion of $f(\epsilon v)$ involving ϵ^2 drop out when we divide by ϵ and take the standard part corresponds to the fact that doubly acyclic complexes yield 0 when we take the secondary Euler characteristic, and the two facts arise in the same way in the proof of additivity. This suggests that we regard doubly acyclic complexes as being doubly infinitesimal in size when compared to arbitrary complexes. It also suggests that we regard the Adams operation ψ^k as being the differential of the functor $N \mapsto S^k N$ from the category of finitely generated projective modules to itself; the differential is formed by first extending the domain of the functor from modules to complexes of modules, which is somehow analogous to first extending the domain of f from M to a nonstandard model of M.

4. The multi-relative S.-construction.

We let [1] denote the ordered set $\{0 < 1\}$ regarded as a category. By an n-dimensional cube $\mathcal{M}$ of (exact) categories we will mean a functor from $[1]^n$ to the category of (exact) categories.

In this section we show how, given an n-dimensional cube $\mathcal{M}$ of exact categories, we may construct a certain n-fold multisimplicial exact category called $C\mathcal{M}$ to serve as the mapping cone of the cube. In the case $n = 1$, it will be the same as a construction of Waldhausen [15, p. 343] denoted $S.(\mathcal{M}_0 \to \mathcal{M}_1)$; in [14, p. 182–184] the same construction is called $F.(\mathcal{M}_0 \to \mathcal{M}_1)$.

If $\mathcal{M}$ and $\mathcal{M}'$ are n-dimensional cubes of exact categories, we let $\operatorname{Exact}(\mathcal{M}, \mathcal{M}')$ denote the set of natural transformations $\mathcal{M} \to \mathcal{M}'$.

If $\mathcal{M}$ is an exact category, we let $[\mathcal{M}]$ denote the corresponding 0-dimensional cube of exact categories. We will often simply identify $\mathcal{M}$ with $[\mathcal{M}]$.

Given n-dimensional cubes $\mathcal{M}$ and $\mathcal{M}'$ of exact categories, and an exact functor $g \in \operatorname{Exact}(\mathcal{M}, \mathcal{M}')$, we may assemble $\mathcal{M}'$ and $\mathcal{M}$ into an $n+1$-dimensional cube of exact categories; we will use the symbol $[\mathcal{M}' \to \mathcal{M}]$ to denote it. We will also use square brackets enclosing a commutative square of n-dimensional cubes of exact categories to denote the corresponding $n+2$ dimensional cube.

If $\mathcal{M}$ is an n-dimensional cube of categories and $\mathcal{M}'$ is an n'-dimensional cube of categories, then we let $\mathcal{M} \boxtimes \mathcal{M}'$ denote the evident $n+n'$-dimensional cube of categories defined by

$$(\mathcal{M} \boxtimes \mathcal{M}')(\epsilon_1, \dots, \epsilon_{n+n'}) = \mathcal{M}(\epsilon_1, \dots, \epsilon_n) \times \mathcal{M}'(\epsilon_{n+1}, \dots, \epsilon_{n+n'}).$$

We let Δ denote the category of finite nonempty totally ordered sets. If $\mathcal{C}$ is a category, let $\operatorname{Ar}(\mathcal{C})$ denote the category of arrows in $\mathcal{C}$, where an arrow in this category is a commutative square. If A is an ordered set regarded as a category, we will use j/i to denote the arrow from i to j in A, if $i \leq j$.

Given an exact category $\mathcal{M}$ with a chosen zero object 0 and an ordered set A, we call a functor $F : \mathrm{Ar}(A) \to \mathcal{M}$ *exact* if $F(i/i) = 0$ for all i, and $0 \to F(j/i) \to F(k/i) \to F(k/j) \to 0$ is exact for all $i \leq j \leq k$. The set of such exact functors is denoted by $\mathrm{Exact}(\mathrm{Ar}(A), \mathcal{M})$. Given ordered sets $A_1, \ldots, A_n$, we let $\mathrm{Exact}(\mathrm{Ar}(A_1) \times \cdots \times \mathrm{Ar}(A_n), \mathcal{M})$ denote the set of *multi-exact* functors, i.e., functors that are exact in each variable.

Given $A, B \in \Delta$ let AB denote the totally ordered set constructed from A and B by concatenation, i.e., as the disjoint union of A and B with every element of A declared to be less than every element of B.

Now let L be a symbol, and consider $\{L\}$ to be an ordered set. Given an n-dimensional cube of exact categories $\mathcal{M}$, we define an n-fold multisimplicial exact category $C\mathcal{M}$ as a functor from $(\Delta^n)^{\mathrm{op}}$ to the category of exact categories by letting $C\mathcal{M}(A_1, \ldots, A_n)$ be the set

$$\mathrm{Exact}([\mathrm{Ar}(A_1) \to \mathrm{Ar}(\{L\}A_1)] \boxtimes \cdots \boxtimes [\mathrm{Ar}(A_n) \to \mathrm{Ar}(\{L\}A_n)], \mathcal{M})$$

of multi-exact natural transformations. When $n = 0$, we may identify $C\mathcal{M}$ with $\mathcal{M}$. We define $S.\mathcal{M}$ to be $S.C\mathcal{M}$, the result of applying the $S.$ construction of Waldhausen degreewise. The construction $S.\mathcal{M}$ is a $n+1$-fold multisimplicial set; to make that explicit we write the new argument to the left of the other ones, and see that

$$S.\mathcal{M}(A_0, A_1, \ldots, A_n) = \mathrm{Exact}([\mathrm{Ar}(A_0)] \boxtimes [\mathrm{Ar}(A_1) \to \mathrm{Ar}(\{L\}A_1)] \boxtimes \cdots \boxtimes [\mathrm{Ar}(A_n) \to \mathrm{Ar}(\{L\}A_n)], \mathcal{M})$$

for $A_0, \ldots, A_n \in \Delta$.

Lemma 4.1. *Suppose we are given $\mathcal{M}' \to \mathcal{M}$ as above.*

(a) There is a fibration sequence $S.[0 \to \mathcal{M}] \to S.[\mathcal{M}' \to \mathcal{M}] \to S.[\mathcal{M}' \to 0]$.

(b) In the case where g is the identity map, the space $S.[\mathcal{M} \to \mathcal{M}]$ is contractible.

(c) $S.[0 \to \mathcal{M}]$ is homotopy equivalent to $S.\mathcal{M}$.

(d) $S.[\mathcal{M} \to 0]$ is a delooping of $S.\mathcal{M}$.

(a) There is a fibration sequence $S.\mathcal{M}' \to S.\mathcal{M} \to S.[\mathcal{M}' \to \mathcal{M}]$.

Proof. One uses the additivity theorem of Waldhausen, just as in [15, p. 343] or [14, p. 182–184]. □

We remark that if

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ T & \longrightarrow & U \end{array}$$

is a square of commutative rings, then tensor product of projective modules leads immediately to a map

$$|S.[\mathcal{P}(R) \to \mathcal{P}(S)]| \wedge |S.[\mathcal{P}(R) \to \mathcal{P}(T)]| \to \left| S.S. \begin{bmatrix} \mathcal{P}(R) & \longrightarrow & \mathcal{P}(S) \\ \downarrow & & \downarrow \\ \mathcal{P}(T) & \longrightarrow & \mathcal{P}(U) \end{bmatrix} \right|$$

which can be used to define products on relative K-groups.

5. The construction of the Adams operations.

For the construction of the Adams operation Ψ^k on the K-groups of an exact category $\mathcal{P}$ we will need to consider k-dimensional multi-complexes N of length one in each direction, and to take total complexes of them in a certain partial way. These "partial" total complexes will have fewer dimensions than N has, and their lengths will be greater; the "total" total complex of N will be of dimension 1 and length k.

We describe now what sort of "partial" total complexes we have in mind.

An equivalence relation φ on a totally ordered set $A \in \Delta$ is *compatible with the ordering* if the quotient set A/φ inherits an ordering from A so that the quotient map is order-preserving. If we denote the equivalence classes by $A_1, \dots, A_t$, then we may write A as the concatenation $A = A_1 A_2 \cdots A_t$, and the quotient A/φ as the ordered set $A/\varphi = \{A_1, \dots, A_t\}$.

Let N be a multi-complex whose directions are indexed by the elements of A. We assume that the differentials anti-commute with each other, i.e., $\partial_i \partial_j + \partial_j \partial_i = 0$; this ensures that when taking total complexes, the sum of the differentials immediately provides a differential. A homogeneous element $x \in N$ has a *multi-degree* $p : A \to \mathbb{Z}$ which is a sequence of integers indexed by the set A, and we let N_p denote the set of homogeneous elements of N of multi-degree p, together with 0. Define $B = A/\varphi$, and let $\pi : A \to B$ be the quotient map. We may define a multi-complex $N' = \operatorname{Tot}_\varphi N$ whose directions are indexed by B by specifying $\pi_* p = q : B \to \mathbb{Z}$, the degree of x as an element of N'. It will be given by the formula $q(b) = \sum_{a \in \varphi^{-1}(b)} p(a)$. This corresponds to setting

$$N'_q = \sum_{\pi_*(p) = q} N_p.$$

Let's use $[1, k]$ as notation for the ordered set $\{1, 2, \dots, k\}$. The number of equivalence relations on $[1, k]$ compatible with the ordering is 2^{k-1}, as such relations are freely and completely specified by the truth or falsity of the statements $i \approx_\varphi i+1$ for $i = 1, \dots, k-1$. We may consider the set of equivalence relations on $[1, k]$ compatible with the ordering to be a set of subsets of $[1, k] \times [1, k]$, and order it by inclusion. It is isomorphic, as a partially ordered set, to $[1]^{k-1}$. We use the isomorphism that associates $(\epsilon_1, \dots, \epsilon_{k-1})$ to φ, where

$$\epsilon_i = \begin{cases} 0 & \text{if } i \not\approx_\varphi i+1 \\ 1 & \text{if } i \approx_\varphi i+1. \end{cases}$$

For each equivalence relation φ on $[1, k]$ compatible with the ordering, we let $\ell_1, \dots, \ell_t$ denote the cardinalities of the equivalence classes, in sequence. Consider the category $\mathcal{M}_\varphi$ of t-dimensional chain-complexes that are, for each i, of length ℓ_i in the i-th direction, and that are acyclic in direction 1 and in direction t. There is a total-complex functor $\mathcal{M}_\varphi \to \mathcal{M}_\psi$ if $\varphi \subseteq \psi$, because the lengths add when total complexes are constructed, and because the total complex of a multi-complex that is acyclic in one direction is acyclic. Using these total-complex functors we may assemble the categories $\mathcal{M}_\varphi$ into a $k-1$ dimensional cube $\tilde{G}^{(k)} \mathcal{P} = \mathcal{M}$ of exact categories which will serve as the target for our Adams operation map ψ^k.

Actually, there is a little problem with getting $\tilde{G}^{(k)}\mathcal{P}$ to be a *functor* from $[1]^{k-1}$ to the category of exact categories, because the composition of two total-complex functors is perhaps only isomorphic to the combined total-complex functor; this is something like failure of strict associativity for direct sums, and can be cured with an easy set-theoretic trick, or by considering $\tilde{G}^{(k)}\mathcal{P}$ instead to be a category cofibered over $[1]^{k-1}$ in exact categories.

We remark that there is homotopy equivalence $\tilde{G}^{(2)}\mathcal{P} \to G\mathcal{P}$ that associates to an acyclic complex of length 2, the images of the two differentials in it. The map ψ^2 can be viewed as a map $|S.\mathcal{P}| \to |S.G\mathcal{P}|$, and it was this version which was found first, and motivated the more general construction described in this paper.

Lemma 5.1. *$S.\tilde{G}^{(k)}\mathcal{P}$ is homotopy equivalent to $S.\mathcal{P}$*

Proof. Consider the edges of the cube $\tilde{G}^{(k)}\mathcal{P}$ that lie in direction 1. These edges are total complex functors $\mathcal{M}_\varphi \to \mathcal{M}_\psi$ where the only difference between φ and ψ is that $1 \not\approx_\varphi 2$ and $1 \approx_\psi 2$.

Consider first the case where $2 \approx_\varphi 3 \approx_\varphi \cdots \approx_\varphi k$. The category $\mathcal{M}_\varphi$ is the category of bicomplexes of length 1 in direction 1, of length $k-1$ in direction 2, and acyclic in both directions. It is equivalent to the category of acyclic complexes of length $k-1$, and the functor $\mathcal{M}_\varphi \to \mathcal{P}^{k-2}$ that assigns to an acyclic complex the collection of images of its differentials yields a homotopy equivalence on K-theory, by the additivity theorem. The category $\mathcal{M}_\psi$ is the category of acyclic complexes of length k. The functor $\mathcal{M}_\psi \to \mathcal{P}^{k-1}$ that assigns to an acyclic complex the collection of images of its differentials yields a homotopy equivalence on K-theory, by the additivity theorem. Let $C : \mathcal{P} \to \mathcal{M}_\psi$ be the functor that assigns CP to $P \in \mathcal{P}$, regarded as an acyclic complex of length k. Then the map $S.[0 \to \mathcal{P}] \to S.[\mathcal{M}_\varphi \to \mathcal{M}_\psi]$ is a homotopy equivalence.

Consider now the other case, where there exists $j \geq 2$ so that $j \not\approx_\varphi j+1$; we claim that $S.\mathcal{M}_\varphi \to S.\mathcal{M}_\psi$ is a homotopy equivalence. This again is a straightforward application of the additivity theorem, just as in the previous paragraph. It is enlightening to regard the additivity theorem itself as a statement something like the one at hand: it says that the total complex functor from the category of one-by-one bicomplexes, acyclic in direction 1, to the category of acyclic complexes of length 2, gives a homotopy equivalence on K-theory.

Combining both cases, we see that we have a map $S.\mathcal{P} \to S.\tilde{G}^{(k)}\mathcal{P}$, obtained by adding additional trivial simplicial directions to $S.\mathcal{P}$, which is a homotopy equivalence. □

6. The construction of the map.

In this section we give the formula for the combinatorial Adams operation map

$$\Psi^k : \operatorname{Sub}_k S.\mathcal{P} \to S.\tilde{G}^{(k)}\mathcal{P}.$$

Here Sub_k is the k-fold subdivision introduced in [5]: if X is a simplicial set, then $\operatorname{Sub}_k X$ is the k-fold multisimplicial set defined by

$$\operatorname{Sub}_k X(A_1, \ldots, A_k) = X(A_1 \cdots A_k).$$

There is a natural homeomorphism $|X| \simeq |\operatorname{Sub}_k X|$, presented in [5]. Here is a way to see how that homeomorphism works. Let V be an affine space of dimension n (torsor under $\mathbb{R}^n$). Given points $v_1, \dots, v_k \in V$ define their *barycenter* $v_1 * \cdots * v_k$ to be the point $(v_1 + \cdots + v_k)/k$. If $S_1, \dots, S_k$ are subsets of V, then we let $S_1 * \cdots * S_k$ denote the set $\{v_1 * \cdots * v_k \mid v_i \in S_i\}$.

If A is a set $\{v_0, \dots, v_p\} \subseteq V$, let $\overline{A}$ denote the convex hull of A. If the vectors in A are affinely independent, then $\overline{A}$ is a p-simplex. Let B and C be subsets of A; we write $B \mid C$ if $i \leq j$ for all $v_i \in B$ and all $v_j \in C$. Given subsets $B_1 \mid \cdots \mid B_k$ of A, the set $\overline{B_1} * \cdots * \overline{B_k}$ is a product of simplices, and such sets subdivide $\overline{A}$ in exactly the same way that $|\operatorname{Sub}_k X|$ subdivides each simplex of $|X|$.

Given

$$M \in \operatorname{Sub}_k S.\mathcal{P}(A_1, \dots, A_k) = \operatorname{Exact}(\operatorname{Ar}(A_1 \cdots A_k), \mathcal{P})$$

we define

$$\Psi^k M \in \operatorname{Exact}([\operatorname{Ar}(A_1)] \boxtimes [\operatorname{Ar}(A_2) \to \operatorname{Ar}(\{L\}A_2)] \boxtimes \cdots \boxtimes [\operatorname{Ar}(A_k) \to \operatorname{Ar}(\{L\}A_k)], \tilde{G}^{(k)}\mathcal{P})$$

by the formula

$$(\Psi^k M)(i_1/j_1, \dots, i_k/j_k) = CM(i_1/\ell_1) *_2 CM(i_2/\ell_2) *_3 \cdots *_k CM(i_k/\ell_k). \tag{6.1}$$

Here $i_1/j_1 \in \operatorname{Ar}(A_1)$, and $i_r/j_r \in \operatorname{Ar}(\{L\}A_r)$ for $2 \leq r \leq k$. We define

$$*_r = \begin{cases} \cdot & \text{if } j_r = L \\ \otimes & \text{if } j_r \neq L \end{cases}$$

and

$$\ell_r = \begin{cases} \ell_{r-1} & \text{if } j_r \notin A_r \text{ and } r > 1 \\ j_r & \text{if } j_r \in A_r \text{ or } r = 1 \end{cases}$$

We spell out the needless conditions concening $r = 1$ and $r > 1$ so the same definition will work below, in a context where $j_1 \notin A_1$ is possible. Notice that $j_r \notin A_r$ is equivalent to $j = L$, for $r > 1$. Finally, one must interpret the symbols $\otimes$ arising in (6.1) as instances of the symbol $*_r$ correctly: they are tensor products of acyclic complexes, but are to be interpreted as yielding bicomplexes if we are looking at $\operatorname{Ar}(A_r)$, or as yielding complexes if we are looking at $\operatorname{Ar}(\{L\}A_r)$; this builds into the notation the business with all the total-complex functors. One checks that $\Psi^k M$ is exact in the variables i_r/j_r using the multilinearity property (2.1), just as in [5].

On the level of the Grothendieck group, the secondary Euler characteristic gives the inverse to the isomorphism $K_0\mathcal{P} \to K_0\tilde{G}^{(k)}\mathcal{P}$. Combining this with the formula (3.1) we see that our map Ψ^k agrees with the usual Adams operation on $K_0\mathcal{P}$.

We now check that our Adams operations agree with the usual ones on the higher K-groups of a ring R. Consider the fibration sequence

$$G.\mathcal{P} \to P.\mathcal{P} \to S.\mathcal{P}$$

from [3] which holds for any exact category $\mathcal{P}$. For reference, we state the definitions, where $A \in \Delta$.

$$G.\mathcal{P}(A) = \operatorname{Exact}(\operatorname{Ar}(\{L,R\}A), \mathcal{P})$$
$$P.\mathcal{P}(A) = \operatorname{Exact}(\operatorname{Ar}(\{L\}A), \mathcal{P})$$
$$S.\mathcal{P}(A) = \operatorname{Exact}(\operatorname{Ar}(A), \mathcal{P})$$

Here we regard $\{L, R\}$ as an partially ordered set where L and R are incomparable symbols, and interpret the concatenation $\{L, R\}A$ for $A \in \Delta$ as a concatenation of partially ordered sets, yielding a partially ordered set; it was called $\gamma(A)$ in [5]. The definition of Ψ^k given in (6.1) applies unchanged to each term of this fibration sequence, except that now $j_1 \notin A_1$ becomes a possibility, for we may have $j_1 = L$ or $j_1 = R$. The result is the following map of fibrations.

$$\begin{array}{ccccc} \mathrm{Sub}_k\, G.\mathcal{P} & \longrightarrow & \mathrm{Sub}_k\, P.\mathcal{P} & \longrightarrow & \mathrm{Sub}_k\, S.\mathcal{P} \\ {\scriptstyle \Psi^k}\downarrow & & {\scriptstyle \Psi^k}\downarrow & & {\scriptstyle \Psi^k}\downarrow \\ G.\tilde{G}^{(k)}\mathcal{P} & \longrightarrow & P.\tilde{G}^{(k)}\mathcal{P} & \longrightarrow & S.\tilde{G}^{(k)}\mathcal{P} \end{array}$$

Having transferred our construction of Ψ^k to the level of the G-construction, one may use methods just like those of [5] to prove that our Adams operation agrees with the one defined by Quillen in [8], or those defined in [9].

One should be able to show directly, for any exact category $\mathcal{P}$ with suitable tensor products and exterior power operations, that the Adams operations on the K-groups constructed here agree with those deduced from the lambda operations constructed in [5].

One striking feature of the construction of Ψ^k is the definition of the category $\tilde{G}^{(k)}\mathcal{P}$, in which the multi-dimensional complexes are required only to be acyclic in the first direction and the last direction. The map Ψ^k, on the other hand, involves only tensor products of generalized Koszul complexes, so the multi-dimensional complexes occurring in it are acyclic in every direction. One might imagine refining the map by changing the definition of $\tilde{G}^{(k)}\mathcal{P}$ accordingly. This would lead to a space which contains various deloopings of the K-theory space for $\mathcal{P}$, and thus might lead to Adams operations maps that decrease the degree, $K_i\mathcal{P} \to K_{i-j}\mathcal{P}$. I think that all such maps might well be zero, and so these maps will turn out to be simply spurious.

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