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The K-Theory of Semilinear Endomorphisms

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In this paper we study the K-theory of semilinear endomorphisms and automorphisms over noncommutative rings. For commutative rings and linear endomorphisms we did this in [G3].

In Section 4 we produce an exact sequence (4.6) involving the K-groups of semilinear automorphisms over a field. The main tool is the introduction of the "twisted projective line," together with the fact that it admits an interesting localization at $\{0, \infty\}$. In Section 5 we use the Frobenius on an algebraically closed field to produce an example of a semilocal domain B with nonzero radical J so that $K_i(B) \cong K_i(B/J)$, $i > 0$.

In Sections 1 and 2 we give another application of the twisted projective line: we prove the natural generalization (2.3) to the higher K-groups of the results of Farrell and Hsiang [FH] about Whitehead groups of twisted Laurent polynomial rings. The proof is a straightforward rewriting of Quillen's proof of the Fundamental Theorem [G2] (in which the adjoined variable was central). The difference between our proof and Ranicki's proof in [R, pp. 427–428] is that we emphasize the role of the twisted projective line, and we identify the group $F_i(\varphi)$ as the homotopy group of the homotopy fiber of the map $1-\varphi^*$.

Other proofs are available. When the ground ring is regular noetherian, the theorem is an exercise in [Q1, pp. 114–122]. One could also obtain a proof by rewriting the proof of Theorem 18.1 of [W], which is much more general.

1. THE TWISTED PROJECTIVE LINE

A *right denominator set* S in a ring R is a subset with the following properties [St, p. 52]:

(S1) $1 \in S$,

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(S2) $s_1, s_2 \in S \Rightarrow s_1 s_2 \in S$,

(S3) $s_1 \in S,\ a \in R \Rightarrow \exists b \in R,\ s_2 \in S\colon s_1 b = a s_2$,

(S4) $s_1 \in S,\ a \in R,\ s_1 a = 0 \Rightarrow \exists s_2 \in S\colon a s_2 = 0$.

These conditions are the most general which ensure that the ring of right fractions RS^{-1} exists. If the elements of S are nonzero divisors (as will be the case here) then (S4) can be omitted. If $S = \{s^n : n \geqslant 0\}$ for some s, we write $R[s^{-1}] = RS^{-1}$.

The axioms for a *left denominator set* are analogous. If S is both a right and a left denominator set, we will call it a *denominator set*; in this case the ring of left fractions $S^{-1}R$ is isomorphic to RS^{-1}.

We let k be a (not necessarily commutative) ring, and φ an automorphism of k. The twisted polynomial ring $R^+ := k[T; \varphi]$ is the ring of polynomials $a_n T^n + \cdots + a_0$, $a_i \in k$, where multiplication satisfies a $T = T\varphi(a)$. The multiplicative set generated by T is a denominator set, so the localization $R^{\pm} := k[T, T^{-1}; \varphi] := k[T; \varphi][T^{-1}]$ is defined; we see that $k[T^{-1}, (T^{-1})^{-1}; \varphi^{-1}] = k[T, T^{-1}; \varphi]$, so $R^{\pm}$ is also a localization of $R^- := k[T^{-1}; \varphi^{-1}]$.

We define a *right X-module M* to be a triple $M = (M^+, M^-, \theta_M)$, where M^+ is a right R^+-module, M^- is a right R^--module, and $\theta_M = M^+[T^{-1}] \simeq M^-[(T^{-1})^{-1}]$ is an isomorphism of right $R^{\pm}$-modules. Here $X = \mathbb{P}^1(\varphi)$ denotes the "twisted projective line" with respect to k and φ and remains undefined. A *map* $f\colon M_1 \to M_2$ of X-modules is a pair $f^+\colon M_1^+ \to M_2^+$ $f^-\colon M_1^- \to M_2^-$ of homomorphisms with $\theta_M \cdot f^+ = f^- \cdot \theta_M$.

The category of right X-modules is an abelian category. Let $\mathscr{M}_x$ denote the exact category of right X-modules M for which M^+ and M^- are finitely generated; it is an abelian category if R^+ and R^- are noetherian, and thus if k is noetherian (according to [FH, Lemma 24]). Let $\mathscr{P}_R$ denote the exact category of finitely generated projective right R-modules, and let $\mathscr{P}_X$ be the exact category of "vector bundles on X," i.e., those X-modules M where $M^+ \in \mathscr{P}_{R^+}$ and $M^- \in \mathscr{P}_{R^-}$. Let

$$K_* X := K_* \mathscr{P}_X.$$

If R is R^+, R^-, or $R^{\pm}$, then φ extends to an automorphism of R by setting $\varphi(T) = T$. Tensor product gives an exact functor $\varphi^*\colon \mathscr{P}_R \to \mathscr{P}_R$. Define $N\langle n\rangle = (\varphi^{-n})^*(N)$ for $N \in \mathscr{P}_R$ and $n \in \mathbb{Z}$. One may also obtain $N\langle n\rangle$ from N by replacing the scalar multiplication with $x * f = x\varphi^n(f)$ for $x \in N$ and $f \in R$. If M is an X-module, we let $M\langle n\rangle = (M^+\langle n\rangle, M\ \ \langle n\rangle, \theta_M)$.

For k-modules V and W, a φ-semilinear map $f\colon V \to W$ is an additive map satisfying $f(va) = f(v)\,\varphi(a)$ for $v \in V$, $a \in k$. This is the same as a

k-linear map $V \to W\langle 1\rangle$. If M is an R^+-module, then right multiplication by T on M is a φ-semilinear endomorphism of the k-module underlying M, and all φ-semilinear endomorphisms of k-modules arise this way.

If M is an X-module, we define $M(n) := (M^+, M^-\langle -n\rangle, \theta_M \circ \rho(T^{-n}))$, where $\rho(T^{-n})$ denotes right multiplication by T^{-n}. One checks that $M(n) \in \mathscr{P}_X$, and $M(m)(n) = M(m+n)$.

If V is a k-module, define an X-module $V(0) := (V \otimes_k R^+, V \otimes_k R^-, 1)$, and X-modules $V(n) := V(0)(n)$. Let $h_n\colon \mathscr{P}_k \to \mathscr{P}_X$ denote the exact functor $h_n(V) = V(n)$.

THEOREM 1.1. *The map*

$$(h_{0*}, h_{1*})\colon K_i k \oplus K_i k \to K_i X$$

is an isomorphism. The relation $h_{m*} + h_{m+2}\langle 1\rangle_* = h_{m+1,*} + h_{m+1}\langle 1\rangle_*$ *holds for all* $m \in \mathbb{Z}$.

Proof. The proof can be done essentially as in [Q1, Theorem 3.1, Sect. 8, p. 143]; the only change is that T is no longer central. Multiplication by T on an R^+-module N is no longer an R-linear endomorphism of N, but is an R-linear map $N \to N\langle 1\rangle$. Thus, one rewrites Quillen's proof by inserting notations like "$\langle n\rangle$" in appropriate spots to preserve linearity of the maps involved. For example, the canonical exact sequence

$$0 \to \mathcal{O}(m) \to \mathcal{O}(m+1)^2 \to \mathcal{O}(m+2) \to 0$$

becomes

$$0 \to V(m) \to V(m+1) \oplus V(m+1)\langle 1\rangle \to V(m+2)\langle 1\rangle \to 0$$

for any $V \in \mathscr{P}_k$, $m \in \mathbb{Z}$. Q.E.D.

2. LOCALIZATION

In this section we discuss localization theorems for K-theory in the twisted projective line. This allows us to relate the K-groups of the projective line with those of R^+, R^-, and $R^\pm$. In the commutative case, the result obtained is the "Fundamental Theorem" of Bass, generalized by Quillen to the higher K-groups. In the case at hand, we obtain the result of Farrell and Hsiang and generalize it to apply to the higher K-groups.

Let $\mathscr{H}^+$ denote the exact category of X-modules M which admit a resolution of length 1 by vector bundles of X and for which $M^- = 0$. This category is equivalent to the category of finitely generated R^+-modules N

of projective dimension 1 such that $N[T^{-1}]=0$, for a resolution of N may be begun with a free R^+-module (which extends to X). Observe that for any right R^+-module P, the subgroup $P \cdot T^i$ is an R^+-submodule; moreover, if $P=R^+$, then $P/P \cdot T^i$ is a free k-module on the generators $1, T, ..., T^{i-1}$. Now the argument of [G2, p. 236] shows that any N in $\mathscr{H}^+$ is projective as k-module, so $\mathscr{H}^+$ is equivalent to the category $\underline{\text{Nil}}(\varphi)$ whose objects are pairs (V, f) with $V \in \mathscr{P}_k$ and $f\colon V \to V$ a nilpotent φ-semilinear endomorphism, $f(va)=f(v)\,\varphi(a)$. (In the untwisted case $\varphi=1$ this equivalence is implicit in [B, proof of the fundamental theorem] and explicit in [Si].)

The exact functors

$$\begin{array}{ll} \mathscr{P}_k \to \underline{\text{Nil}}(\varphi), & \underline{\text{Nil}}(\varphi) \to \mathscr{P}_k \\ V \mapsto (V, 0), & (V, f) \mapsto V \end{array}$$

allow one to split

$$K_i \underline{\text{Nil}}(\varphi) = K_i k \oplus \text{Nil}_i(\varphi),$$

defining $\text{Nil}_i(\varphi)$.

The ring homomorphisms

$$\begin{array}{ll} k \to R^+, & R^+ \to k \\ a \to a, & f(T) \to f(0) \end{array}$$

allow one to split

$$K_i R^+ = K_i k \oplus NK_i(\varphi),$$

defining $NK_i(\varphi)$. Similarly,

$$K_i R^- = K_i k \oplus NK_i(\varphi^{-1}).$$

THEOREM 2.1. *There are localization exact sequences*

(a) $\cdots K_{i+1} R^{\pm} \to K_i \mathscr{H}^+ \to K_i R^+ \to K_i R^{\pm} \cdots$,

(b) $\cdots K_{i+1} R^{-} \to K_i \mathscr{H}^+ \to K_i X \to K_i R^{-} \cdots$, *and*

(c) $NK_i(\varphi^{-1}) = \text{Nil}_{i-1}(\varphi)$.

Proof. Part (a) was proved in [G1]. For part (b) one checks that the proof in [G2, Theorem on p. 222] can be carried over into this context, using the preliminary material about the twisted projective line presented above. One interprets the notation from [G2] as follows:

$$\begin{aligned} j^*M &:= M^- \\ j_*M^- &:= (M^- \otimes R^\pm, M^-, 1) \\ I^{-n}M &:= (M^+ \cdot T^{-n}, M^-, 1) \\ &\subseteq j_* j^*M. \end{aligned}$$

Part (c) follows from (b) as in [G2]. Q.E.D.

Remark 2.2. If k is commutative, or if we are given an isomorphism $k \cong k^{\mathrm{op}}$, then there is an isomorphism $(R^+)^{\mathrm{op}} \cong R^-$. It follows from [Q1, (13) on p. 104] that $K_iR^+ = K_iR^-$, and thus $NK_i(\varphi) \cong NK_i(\varphi^{-1})$ and $\mathrm{Nil}_i(\varphi) \cong \mathrm{Nil}_i(\varphi^{-1})$. There is also an equivalence $\underline{\mathrm{Nil}}(\varphi)^{\mathrm{op}} \simeq \mathrm{Nil}(\varphi^{-1})$ defined by $(V, f) \mapsto (V^*, f^t)$, where $V^* = \mathrm{Hom}_k(V, k)$ and $f^t = \varphi_*^{-1} \circ f^*$. The isomorphism $\mathrm{Nil}_i(\varphi) \cong \mathrm{Nil}_i(\varphi_*^{-1})$ that this equivalence provides is probably the same as the other one.

Remark. One can use Quillen's dévissage and resolution theorems to prove that $\mathrm{Nil}_*(\varphi) = 0$ when k is regular noetherian, thereby recovering his result that $K_iR^+ \cong K_ik$.

Define $F_i(\varphi) = \pi_i\Omega(K(k) \to^{1-\varphi^*} K(k))$, where $K(k)$ is the space $\Omega BQ\mathscr{P}_k$, whose homotopy groups are the K-groups, and where $\Omega(X \to Y)$ denotes the homotopy fiber of a map. If $\varphi = 1$, then $F_i(\varphi) = K_i(k) \oplus K_{i+1}(k)$. Notice, also, that $F_i(\varphi^{-1}) = F_i(\varphi)$.

THEOREM 2.3. *There is, for $i \geqslant 1$, a canonical isomorphism*

$$K_iR^\pm \cong F_{i-1}(\varphi) \oplus \mathrm{Nil}_{i-1}(\varphi) \oplus \mathrm{Nil}_{i-1}(\varphi^{-1}).$$

Remark. For $i = 1$, this theorem was proved by Farrell and Hsiang and by Siebenmann.

Proof. There is a restriction map from the sequence 2.1(b) to 2.1(a), which is the identity on $K_i\mathscr{H}^+$. A diagram chase yields a Mayer–Vietoris-type exact sequence

$$\cdots K_{i+1}R^\pm \to K_iX \to K_iR^+ \oplus K_iR^- \to K_iR^\pm \cdots.$$

We rewrite the terms using (1.1) and 2.1(c) yielding

$$\begin{array}{c} \vdots \\ K_{i+1}R^{\pm} \\ \big\downarrow{\scriptstyle B} \\ K_ik \oplus K_ik \\ \big\downarrow{\scriptstyle A} \\ K_ik \oplus \mathrm{Nil}_{i-1}(\varphi^{-1}) \oplus K_ik \oplus \mathrm{Nil}_{i-1}(\varphi) \\ \big\downarrow \\ K_iR^{\pm} \\ \vdots \end{array}$$

The matrix of the map A is seen to be

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 1 & \varphi^* \\ 0 & 0 \end{pmatrix}$$

and because $AB=0$, we see that the matrix of B is

$$\begin{pmatrix} g \\ -g \end{pmatrix}$$

for some map g. This allows us to split off a K_ik factor, yielding

$$\cdots K_{i+1}R^{\pm} \xrightarrow{g} K_ik \xrightarrow{\begin{pmatrix} 1-\varphi^* \\ 0 \\ 0 \end{pmatrix}} K_ik \oplus \mathrm{Nil}_{i-1}(\varphi^{-1}) \oplus \mathrm{Nil}_{i-1}(\varphi) \to K_iR^{\pm} \cdots.$$

Consider the diagram

$$\begin{array}{ccccc} & & K_{i+1}(R^-) & \to & K_i(\mathscr{H}^+) \\ & & \downarrow & & \| \\ K_{i+1}(R^+) & \to & K_{i+1}(R^{\pm}) & \to & K_i(\mathscr{H}^+) \\ \downarrow & & \downarrow & & \\ K_i(\mathscr{H}^-) & = & K_i(\mathscr{H}^-) & & \end{array}$$

with exact rows and columns. Application of the decompositions we know so far gives

$$\begin{array}{ccccc}
 & & K_{i+1}k \oplus \mathrm{Nil}_i(\varphi) & \xrightarrow{\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)} & K_i k \oplus \mathrm{Nil}_i(\varphi) \\
 & & \downarrow & & \| \\
K_{i+1}k \oplus \mathrm{Nil}_i(\varphi^{-1}) & \to & K_{i+1}R^{\pm} & \longrightarrow & K_i k \oplus \mathrm{Nil}_i(\varphi) \\
{\scriptstyle \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)}\downarrow & & \downarrow & & \\
K_i k \oplus \mathrm{Nil}_i(\varphi^{-1}) & = & K_i k \oplus \mathrm{Nil}_i(\varphi^{-1}) & &
\end{array}$$

It follows that

$$K_{i+1}R^{\pm} \cong ? \oplus \mathrm{Nil}_i(\varphi) \oplus \mathrm{Nil}_i(\varphi^{-1})$$

and we get an exact sequence

$$\cdots \to ? \to K_i k \xrightarrow{1-\varphi^*} K_i k \to \cdots.$$

In order to identify '?" with $F_i(\varphi)$, we argue with the underlying spaces. We get a map of fibrations

$$\begin{array}{ccccc}
\Omega K(R^{\pm}) & \to K(k) & \xrightarrow{\left(\begin{smallmatrix}1-\varphi^* \\ \text{pt.} \\ \text{pt.}\end{smallmatrix}\right)} & K(k) \times NK(\varphi^{-1}) \times NK(\varphi) \\
{\scriptstyle s}\uparrow\downarrow{\scriptstyle t} & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle pr_1} \\
F(\varphi) & \to K(k) & \xrightarrow{(1-\varphi^*)} & K(k)
\end{array}$$

where the notations NK and F for spaces ought to be self-explanatory. The existence of the section s follows from Lemma 2.4 below. The spaces here are homotopy-everything H-spaces with additive inverses, so we may split

$$\Omega K(R^{\pm}) \cong F(\varphi) \times \Omega(t).$$

Moreover, the homotopy fibers of the three vertical maps above form a fibration which tells us that

$$\Omega(t) \cong \Omega NK(\varphi^{-1}) \times \Omega NK(\varphi).$$

Thus

$$\Omega K(R^{\pm}) \cong F(\varphi) \times \Omega NK(\varphi^{-1}) \times \Omega NK(\varphi).$$

Taking homotopy groups yields the result. Q.E.D.

LEMMA 2.4. *Given maps of pointed spaces $f\colon A \to X$ and $g\colon A \to Y$, let*

$G=\Omega(A \to^g Y)$ and $F=\Omega(A \to^{(f,g)} X\times Y)$. A null homotopy $f\sim pt$ provides a section for the projection $F\to G$.

Proof. This follows immediately from the definition of the homotopy fiber, namely $G=A\times_Y Y^I\times_Y y_0$, where $\{y_0\}$ is the base point of Y. Q.E.D.

3. Defining Modules Locally

In this section, we prove a version of the theorem from commutative algebra that says quasicoherent sheaves may be defined locally.

Suppose S and T are right denominator sets in R, and let $U := \langle S, T\rangle$ denote the multiplicative set they generate. It is easy to see that U is also a right denominator set.

It follows from the universal property for localizations that RU^{-1} is the pushout (in the category of rings) of the diagram $RS^{-1}\leftarrow R\rightarrow RT^{-1}$.

We call S and T *compatible* if $ST=TS$ $(=U)$, or equivalently, the following axioms are satisfied:

(ST1) $s_1\in S,\ t_1\in T\Rightarrow \exists t_2\in T,\ s_1\in S\ s_1t_2=t_2s_2$,

(ST2) $t_1\in T,\ s_1\in S\Rightarrow \exists s_2\in S,\ t_2\in T\ t_1s_1=s_2t_2$.

Lemma 3.1. *If S and T are compatible, then $(RS^{-1})\,T^{-1}\cong R(ST)^{-1}\cong (RT^{-1})\,S^{-1}$ are all isomorphic as rings.*

Proof. One checks that the image of T in RS^{-1} is a right denominator set, then the statement follows from the universal property of localization. Q.E.D.

We introduce the following covering axiom for S and T.

(ST3) $s\in S$ and $t\in T\Rightarrow sR+tR=R$.

This axiom implies that $RS^{-1}\times RT^{-1}$ is faithfully flat as left R-module. For if $0=M\otimes_R(RS^{-1}\times RT^{-1})=MS^{-1}\oplus MT^{-1}$ and $m\in M$, then $ms=mt=0$ for some $s\in S$, $t\in T$, thus $m=0$, and $M=0$. Then one proves the following in the usual way.

Proposition 3.2. *Suppose S, $T\subseteq R$ are right denominator sets which are compatible and satisfy the covering axiom. Then the category of (right) R-modules M is equivalent to the category of triples (P, Q, θ), where P is an RS^{-1}-module, Q is an RT^{-1}-module, and $\theta\colon PT^{-1}\to QS^{-1}$ is an $R(ST)^{-1}$-isomorphism.*

COROLLARY 3.3. *In the equivalence of Proposition 3.2, M is finitely generated (resp. finitely presented) iff P and Q are. If S and T are also left denominator sets, then M is finitely generated projective iff P and Q are.*

Proof. The proof of the first assertion is standard. For the second we consider the sequence

$$0 \to M \to MS^{-1} \oplus MT^{-1} \to M(ST)^{-1} \to 0,$$

which is exact because it becomes exact under localization by S or by T. The hypothesis implies that RS^{-1}, RT^{-1}, and $R((ST)^{-1}$ are all right and left flat over R, so M is also. Since M is flat and finitely presented, it follows from Lazard's theorem [La, Corollary 1.4] that M is projective.

Q.E.D.

4. A LOCALIZATION OF THE PROJECTIVE LINE

We now make the blanket assumption that k is a (skew) field. Let $S^+ \subseteq R^+$ be the multiplicative set of all nonzero polynomials, and let $S_0^+ \subseteq R^+$ be the multiplicative set of all polynomials with nonzero constant term.

LEMMA 4.1. *S^+ and S_0^+ are denominator sets (consisting only of nonzero divisors).*

Proof. First prove it for S^+. Let R_j denote the polynomials of degree $\leqslant j$. Given $f \in R^+$ and $s \in S^+$, let $m = \deg f$, $n = \deg s$, and consider the map $R_m \oplus R_n \to R_{m+n}$ defined by $(u, v) \to fu - sv$. This k-linear map has nonzero kernel for dimension reasons. When $fu - sv = 0$, then $u \in S^+$ unless $u = v = 0$, for R^+ is an integral domain.

Next prove it for S_0^+. Proceed as before: if $u(0) = 0$, then $v(0) = 0$ (because $s(0) \neq 0$), so we may divide u and v by a suitable power of T to achieve $u \in S_0^+$.

We've given the proof on the right side: the left side goes the same way.

Q.E.D.

In the ring $B^+ := (S_0^+)^{-1} R^+ = R^+(S_0^+)^{-1}$, the multiplicative set generated by T still is a denominator set, so letting $B^\pm := B^+[T^{-1}]$, we see that $B^\pm = R^+(S^+)^{-1}$ is a skew field. Using by now obvious notation, we also have the ring $B^- := R^-(S_0^-)^{-1}$, and $B^\pm = B^-[(T^{-1})^{-1}]$. Define $B := B^+ \cap B^- \subseteq B^\pm$.

LEMMA 4.2. *B consists of all fractions fg^{-1}, with f and $g \in R^+$, $g(0) \neq 0$, and* $\deg g \geqslant \deg f$.

Proof. Write a typical element of $B \subseteq B^+$ in the form fg^{-1} with $f, g \in R^+$, $g \in S_0^+$. Let $n = \max(\deg f, \deg g)$, and let $G(T^{-1}) := g(T)\, T^{-n}$, $F(T^{-1}) := f(T)\, T^{-n}$ so that $fg^{-1} = FG^{-1}$, and $G, F \in R^-$. Since $FG^{-1} \in B^-$, we may write $FG^{-1} = JH^{-1}$ with $H, J \in R^-$ and $H \in S_0^-$. By definition of fractions, we may find K, L nonzero in R^- so that $GK = HL$ and $FK = JL$.

We may assume T^{-1} does not divide both K and L. Then if T^{-1} divides G, it follows that $T^{-1} \mid L$ and $T^{-1} \nmid K$, and so $T^{-1} \mid F$. But T^{-1} does not divide both F and G, so $T^{-1} \nmid G$, and $n = \deg g \geqslant \deg f$. Q.E.D.

Let $p_0 \colon B^+ \to k$ be the ring homomorphism with $p_0(T) = 0$, and let $p_\infty \colon B^- \to k$ be the homomorphism with $p_\infty(T^{-1}) = 0$. If $p_0(fg^{-1}) \neq 0$, then $f(0) \neq 0$, so $gf^{-1} \in B^+$ and fg^{-1} is a unit in B^+. Thus $I_0 = \ker p_0$ is a maximal (left, right, or 2-sided) ideal whose complement consists of units, and is the only maximal (left or right) ideal. The same remarks apply to $I_\infty = \ker p_0 \subseteq B^-$. Thus the rings B^+, B^- are *local.*

Let $J_0 := I_0 \cap B$, $J_\infty := I_\infty \cap B$. If $fg^{-1} \in B$, and $p_0(fg^{-1}) \neq 0$, $p_\infty(fg^{-1}) \neq 0$, then it follows that $f(0) \neq 0$ and $\deg g = \deg f$, so fg^{-1} is a unit in B (by Lemma 4.2). It follows that J_0, J_∞ are the only maximal (left or right) ideals of B. For if C is another maximal left ideal, take $\beta \in C \backslash J_0$ and $\gamma \in C \backslash J_\infty$; one of β, γ, $\beta + \gamma$ is in $C \backslash (J_0 \cup J_\infty) = C \cap B^\times$, a contradiction. We conclude that the radical $J := \operatorname{rad}(B) = J_0 \cap J_\infty$ and is the kernel of the surjective homomorphism

$$p = (p_0, p_\infty) \colon B \to k \times k.$$

LEMMA 4.3. *B^+, B^-, and $B^\pm$ are all left (or right) rings of fractions of B.*

Proof. Suppose $fg^{-1} \in B^+$, with $g, f \in R^+$ and $g(0) \neq 0$. Let $b := \max\{0, \ \deg f - \deg g\}$, and $h = (1+T)^b \cdot g$. Then $fg^{-1} = (fh^{-1})((1+T)^{-b})^{-1}$, and $fh^{-1} \in B$, $(1+T)^{-b} \in B$, which shows B^+ is a right ring of fractions of B. The proof for B^- is similar (replace T by T^{-1}), as is the proof on the left side. Since we didn't use the condition $g(0) \neq 0$ in arranging $\deg h \geqslant \deg f$, the proofs for B^+ and B^- combine to show $B^\pm$ is a localization of B. Q.E.D.

According to the lemma, we may write

$$T^+ := B \cap (B^+)^\times = \{ fg^{-1} \mid g(0) \neq 0, f(0) \neq 0, \deg f \leqslant \deg g \}$$

$$T^- := B \cap (B^-)^\times = \{ fg^{-1} \mid g(0) \neq 0, \deg f - \deg g \}$$

$$T^\pm := B \cap (B^\pm)^\times = B \backslash \{0\}$$

$$B^+ = (T^+)^{-1} B$$
$$B^- = (T^-)^{-1} B$$
$$B^\pm = (T^\pm)^{-1} B.$$

LEMMA 4.4. *Proposition 3.2 and all of Corollary 3.3 apply to the multiplicative sets T^+ and T^- in the ring B.*

Proof. Given $fg^{-1} \in T^\pm$ with $f, g \in R^+$, we may write $fg^{-1} = (1+T)^{-a} (((1+T)^a f) g^{-1})$ with $a = \deg g - \deg f \geqslant 0$. This makes $(1+T)^a fg^{-1} \in T^-$ and $(1+T)^{-a} \in T^+$, so $T^\pm = T^+ T^-$. By symmetry (writing denominators on the other side) we see that $T^\pm = T^- \cdot T^+$, and thus T^+ and T^- are compatible.

The covering condition follows from $T^+ = B \backslash J_0$, $T^- = B \backslash J_\infty$, and the fact that J_0 and J_∞ are the only maximal right ideals of B. Q.E.D.

Remark. It follows that B is a ring of global dimension 1, because B^+ and B^- are

COROLLARY 4.5. *There are exact functors $\mathcal{M}_X \to \mathcal{M}_B$ and $\mathcal{P}_X \to \mathcal{P}_B$ defined by $(M^+, M^-, \theta) \to$ pullback of $(M^+ \otimes_{R^+} B^+, M^- \otimes_{R^-} B^-, \theta \otimes 1)$.*

We may think of this functor as a localization functor. Indeed, as in the commutative case, we may think of B as the semilocal ring at $\{0, \infty\}$ in the projective line.

We denote the functors of (4.5) with $M \to M \otimes_X B$, for $M \in \mathcal{M}_X$. Let $\mathcal{H}$ denote the exact category of all those X-modules M which have a resolution of length 1 on X by vector bundles on X, and for which $M \otimes_X B = 0$.

Define $\mathrm{Aut}(\varphi)$ to be the exact category consisting of all pairs (V, f) with $V \in \mathcal{P}_k$ and $f\colon V \to V$ a φ-semilinear automorphism, $f(va) = f(v)\, \varphi(a)$. An arrow $(V, f) \to (V', f')$ is a map $g\colon V \to V'$ with $gf = f'g$, as usual.

THEOREM 4.6. (a) *There is a long exact "localization" sequence*

$$\cdots K_i \mathcal{H} \to K_i X \to K_i B \to K_{i-1} \mathcal{H} \cdots.$$

(b) *There is an equivalence $\mathcal{H} \cong \mathrm{Aut}(\varphi)$ of exact categories.*

Proof. (a) We reread Quillen's proof of the localization theorem for projective modules [G2, p. 229] to verify that it works in our context. The crucial Lemma 2 there is rephrased as follows: for each $N \in \mathcal{P}_B$, the category C_N of pairs (M, β), with $M \in \mathcal{P}_X$, and β an isomorphism $\beta\colon M \otimes_X B \xrightarrow{\sim} N$, is equivalent to a filtering ordered set. (One may compare C_N with $\mathcal{L}_W$ of [G1].) To convince ourselves of this statement, we first, for each M, replace M^+ by its isomorphic image in N^+, and similarly for M^-. This gives a retraction of C_N onto a partially ordered set $\mathscr{D}_N$, consisting of certain "submodules" of N. Since $M_1+M_2\in\mathscr{D}_N$ when $M_1, M_2\in\mathscr{D}_N$, we see that $\mathscr{D}_N$ is filtering. (The reason $M_1+M_2\in\mathscr{D}_N$ is that R^+ and R^- are (noncommutative) Euclidean domains, for which any finitely generated torsion free module is projective.)

(b) A functor $F:\mathscr{H}\to\mathrm{Aut}(\varphi)$ can be defined by $M\to(M^+$, multiplication by $T)$. A functor $G:\mathrm{Aut}(\varphi)\to\mathscr{H}$ can be defined by $(V,f)\to(V_f, V_f, 1)$, where V_f denotes the R^+-module whose underlying k-module is V, but on which T acts as f, and where V_f also denotes the R^--module which is V with T^{-1} acting as f^{-1}. Certain details must be checked, the only obvious one being that $F\circ G=1$.

To see that $G\circ F=1$, we must verify that for $M\in\mathscr{H}$, T acts invertibly on M^+ and T^{-1} acts invertibly on M^-, so that $M^+\cong M^+[T^{-1}]\cong M^-[(T^{-1})^{-1}]\cong M^-$. From $M^+(S_0^+)^{-1}=0$ it follows that for any $x\in M^+$ there exists $s\in S_0^+$ with $xs=0$. Writing $s=a_0+a_1T+\cdots+a_nT^n$ $(a_0\neq 0)$ we see that $x=(-x(a_1+\cdots+a_nT^{n-1})\,\varphi^{-1}(a_0^{-1}))\,T$, showing that multiplication by T is surjective. For injectivity, the assumption $xT=0$ implies $xa_0=0$, whence $x=0$.

To see that F is well-defined, we must check that if $M\in\mathscr{H}$, then M^+ is a finite-dimensional k-vector space; this is clear, for we may express M^+ as a quotient of $(R/sR)^j$, some $s\in S_0^+$, some j.

To see that G is well-defined we must, given $(V,f)\in\mathrm{Aut}(\varphi)$ and $v\in V_f$, locate $s\in S_0^+$ so that $v\cdot s=0$. This is done in the usual way, by considering $\{v, vT, vT^2,...\}\subseteq V$. We must also check that $G(V,f)$ has a resolution of length one by X-vector bundles; it is easy to establish the exactness of the sequence

$$0\to V(-1)\xrightarrow{g} V(0)\xrightarrow{k} G(V,f)\to 0,$$

where k is the obvious map, and g consists of

$$\begin{aligned}&V\otimes R^+\to V\otimes R^+\\&\qquad v\otimes p\to f^{-1}(v)\otimes Tp\quad v\otimes p\end{aligned}$$

and

$$\begin{aligned}&V\otimes R^-\langle 1\rangle\to V\otimes R^-\\&\qquad v\otimes q\to f^{-1}(v)\otimes\varphi^{-1}(q)-v\otimes qT^{-1}.\end{aligned}$$

This is the characteristic sequence of the semilinear automorphism f (cf. B, p. 630; G3, p. 442; and FH, Lemma 9]. Q.E.D.

Remark. If $\varphi=1$, then as in [G3], $K_i\,\mathrm{Aut}(\varphi)$ contains K_ik as a direct factor, because $(V, 1_V) \in \operatorname{Aut}(\varphi)$ for any $V \in \mathscr{P}_k$. If $\varphi \neq 1$ then this no longer works. For this reason, it is not possible to describe $K_i \operatorname{Aut}(\varphi)$ as in [G3, Theorem 2]. The localization sequence for $R^+ \to B^+$ does, however, split into short exact sequences, yielding the decomposition

$$K_i B^+ = K_i R^+ \oplus K_{i-1} \operatorname{Aut}(\varphi).$$

5. The Case φ = Frobenius

In this final section we assume k is an algebraically closed (commutative) field of characteristic p, and φ is the Frobenius, $\varphi(a) = a^p$. We let $\mathbb{F}_p$ denote the prime field. The functor $L: \mathscr{P}_{\mathbb{F}_p} \to \operatorname{Aut}(\varphi)$ defined by $W \to (W \otimes k, 1_W \otimes \varphi)$ is know to be an equivalence of categories [Q1, p. 115] or [L]. This "deep descent" presents the possibility of using (4.6) and (1.1) to compute $K_i B$.

Remark. To extract the statement that L is an equivalence one proceeds as follows. Given $(V, f) \in \operatorname{Aut}(\varphi)$, choose a basis of V and let A be the matrix of f with respect to that basis; then [L] provides a matrix B with $B^{(p)} B^{-1} = A$. One can check that B provides a change of basis for V so that f fixes each element of the basis. This shows that the functor $\operatorname{Aut}(\varphi) \to \mathscr{P}_{\mathbb{F}_p}$ defined by $(V, f) \to \{v \in V \mid f(v) = v\}$ is well-defined and an inverse equivalence for L.

THEOREM 5.1. *Under the assumptions made above, the map $K_i(B) \to K_i(B/J)$ is an isomorphism for $i > 0$, and thus $K_i B \simeq K_i k \times K_i k$.*

Proof. We make explicit the dotted arrow in the following diagram:

$$\begin{array}{ccc} K_i \mathscr{H} & \xrightarrow{i^*} & K_i X \\ {\scriptstyle L^*} \uparrow \wr & & \uparrow \wr \; {\scriptstyle (k_0^*, k_1^*)} \\ K_i \mathbb{F}_p & \cdots\cdots\to & K_i k \times K_i k \end{array}$$

The characteristic sequence of (4.6) is natural in V, so we find that $i^* \circ L^* = (h_0^* - h_{-1}^*) \circ j^*$, where we let j denote the inclusion $\mathbb{F}_p \to k$. The natural exact sequence of (1.1) yields $h_0^* - h_{-1}^* = (\varphi^{-1})^* (h_1^* - h_0^*)$. The isomorphism $V\langle 1 \rangle \otimes R^+ \to V \otimes R^+ \langle 1 \rangle$ defined by $v \otimes p \to v \otimes \varphi(p)$, with a similar one for R^-, provides an isomorphism $V(n)\langle m \rangle = V\langle m \rangle(n)$; thus $\varphi^* h_n^* = h_n^* \varphi^*$. Since $\varphi^* j^* = j^*$ we get $i^* \circ L^* = (h_1^* - h_0^*) \circ j^*$, so the dotted arrow is $\binom{-j^*}{j^*}$. The matrix of the composite map

$$K_i k \times K_i k \to K_i X \to K_i B \to K_i B/J = K_i k \times K_i k$$

is easily seen to be

$$C=\begin{pmatrix}1 & 1\\ 1 & \varphi^*\end{pmatrix}.$$

Quillen has shown [Q2, pp. 583–585] that j^* is injective when k is an algebraic closure of $\mathbb{F}_p$; commutativity of K-theory with filtering direct limits and the Hilbert Nullstellensatz extend this result to arbitrary k. Thus from (4.6) one obtains the diagram

$$\begin{array}{ccccccccc} 0 \to & K_i\mathbb{F}_p & \to & K_ik\times K_ik & \longrightarrow & K_iB & \to 0 \\ & \| & & \| & & \downarrow & \\ 0 \to & K_i\mathbb{F}_p & \to & K_ik\times K_ik & \xrightarrow[C]{} & K_ik\times K_ik & \to 0 \end{array}$$

in which the upper row is known to be exact. The exactness of the lower row would follow from the exactness of

$$0\to K_i\mathbb{F}_p\to K_ik \xrightarrow{1-\varphi^*} K_ik\to 0 \qquad (*)$$

by a simple diagram chase. Quillen has shown [H, Corollary 5.2] that $\varphi^*=\psi^p$, the pth Adams operation. The exactness of $(*)$ is Quillen's conjecture, shown by Hiller [H, Theorem 7.2] to be equivalent to Lichtenbaum's conjecture that $K_i(\overline{\mathbb{F}}_p)\to K_i(k)$ has cokernel a rational vector space (here $\overline{\mathbb{F}}_p$ = algebraic closure of $\mathbb{F}_p$). The latter conjecture was proved by Suslin [Su]. Q.E.D.

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