The $K$-Theory of Semilinear Endomorphisms

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In this paper we study the $K$-theory of semilinear endomorphisms and automorphisms over noncommutative rings. For commutative rings and linear endomorphisms we did this in [G3].

In Section 4 we produce an exact sequence (4.6) involving the $K$-groups of semilinear automorphisms over a field. The main tool is the introduction of the “twisted projective line,” together with the fact that it admits an interesting localization at $\{0, \infty\}$. In Section 5 we use the Frobenius on an algebraically closed field to produce an example of a semilocal domain $B$ with nonzero radical $J$ so that $K_i(B) \cong K_i(B/J)$, $i > 0$.

In Sections 1 and 2 we give another application of the twisted projective line: we prove the natural generalization (2.3) to the higher $K$-groups of the results of Farrell and Hsiang [FH] about Whitehead groups of twisted Laurent polynomial rings. The proof is a straightforward rewriting of Quillen’s proof of the Fundamental Theorem [G2] (in which the adjoined variable was central). The difference between our proof and Ranicki’s proof in [R, pp. 427–428] is that we emphasize the role of the twisted projective line, and we identify the group $F_i(\varphi)$ as the homotopy group of the homotopy fiber of the map $1 - \varphi^*$.

Other proofs are available. When the ground ring is regular noetherian, the theorem is an exercise in [Q1, pp. 114–122]. One could also obtain a proof by rewriting the proof of Theorem 18.1 of [W], which is much more general.

1. The Twisted Projective Line

A right denominator set $S$ in a ring $R$ is a subset with the following properties [St, p. 52]:

(S1) $1 \in S$,

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(S2) \( s_1, s_2 \in S \Rightarrow s_1 s_2 \in S \),
(S3) \( s_1 \in S, a \in R \Rightarrow \exists b \in R, s_2 \in S: s_1 b = as_2 \),
(S4) \( s_1 \in S, a \in R, s_1 a = 0 \Rightarrow \exists s_2 \in S: as_2 = 0 \).

These conditions are the most general which ensure that the ring of right fractions \( RS^{-1} \) exists. If the elements of \( S \) are nonzero divisors (as will be the case here) then (S4) can be omitted. If \( S = \{ s^n : n \geq 0 \} \) for some \( s \), we write \( R[s^{-1}] = RS^{-1} \).

The axioms for a left denominator set are analogous. If \( S \) is both a right and a left denominator set, we will call it a denominator set; in this case the ring of left fractions \( S^{-1}R \) is isomorphic to \( RS^{-1} \).

We let \( k \) be a (not necessarily commutative) ring, and \( \varphi \) an automorphism of \( k \). The twisted polynomial ring \( R^+ := k[T; \varphi] \) is the ring of polynomials \( a_n T^n + \cdots + a_0, a_i \in k \), where multiplication satisfies a \( T = T \varphi(a) \). The multiplicative set generated by \( T \) is a denominator set, so the localization \( R^\pm := k[T, T^{-1}; \varphi] := k[T; \varphi][T^{-1}] \) is defined; we see that \( k[T^{-1}, (T^{-1})^{-1}, \varphi^{-1}] = k[T, T^{-1}, \varphi] \), so \( R^\pm \) is also a localization of \( R^{-} := k[T^{-1}; \varphi^{-1}] \).

We define a right \( X \)-module \( M \) to be a triple \( M = (M^+, M^-, \theta_M) \), where \( M^+ \) is a right \( R^+ \)-module, \( M^- \) is a right \( R^- \)-module, and \( \theta_M = M^+[T^{-1}] \cong M^-[(T^{-1})^{-1}] \) is an isomorphism of right \( R^\pm \)-modules. Here \( X = \mathbb{P}^1(\varphi) \) denotes the "twisted projective line" with respect to \( k \) and \( \varphi \) and remains undefined. A map \( f: M_1 \to M_2 \) of \( X \)-modules is a pair \( f^+: M_1^+ \to M_2^+ \ f^-: M_1^- \to M_2^- \) of homomorphisms with \( \theta_M \cdot f^+ = f^- \cdot \theta_M \).

The category of right \( X \)-modules is an abelian category. Let \( \mathcal{M}_x \) denote the exact category of right \( X \)-modules \( M \) for which \( M^+ \) and \( M^- \) are finitely generated; it is an abelian category if \( R^+ \) and \( R^- \) are noetherian, and thus if \( k \) is noetherian (according to [FH, Lemma 24]). Let \( \mathcal{P}_R \) denote the exact category of finitely generated projective right \( R \)-modules, and let \( \mathcal{P}_X \) be the exact category of "vector bundles on \( X \)," i.e., those \( X \)-modules \( M \) where \( M^+ \in \mathcal{P}_{R^+} \) and \( M^- \in \mathcal{P}_{R^-} \). Let

\[ K_* X := K_* \mathcal{P}_X. \]

If \( R \) is \( R^+ \), \( R^- \), or \( R^\pm \), then \( \varphi \) extends to an automorphism of \( R \) by setting \( \varphi(T) = T \). Tensor product gives an exact functor \( \varphi^*: \mathcal{P}_R \to \mathcal{P}_R \). Define \( N\langle n \rangle = (\varphi^{-n})^* (N) \) for \( N \in \mathcal{P}_R \) and \( n \in \mathbb{Z} \). One may also obtain \( N\langle n \rangle \) from \( N \) by replacing the scalar multiplication with \( x \star f = x \varphi^n(f) \) for \( x \in \mathbb{N} \) and \( f \in R \). If \( M \) is an \( X \)-module, we let \( M\langle n \rangle = (M^+\langle n \rangle, M^-\langle n \rangle, \theta_M) \).

For \( k \)-modules \( V \) and \( W \), a \( \varphi \)-semilinear map \( f: V \to W \) is an additive map satisfying \( f(va) = f(v) \varphi(a) \) for \( v \in V, a \in k \). This is the same as a
k-linear map \( V \to W \langle 1 \rangle \). If \( M \) is an \( R^+ \)-module, then right multiplication by \( T \) on \( M \) is a \( \varphi \)-semilinear endomorphism of the \( k \)-module underlying \( M \), and all \( \varphi \)-semilinear endomorphisms of \( k \)-modules arise this way.

If \( M \) is an \( X \)-module, we define \( M(n) := (M^+, M^- \langle -n \rangle, \theta_M \circ \rho(T^{-n})) \), where \( \rho(T^{-n}) \) denotes right multiplication by \( T^{-n} \). One checks that \( M(n) \in \mathcal{P}_X \), and \( M(m)(n) = M(m+n) \).

If \( V \) is a \( k \)-module, define an \( X \)-module \( V(0) := (V \otimes_k R^+, V \otimes_k R^-, 1) \), and \( X \)-modules \( V(n) := V(0)(n) \). Let \( h_n: \mathcal{P}_k \to \mathcal{P}_X \) denote the exact functor \( h_n(V) = V(n) \).

**Theorem 1.1.** The map

\[
(h_0*, h_{1*}): K_i k \oplus K_i k \to K_i X
\]

is an isomorphism. The relation \( h_m* + h_{m+2} \langle 1 \rangle* = h_{m+1}* + h_{m+1} \langle 1 \rangle* \) holds for all \( m \in \mathbb{Z} \).

**Proof.** The proof can be done essentially as in [Q1, Theorem 3.1, Sect. 8, p. 143]; the only change is that \( T \) is no longer central. Multiplication by \( T \) on an \( R^+ \)-module \( N \) is no longer an \( R \)-linear endomorphism of \( N \), but is an \( R \)-linear map \( N \to N \langle 1 \rangle \). Thus, one rewrites Quillen’s proof by inserting notations like “\( \langle n \rangle \)” in appropriate spots to preserve linearity of the maps involved. For example, the canonical exact sequence

\[
0 \to \mathcal{O}(m) \to \mathcal{O}(m + 1)^2 \to \mathcal{O}(m + 2) \to 0
\]

becomes

\[
0 \to V(m) \to V(m + 1) \oplus V(m + 1) \langle 1 \rangle \to V(m + 2) \langle 1 \rangle \to 0
\]

for any \( V \in \mathcal{P}_k, m \in \mathbb{Z} \). Q.E.D.

2. Localization

In this section we discuss localization theorems for \( K \)-theory in the twisted projective line. This allows us to relate the \( K \)-groups of the projective line with those of \( R^+, R^- \), and \( R^\pm \). In the commutative case, the result obtained is the “Fundamental Theorem” of Bass, generalized by Quillen to the higher \( K \)-groups. In the case at hand, we obtain the result of Farrell and Hsiang and generalize it to apply to the higher \( K \)-groups.

Let \( \mathcal{X}^+ \) denote the exact category of \( X \)-modules \( M \) which admit a resolution of length 1 by vector bundles of \( X \) and for which \( M^- = 0 \). This category is equivalent to the category of finitely generated \( R^+ \)-modules \( N \)
of projective dimension 1 such that \( N[T^{-1}] = 0 \), for a resolution of \( N \) may be begun with a free \( R^+ \)-module (which extends to \( X \)). Observe that for any right \( R^+ \)-module \( P \), the subgroup \( P \cdot T^i \) is an \( R^+ \)-submodule; moreover, if \( P = R^+ \), then \( P/P \cdot T^i \) is a free \( k \)-module on the generators \( 1, T, \ldots, T^{i-1} \). Now the argument of [G2, p. 236] shows that any \( N \) in \( \mathcal{H}^+ \) is projective as \( k \)-module, so \( \mathcal{H}^+ \) is equivalent to the category \( \text{Nil}(\varphi) \) whose objects are pairs \((V, f)\) with \( V \in \mathcal{P}_k \) and \( f: V \rightarrow V \) a nilpotent \( \varphi \)-semilinear endomorphism, \( f(va) = f(v) \varphi(a) \). (In the untwisted case \( \varphi = 1 \) this equivalence is implicit in [B, proof of the fundamental theorem] and explicit in [Si].)

The exact functors

\[
\mathcal{P}_k \rightarrow \text{Nil}(\varphi), \quad \text{Nil}(\varphi) \rightarrow \mathcal{P}_k
\]

\[
V \mapsto (V, 0), \quad (V, f) \mapsto V
\]

allow one to split

\[
K_i \text{Nil}(\varphi) = K_i k \oplus \text{Nil}_i(\varphi),
\]

defining \( \text{Nil}_i(\varphi) \).

The ring homomorphisms

\[
k \rightarrow R^+, \quad R^+ \rightarrow k
\]

\[
a \rightarrow a, \quad f(T) \rightarrow f(0)
\]

allow one to split

\[
K_i R^+ = K_i k \oplus NK_i(\varphi),
\]

defining \( NK_i(\varphi) \). Similarly,

\[
K_i R^- = K_i k \oplus NK_i(\varphi^{-1}).
\]

**Theorem 2.1.** There are localization exact sequences

(a) \( \cdots \rightarrow K_{i+1} R^\pm \rightarrow K_i \mathcal{H}^+ \rightarrow K_i R^+ \rightarrow K_i R^\pm \rightarrow \cdots \),

(b) \( \cdots \rightarrow K_{i+1} R^- \rightarrow K_i \mathcal{H}^+ \rightarrow K_i X \rightarrow K_i R^- \rightarrow \cdots \), and

(c) \( NK_i(\varphi^{-1}) = \text{Nil}_{i-1}(\varphi) \).

**Proof.** Part (a) was proved in [G1]. For part (b) one checks that the proof in [G2, Theorem on p. 222] can be carried over into this context,
using the preliminary material about the twisted projective line presented above. One interprets the notation from [G2] as follows:

\[ j^* M := M^- \]

\[ j_* M^- := (M^- \otimes R^\pm, M^-, 1) \]

\[ I^{-n} M := (M^+ \cdot T^{-n}, M^-, 1) \]

\[ \subseteq j_* j^* M. \]

Part (c) follows from (b) as in [G2]. \( \text{Q.E.D.} \)

**Remark 2.2.** If \( k \) is commutative, or if we are given an isomorphism \( k \cong k^{\text{op}} \), then there is an isomorphism \((R^+)^{\text{op}} \cong R^-\). It follows from [Q1, (13) on p. 104] that \( K_i R^+ = K_i R^- \), and thus \( NK_i(\varphi) \cong NK_i(\varphi^{-1}) \) and \( \text{Nil}_i(\varphi) \cong \text{Nil}_i(\varphi^{-1}) \). There is also an equivalence \( \text{Nil}(\varphi)^{\text{op}} \cong \text{Nil}(\varphi^{-1}) \) defined by \((V, f) \mapsto (V^*, f^*)\), where \( V^* = \text{Hom}_k(V, k) \) and \( f^* = \varphi^{-1} \circ f^* \). The isomorphism \( \text{Nil}_i(\varphi) \cong \text{Nil}_i(\varphi^{-1}) \) that this equivalence provides is probably the same as the other one.

**Remark.** One can use Quillen's dévissage and resolution theorems to prove that \( \text{Nil}_i(\varphi) = 0 \) when \( k \) is regular noetherian, thereby recovering his result that \( K_i R^+ \cong K_i k \).

Define \( F_i(\varphi) = \pi_i \Omega(K(k) \to 1 - \varphi^* K(k)) \), where \( K(k) \) is the space \( \Omega BQ^H k \), whose homotopy groups are the \( K \)-groups, and where \( \Omega(X \to Y) \) denotes the homotopy fiber of a map. If \( \varphi = 1 \), then \( F_i(\varphi) = K_i(k) \oplus K_{i+1}(k) \). Notice, also, that \( F_i(\varphi^{-1}) = F_i(\varphi) \).

**Theorem 2.3.** There is, for \( i \geq 1 \), a canonical isomorphism

\[ K_i R^\pm \cong F_{i-1}(\varphi) \oplus \text{Nil}_{i-1}(\varphi) \oplus \text{Nil}_{i-1}(\varphi^{-1}). \]

**Remark.** For \( i = 1 \), this theorem was proved by Farrell and Hsiang and by Siebenmann.

**Proof.** There is a restriction map from the sequence 2.1(b) to 2.1(a), which is the identity on \( K_i H^+ \). A diagram chase yields a Mayer–Vietoris-type exact sequence

\[ \cdots K_{i+1} R^\pm \to K_i X \to K_i R^+ \oplus K_i R^- \to K_i R^\pm \cdots. \]

We rewrite the terms using (1.1) and 2.1(c) yielding
The matrix of the map $A$ is seen to be
\[
\begin{pmatrix}
1 & 1 \\
0 & 0 \\
1 & \varphi^* \\
0 & 0
\end{pmatrix}
\]
and because $AB = 0$, we see that the matrix of $B$ is
\[
\begin{pmatrix}
g \\
-g
\end{pmatrix}
\]
for some map $g$. This allows us to split off a $k_i k$ factor, yielding
\[
\cdots K_{l+1} R^\pm \xrightarrow{g} K_i k \xrightarrow{\begin{pmatrix} 1 - \varphi^* \\ 0 \\ 0 \\ 0 \end{pmatrix}} K_i k \oplus \text{Nil}_{l-1}(\varphi^{-1}) \oplus \text{Nil}_{l-1}(\varphi) \to K_i R^\pm \cdots.
\]
Consider the diagram
\[
\begin{align*}
K_{i+1}(R^-) & \to K_i(\mathcal{H}^+) \\
\downarrow & \\
K_{i+1}(R^+) & \to K_{i+1}(R^\pm) \to K_i(\mathcal{H}^+) \\
\downarrow & \\
K_i(\mathcal{H}^-) & = K_i(\mathcal{H}^-)
\end{align*}
\]
with exact rows and columns. Application of the decompositions we know so far gives
\[ K_{i+1} \mathbb{k} \oplus \text{Nil}_i(\varphi) \xrightarrow{(0 \ 0 \ 1)} K_i \mathbb{k} \oplus \text{Nil}_i(\varphi) \]

\[ K_{i+1} \mathbb{k} \oplus \text{Nil}_i(\varphi^{-1}) \rightarrow K_{i+1} R^\pm \rightarrow K_i \mathbb{k} \oplus \text{Nil}_i(\varphi) \]

\[ (0 \ 0 \ 1) \]

\[ K_i \mathbb{k} \oplus \text{Nil}_i(\varphi^{-1}) = K_i \mathbb{k} \oplus \text{Nil}_i(\varphi^{-1}) \]

It follows that

\[ K_{i+1} R^\pm \cong \mathbb{k} \oplus \text{Nil}_i(\varphi) \oplus \text{Nil}_i(\varphi^{-1}) \]

and we get an exact sequence

\[ \cdots \rightarrow ? \rightarrow K_i \mathbb{k} \xrightarrow{1-\varphi^*} K_i \mathbb{k} \rightarrow \cdots. \]

In order to identify "?" with \( F_i(\varphi) \), we argue with the underlying spaces. We get a map of fibrations

\[ \Omega K(R^\pm) \rightarrow K(k) \xrightarrow{\begin{pmatrix} 1 - \varphi^* \\ \text{pt.} \\ \text{pt.} \end{pmatrix}} K(k) \times NK(\varphi^{-1}) \times NK(\varphi) \]

\[ s \downarrow \quad \downarrow 1 \quad \downarrow \text{pr}_1 \]

\[ F(\varphi) \rightarrow K(k) \xrightarrow{(1 - \varphi^*)} K(k) \]

where the notations \( NK \) and \( F \) for spaces ought to be self-explanatory. The existence of the section \( s \) follows from Lemma 2.4 below. The spaces here are homotopy-everything \( H \)-spaces with additive inverses, so we may split

\[ \Omega K(R^\pm) \cong F(\varphi) \times \Omega(t). \]

Moreover, the homotopy fibers of the three vertical maps above form a fibration which tells us that

\[ \Omega(t) \cong \Omega NK(\varphi^{-1}) \times \Omega NK(\varphi). \]

Thus

\[ \Omega K(R^\pm) \cong F(\varphi) \times \Omega NK(\varphi^{-1}) \times \Omega NK(\varphi). \]

Taking homotopy groups yields the result. Q.E.D.

**Lemma 2.4.** Given maps of pointed spaces \( f: A \rightarrow X \) and \( g: A \rightarrow Y \), let
\( G = \Omega(A \to^g Y) \) and \( F = \Omega(A \to^{f,g} X \times Y) \). A null homotopy \( f \sim pt \) provides a section for the projection \( F \to G \).

Proof. This follows immediately from the definition of the homotopy fiber, namely \( G = A \times_Y Y' \times_Y y_0 \), where \( \{y_0\} \) is the base point of \( Y \).

Q.E.D.

3. Defining Modules Locally

In this section, we prove a version of the theorem from commutative algebra that says quasicoherent sheaves may be defined locally.

Suppose \( S \) and \( T \) are right denominator sets in \( R \), and let \( U := \langle S, T \rangle \) denote the multiplicative set they generate. It is easy to see that \( U \) is also a right denominator set.

It follows from the universal property for localizations that \( RU^{-1} \) is the pushout (in the category of rings) of the diagram \( RS^{-1} \to R \to RT^{-1} \).

We call \( S \) and \( T \) compatible if \( ST = TS \) \( (= U) \), or equivalently, the following axioms are satisfied:

\[(\text{ST1}) \quad s_1 \in S, \ t_1 \in T \Rightarrow \exists t_2 \in T, \ s_1 \in S \ s_1 t_2 = t_2 s_2 ,\]
\[(\text{ST2}) \quad t_1 \in T, \ s_1 \in S \Rightarrow \exists s_2 \in S, \ t_2 \in T \ t_1 s_1 = s_2 t_2 .\]

Lemma 3.1. If \( S \) and \( T \) are compatible, then \( (RS^{-1})^{-1} \cong R(ST)^{-1} \cong (RT^{-1})^{-1} \) are all isomorphic as rings.

Proof. One checks that the image of \( T \) in \( RS^{-1} \) is a right denominator set, then the statement follows from the universal property of localization.

Q.E.D.

We introduce the following covering axiom for \( S \) and \( T \).

\[(\text{ST3}) \quad s \in S \text{ and } t \in T \Rightarrow sR + tR = R .\]

This axiom implies that \( RS^{-1} \times RT^{-1} \) is faithfully flat as left \( R \)-module.

For if \( 0 = M \otimes_R (RS^{-1} \times RT^{-1}) = MS^{-1} \oplus MT^{-1} \) and \( m \in M \), then \( ms = mt = 0 \) for some \( s \in S \), \( t \in T \), thus \( m = 0 \), and \( M = 0 \). Then one proves the following in the usual way.

Proposition 3.2. Suppose \( S, T \subseteq R \) are right denominator sets which are compatible and satisfy the covering axiom. Then the category of (right) \( R \)-modules \( M \) is equivalent to the category of triples \((P, Q, \theta)\), where \( P \) is an \( RS^{-1} \)-module, \( Q \) is an \( RT^{-1} \)-module, and \( \theta: PT^{-1} \to QS^{-1} \) is an \( R(ST)^{-1} \)-isomorphism.
Corollary 3.3. In the equivalence of Proposition 3.2, M is finitely generated (resp. finitely presented) iff P and Q are. If S and T are also left denominator sets, then M is finitely generated projective iff P and Q are.

Proof. The proof of the first assertion is standard. For the second we consider the sequence

$$0 \to M \to MS^{-1} \oplus MT^{-1} \to M(ST)^{-1} \to 0,$$

which is exact because it becomes exact under localization by S or by T. The hypothesis implies that $RS^{-1}$, $RT^{-1}$, and $R((ST)^{-1}$ are all right and left flat over $R$, so $M$ is also. Since $M$ is flat and finitely presented, it follows from Lazard’s theorem [La, Corollary 1.4] that $M$ is projective.

Q.E.D.

4. A Localization of the Projective Line

We now make the blanket assumption that $k$ is a (skew) field. Let $S^+ \subseteq R^+$ be the multiplicative set of all nonzero polynomials, and let $S^+_0 \subseteq R^+$ be the multiplicative set of all polynomials with nonzero constant term.

Lemma 4.1. $S^+$ and $S^+_0$ are denominator sets (consisting only of nonzero divisors).

Proof. First prove it for $S^+$. Let $R_j$ denote the polynomials of degree $\leq j$. Given $f \in R^+$ and $s \in S^+$, let $m = \deg f$, $n = \deg s$, and consider the map $R_m \oplus R_n \to R_{m+n}$ defined by $(u, v) \to fu - sv$. This $k$-linear map has nonzero kernel for dimension reasons. When $fu - sv = 0$, then $u \in S^+$ unless $u = v = 0$, for $R^+$ is an integral domain.

Next prove it for $S^+_0$. Proceed as before: if $u(0) = 0$, then $v(0) = 0$ (because $s(0) \neq 0$), so we may divide $u$ and $v$ by a suitable power of $T$ to achieve $u \in S^+_0$.

We've given the proof on the right side: the left side goes the same way.

Q.E.D.

In the ring $B^+ := (S^+_0)^{-1}R^+ = R^+(S^+_0)^{-1}$, the multiplicative set generated by $T$ still is a denominator set, so letting $B^\pm := B^+[T^{-1}]$, we see that $B^\pm = R^+(S^+)^{-1}$ is a skew field. Using by now obvious notation, we also have the ring $B^- := R^-(S^-_0)^{-1}$, and $B^\pm = B^-[(T^-)^{-1}]$. Define $B := B^+ \cap B^- \subseteq B^\pm$.

Lemma 4.2. $B$ consists of all fractions $fg^{-1}$, with $f$ and $g \in R^+$, $g(0) \neq 0$, and $\deg g \geq \deg f$. 

Proof. Write a typical element of \( B \subseteq B^+ \) in the form \( fg^{-1} \) with \( f, g \in R^+ \), \( g \in S_T^- \). Let \( n = \max(\deg f, \deg g) \), and let \( G(T^{-1}) := g(T) T^{-n} \), \( F(T^{-1}) := f(T) T^{-n} \) so that \( fg^{-1} = FG^{-1} \), and \( G, F \in R^- \). Since \( FG^{-1} \in B^- \), we may write \( FG^{-1} = JH^{-1} \) with \( H, J \in R^- \) and \( H \in S_T^- \). By definition of fractions, we may find \( K, L \) nonzero in \( R^- \) so that \( GK = HL \) and \( FK = JL \).

We may assume \( T^{-1} \) does not divide both \( K \) and \( L \). Then if \( T^{-1} \) divides \( G \), it follows that \( T^{-1} \mid L \) and \( T^{-1} \nmid K \), and so \( T^{-1} \nmid F \). But \( T^{-1} \) does not divide both \( F \) and \( G \), so \( T^{-1} \nmid G \), and \( n = \deg g \geq \deg f \). Q.E.D.

Let \( p_0 : B^+ \to k \) be the ring homomorphism with \( p_0(T) = 0 \), and let \( p_\infty : B^- \to k \) be the homomorphism with \( p_\infty(T^{-1}) = 0 \). If \( p_0(fg^{-1}) \neq 0 \), then \( f(0) \neq 0 \), so \( gf^{-1} \in B^+ \) and \( fg^{-1} \) is a unit in \( B^+ \). Thus \( I_0 = \ker p_0 \) is a maximal (left, right, or 2-sided) ideal whose complement consists of units, and is the only maximal (left or right) ideal. The same remarks apply to \( I_\infty = \ker p_\infty \subseteq B^- \). Thus the rings \( B^+, B^- \) are local.

Let \( J_0 := I_0 \cap B, J_\infty := I_\infty \cap B \). If \( fg^{-1} \in B \), and \( p_0(fg^{-1}) \neq 0 \), \( p_\infty(fg^{-1}) \neq 0 \), then it follows that \( f(0) \neq 0 \) and \( \deg g = \deg f \), so \( fg^{-1} \) is a unit in \( B \) (by Lemma 4.2). It follows that \( J_0, J_\infty \) are the only maximal (left or right) ideals of \( B \). For if \( C \) is another maximal left ideal, take \( \beta \in C \setminus J_0 \) and \( \gamma \in C \setminus J_\infty \); one of \( \beta, \gamma, \beta + \gamma \) is in \( C \setminus (J_0 \cup J_\infty) = C \cap B^\times \), a contradiction. We conclude that the radical \( J := \text{rad}(B) = J_0 \cap J_\infty \) and is the kernel of the surjective homomorphism

\[
p = (p_0, p_\infty) : B \to k \times k.
\]

**Lemma 4.3.** \( B^+, B^- \), and \( B^\pm \) are all left (or right) rings of fractions of \( B \).

*Proof.* Suppose \( fg^{-1} \in B^+ \), with \( g, f \in R^+ \) and \( g(0) \neq 0 \). Let \( b := \max\{0, \deg f - \deg g\} \), and \( h = (1 + T)^b \cdot g \). Then \( fg^{-1} = (fh^{-1})((1 + T)^{-b})^{-1} \), and \( fh^{-1} \in B, (1 + T)^{-b} \in B \), which shows \( B^+ \) is a right ring of fractions of \( B \). The proof for \( B^- \) is similar (replace \( T \) by \( T^{-1} \)), as is the proof on the left side. Since we didn't use the condition \( g(0) \neq 0 \) in arranging \( \deg h \geq \deg f \), the proofs for \( B^+ \) and \( B^- \) combine to show \( B^\pm \) is a localization of \( B \). Q.E.D.

According to the lemma, we may write

\[
T^+ := B \cap (R^+)^\times = \{ fg^{-1} \mid g(0) \neq 0, f(0) \neq 0, \deg f \leq \deg g \}
\]

\[
T^- := B \cap (B^-)^\times = \{ fg^{-1} \mid g(0) \neq 0, \deg f - \deg g \}
\]

\[
T^\pm := B \cap (B^\pm)^\times = B \setminus \{0\}
\]
\[ B^+ = (T^+)^{-1} B \]
\[ B^- = (T^-)^{-1} B \]
\[ B^\pm = (T^\pm)^{-1} B. \]

**Lemma 4.4.** Proposition 3.2 and all of Corollary 3.3 apply to the multiplicative sets \( T^+ \) and \( T^- \) in the ring \( B \).

**Proof.** Given \( fg^{-1} \in T^\pm \) with \( f, g \in R^+ \), we may write \( fg^{-1} = (1 + T)^{-a} (((1 + T)^a f) / g)^{-1} \) with \( a = \deg g - \deg f \geq 0 \). This makes \( (1 + T)^a fg^{-1} \in T^- \) and \( (1 + T)^{-a} \in T^+ \), so \( T^\pm = T^+ T^- \). By symmetry (writing denominators on the other side) we see that \( T^\pm = T^- \cdot T^+ \), and thus \( T^+ \) and \( T^- \) are compatible.

The covering condition follows from \( T^+ = B \setminus J_0 \), \( T^- = B \setminus J_\infty \), and the fact that \( J_0 \) and \( J_\infty \) are the only maximal right ideals of \( B \). Q.E.D.

**Remark.** It follows that \( B \) is a ring of global dimension 1, because \( B^+ \) and \( B^- \) are

**Corollary 4.5.** There are exact functors \( \mathcal{M}_X \to \mathcal{M}_B \) and \( \mathcal{P}_X \to \mathcal{P}_B \) defined by \( (M^+, M^-, \theta) \to \text{pullback of } (M^+ \otimes_R B^+, M^- \otimes_R B^-, \theta \otimes 1) \).

We may think of this functor as a localization functor. Indeed, as in the commutative case, we may think of \( B \) as the semilocal ring at \( \{0, \infty\} \) in the projective line.

We denote the functors of (4.5) with \( M \to M \otimes_X B \), for \( M \in \mathcal{M}_X \). Let \( \mathcal{H} \) denote the exact category of all those \( X \)-modules \( M \) which have a resolution of length 1 on \( X \) by vector bundles on \( X \), and for which \( M \otimes_X B = 0 \).

Define \( \text{Aut}(\varphi) \) to be the exact category consisting of all pairs \( (V, f) \) with \( V \in \mathcal{P}_k \) and \( f: V \to V \) a \( \varphi \)-semilinear automorphism, \( f(va) = f(v) \varphi(a) \). An arrow \( (V, f) \to (V', f') \) is a map \( g: V \to V' \) with \( gf = f'g \), as usual.

**Theorem 4.6.** (a) There is a long exact "localization" sequence

\[ \cdots K_i \mathcal{H} \to K_i X \to K_i B \to K_{i-1} \mathcal{H} \cdots. \]

(b) There is an equivalence \( \mathcal{H} \cong \text{Aut}(\varphi) \) of exact categories.

**Proof.** (a) We re-read Quillen's proof of the localization theorem for projective modules [G2, p. 229] to verify that it works in our context. The crucial Lemma 2 there is rephrased as follows: for each \( N \in \mathcal{P}_B \), the category \( C_N \) of pairs \( (M, \beta) \), with \( M \in \mathcal{P}_X \), and \( \beta \) an isomorphism \( \beta: M \otimes_X B \cong N \), is equivalent to a filtering ordered set. (One may compare \( C_N \) with \( \mathcal{L}_W \) of [G1].) To convince ourselves of this statement, we first, for
each $M$, replace $M^+$ by its isomorphic image in $N^+$, and similarly for $M^-$. This gives a retraction of $C_N$ onto a partially ordered set $D_N$, consisting of certain “submodules” of $N$. Since $M_1 + M_2 \in D_N$ when $M_1, M_2 \in D_N$, we see that $D_N$ is filtering. (The reason $M_1 + M_2 \in D_N$ is that $R^+$ and $R^-$ are (noncommutative) Euclidean domains, for which any finitely generated torsion free module is projective.)

(b) A functor $F: \mathcal{H} \to \text{Aut}(\varphi)$ can be defined by $M \mapsto (M^+, \text{multiplication by } T)$. A functor $G: \text{Aut}(\varphi) \to \mathcal{H}$ can be defined by $(V, f) \mapsto (V_f, V_f, 1)$, where $V_f$ denotes the $R^+$-module whose underlying $k$-module is $V$, but on which $T$ acts as $f$, and where $V_f$ also denotes the $R^-$-module which is $V$ with $T^{-1}$ acting as $f^{-1}$. Certain details must be checked, the only obvious one being that $F \circ G = 1$.

To see that $G \circ F = 1$, we must verify that for $M \in \mathcal{H}$, $T$ acts invertibly on $M^+$ and $T^{-1}$ acts invertibly on $M^-$, so that $M^+ \cong M^+ [T^{-1}] \cong M^- [(T^{-1})^{-1}] \cong M^-$. From $M^+(S_0^-)^{-1} = 0$ it follows that for any $x \in M^+$ there exists $s \in S_0^+$ with $xs = 0$. Writing $s = a_0 + a_1 T + \cdots + a_n T^n \ (a_0 \neq 0)$ we see that $x = (1 - x(a_1 + \cdots + a_n T^{n-1}) \varphi^{-1}(a_0^{-1})) T$, showing that multiplication by $T$ is surjective. For injectivity, the assumption $xT = 0$ implies $xa_0 = 0$, whence $x = 0$.

To see that $F$ is well-defined, we must check that if $M \in \mathcal{H}$, then $M^+$ is a finite-dimensional $k$-vector space; this is clear, for we may express $M^+$ as a quotient of $(R/sR)^j$, some $s \in S_0^+$, some $j$.

To see that $G$ is well-defined we must, given $(V, f) \in \text{Aut}(\varphi)$ and $v \in V_f$, locate $s \in S_0^+$ so that $v \cdot s = 0$. This is done in the usual way, by considering $\{v, vT, vT^2, \ldots\} \subseteq V$. We must also check that $G(V, f)$ has a resolution of length one by $X$-vector bundles; it is easy to establish the exactness of the sequence

$$0 \to V(-1) \xrightarrow{g} V(0) \xrightarrow{k} G(V, f) \to 0,$$

where $k$ is the obvious map, and $g$ consists of

$$V \otimes R^+ \to V \otimes R^+$$

$$v \otimes p \mapsto f^{-1}(v) \otimes Tp \quad v \otimes p$$

and

$$V \otimes R^- \langle 1 \rangle \to V \otimes R^-$$

$$v \otimes q \mapsto f^{-1}(v) \otimes \varphi^{-1}(q) - v \otimes qT^{-1}.$$  

This is the characteristic sequence of the semilinear automorphism $f$ (cf. B, p. 630; G3, p. 442; and FH, Lemma 9].

Q.E.D.

Remark. If $\varphi = 1$, then as in [G3], $K_i \text{Aut}(\varphi)$ contains $K_i k$ as a direct
factor, because \((V, 1) \in \text{Aut}(\varphi)\) for any \(V \in \mathcal{P}_k\). If \(\varphi \neq 1\) then this no longer works. For this reason, it is not possible to describe \(K_i \text{Aut}(\varphi)\) as in [G3, Theorem 2]. The localization sequence for \(R^+ \to B^+\) does, however, split into short exact sequences, yielding the decomposition

\[
K_i B^+ = K_i R^+ \oplus K_{i-1} \text{Aut}(\varphi).
\]

5. THE CASE \(\varphi = \text{FRObenius}\)

In this final section we assume \(k\) is an algebraically closed (commutative) field of characteristic \(p\), and \(\varphi\) is the Frobenius, \(\varphi(a) = a^p\). We let \(\mathbb{F}_p\) denote the prime field. The functor \(L: \mathcal{P}_{\mathbb{F}_p} \to \text{Aut}(\varphi)\) defined by \(W \to (W \otimes k, 1_W \otimes \varphi)\) is know to be an equivalence of categories [Q1, p. 115] or [L]. This "deep descent" presents the possibility of using (4.6) and (1.1) to compute \(K_i B\).

**Remark.** To extract the statement that \(L\) is an equivalence one proceeds as follows. Given \((V, f) \in \text{Aut}(\varphi)\), choose a basis of \(V\) and let \(A\) be the matrix of \(f\) with respect to that basis; then \([L]\) provides a matrix \(B\) with \(B^{(p)} B^{-1} = A\). One can check that \(B\) provides a change of basis for \(V\) so that \(f\) fixes each element of the basis. This shows that the functor \(\text{Aut}(\varphi) \to \mathcal{P}_{\mathbb{F}_p}\) defined by \((V, f) \to \{v \in V \mid f(v) = v\}\) is well-defined and an inverse equivalence for \(L\).

**Theorem 5.1.** Under the assumptions made above, the map \(K_i(B) \to K_i(B/J)\) is an isomorphism for \(i > 0\), and thus \(K_i B \cong K_i k \times K_i k\).

**Proof.** We make explicit the dotted arrow in the following diagram:

\[
\begin{array}{ccc}
K_i \mathcal{P} & \xrightarrow{i^*} & K_i X \\
L^* \downarrow & & \downarrow (h_0^*, h_1^*) \\
K_i \mathbb{F}_p & \cdots \cdots \to & K_i k \times K_i k
\end{array}
\]

The characteristic sequence of (4.6) is natural in \(V\), so we find that \(i^* \circ L^* = (h_0^* - h_{-1}^*) \circ j^*\), where we let \(j\) denote the inclusion \(\mathbb{F}_p \to k\). The natural exact sequence of (1.1) yields \(h_0^* - h_{-1}^* = (\varphi^{-1})^* (h_1^* - h_0^*)\). The isomorphism \(V \langle 1 \rangle \otimes R^+ \to V \otimes R^+ \langle 1 \rangle\) defined by \(v \otimes p \to v \otimes \varphi(p)\), with a similar one for \(R^+\), provides an isomorphism \(V(n) \langle m \rangle = V \langle m \rangle(n)\); thus \(\varphi^* h_{n}^* = h_{n}^* \varphi^*\). Since \(\varphi \circ j^* = j^*\) we get \(i^* \circ L^* = (h_1^* - h_0^*) \circ j^*\), so the dotted arrow is \((j^*)^*\). The matrix of the composite map

\[
K_i k \times K_i k \to K_i X \to K_i B \to K_i B/J = K_i k \times K_i k
\]
is easily seen to be

\[ C = \begin{pmatrix} 1 & 1 \\ 1 & \varphi^* \end{pmatrix}. \]

Quillen has shown [Q2, pp. 583–585] that \( j^* \) is injective when \( k \) is an algebraic closure of \( \mathbb{F}_p \); commutativity of \( K \)-theory with filtering direct limits and the Hilbert Nullstellensatz extend this result to arbitrary \( k \). Thus from (4.6) one obtains the diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & K_i \mathbb{F}_p & \rightarrow & K_i k \times K_i k & \rightarrow & K_i B & \rightarrow & 0 \\
| & & | & | & | & | & | & \\
0 & \rightarrow & K_i \mathbb{F}_p & \rightarrow & K_i k \times K_i k & \rightarrow & K_i k \times K_i k & \rightarrow & 0
\end{array}
\]

in which the upper row is known to be exact. The exactness of the lower row would follow from the exactness of

\[ 0 \rightarrow K_i \mathbb{F}_p \rightarrow K_i k \xrightarrow{1-\varphi^*} K_i k \rightarrow 0 \]  

by a simple diagram chase. Quillen has shown [H, Corollary 5.2] that \( \varphi^* = \psi^p \), the \( p \)th Adams operation. The exactness of (\( \ast \)) is Quillen’s conjecture, shown by Hiller [H, Theorem 7.2] to be equivalent to Lichtenbaum’s conjecture that \( K_i(\mathbb{F}_p) \rightarrow K_i(k) \) has cokernel a rational vector space (here \( \mathbb{F}_p = \) algebraic closure of \( \mathbb{F}_p \)). The latter conjecture was proved by Suslin [Su].

Q.E.D.

REFERENCES


