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SK_1 OF AN INTERESTING PRINCIPAL IDEAL DOMAIN

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Suppose f is a diffeomorphism of a compact smooth manifold M. A point x in M is said to *wander* if it has a neighborhood U so $U \cap f^n U = \emptyset$ for all $n \neq 0$. The condition that f be *Morse–Smale* is equivalent to

(1) all but a finite number of points of M wander,

(2) f is structurally stable (i.e. a small change in f does not change the topology).

The Morse–Smale diffeomorphisms are considered to be simple from the point of view of dynamical systems. Easy examples can be constructed from the gradient flow of the height function on the sphere or the (non-vertical and non-horizontal) torus in 3-space, [7, 8].

It turns out that the condition that f be isotopic to a Morse–Smale diffeomorphism is equivalent to a condition on the action of f on integral homology (provided $\dim M > 5$ and $\pi_1 M = 0$). If we assume that the eigenvalues of f on homology are roots of unity, then an explicit obstruction to fulfillment of this condition (the "Lefschetz invariant") lies in a quotient, *SSF*, of the Grothendieck group constructed from abelian group endomorphisms with eigenvalues roots of unity, [8, 4].

Bass has studied this group and given an interesting presentation for it in terms of class groups of cyclotomic fields [2]. This presentation was later used by Lenstra to obtain an effective calculation of the group *SSF* (see [6] and [3]).

M. Stein and J. Franks noticed that a result from [5] implies $SSF = SK_1 R$, where R is obtained from $\mathbb{Z}[T]$ by inverting T and all polynomials $T^m - 1$ for $m \geq 1$ [2]. Such a result is natural because R is the smallest localization of $\mathbb{Z}[T]$ in which every polynomial, whose roots are roots of unity or zero, is invertible. Indeed, the notion of *characteristic polynomial* can be generalized slightly to yield an element of $K_1 R = R^* \oplus SK_1 R$ for any abelian group endomorphism whose eigenvalues are roots of 1.

The purpose of this paper is to present the following observations.

(i) R is a principal ideal domain. An explicit *presentation* for $SK_1 R$ in terms of elements of R (Mennicke symbols) is the result.

(ii) The consideration of the appropriate categories of $\mathbb{Z}[T]$ modules and

Quillen's localization theorem for abelian categories explains computations in [5] and [2].

First we summarize the prime ideal structure of $A=\mathbb{Z}[T]$, a factorial Jacobson domain of dimension two.

There is one prime ideal of height 0, namely (0).

The height one primes are not maximal and are generated by an irreducible element f of A. This f is either a rational prime p or a primitive irreducible polynomial. Among these we single out the cyclotomic primes — those generated by a cyclotomic polynomial φ_n, $n\geq 0$. We adopt the convention $\varphi_0=T$, and let S be the multiplicative set generated by $\{\varphi_n\}$.

The height two primes are all maximal and have finite residue fields.

In a similar way we distinguish among $\mathbb{Z}[T]$-modules M by looking at their supports. Let $\mathfrak{M}^i$, where $i\in\{0,1,C,2,\infty\}$, denote the category of finitely generated $\mathbb{Z}[T]$-modules M satisfying the appropriate additional condition:

0:	$\operatorname{codim} M\geq 0$	(i.e. the null condition),
1:	$\operatorname{codim} M\geq 1$	(i.e. M is torsion),
C:	$S^{-1}M=0$	(i.e. M is S-torsion),
2:	$\operatorname{codim} M\geq 2$	(i.e. M is finite),
∞:	$M=0$.	

Proposition 1. $\mathfrak{M}^2\subseteq\mathfrak{M}^C$.

Proof. Suppose $M\in\mathfrak{M}^2$. To show $M\in\mathfrak{M}^C$ we may assume M is a simple module, i.e. $M\cong A/m$ where m is a maximal ideal. Since A/m is a finite field, T is 0 or a root of unity mod m. Thus m contains an element of S, and $S^{-1}M=0$.

Corollary 2. *Let $R=S^{-1}\mathbb{Z}[T]$. Then R is a principal ideal domain.*

Proof. The proposition shows that R has no primes of height two, so $\dim R=1$. Since R is factorial, it is a principal ideal domain.

Corollary 3. *SK_1R has the following presentation as abelian group.*

generators: $\begin{bmatrix} f \\ g \end{bmatrix}$, *$f$, g relatively prime elements of R;*

relations: $\begin{bmatrix} f+\lambda g \\ g \end{bmatrix}=\begin{bmatrix} f \\ g \end{bmatrix}$,

$$\begin{bmatrix} f \\ g \end{bmatrix}=\begin{bmatrix} g \\ f \end{bmatrix},$$

$$\begin{bmatrix} fg \\ h \end{bmatrix}=\begin{bmatrix} f \\ h \end{bmatrix}\begin{bmatrix} g \\ h \end{bmatrix}.$$

This presentation is equivalent to the one deducible from [1, p. 298, Theorem 2.3].

Since R is a fraction ring of $\mathbb{Z}[T]$ we also have the following equivalent presentation.

$$\text{generators: } \begin{bmatrix} f \\ g \end{bmatrix}, \quad f, g \in \mathbb{Z}[T], \text{ g.c.d.}(f,g) \in S;$$

$$\text{relations: } \begin{bmatrix} f+\lambda g \\ g \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix},$$

$$\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ f \end{bmatrix},$$

$$\begin{bmatrix} fg \\ h \end{bmatrix} = \begin{bmatrix} f \\ h \end{bmatrix}\begin{bmatrix} g \\ h \end{bmatrix},$$

$$\begin{bmatrix} \varphi_n \\ g \end{bmatrix} = 1, \quad n \geq 0.$$

Now $[{}^f_g] = 1$ if $f, g \in \mathbb{Z}$, by the Euclidean algorithm in $\mathbb{Z}$. The Euclidean algorithm in $\mathbb{Q}(T)$ shows that any symbol $[{}^f_g]$ can be expressed as a product of symbols $[{}^u_p]^{-1}$ with $u \in \mathbb{Z}[T]$ and $p \in \mathbb{Z}$, representing the denominators $1/p$ introduced by the algorithm.

Corollary 4. *SK_1R is generated by symbols $[{}^f_p]$, where $f \in \mathbb{Z}[T] - \mathbb{Z}$, $p \in \mathbb{Z}$, and f and p are irreducible.*

If $\mathfrak{p} \in \operatorname{Spec} R$, then a map $k(\mathfrak{p})^* \to SK_1R$ is defined by $\bar{g} \mapsto [{}^f_g]$, where $\bar{g}$ is the residue mod $\mathfrak{p}$ of $g \in R$, and f is a generator for $\mathfrak{p}$. The Mennicke symbol presentation amounts to expressing SK_1R as a particular quotient of $\bigoplus_{\mathfrak{p}} k(\mathfrak{p})^*$.

Corollary 4 implies that $\bigoplus k(\mathfrak{p})^* \to SK_1R$ is still surjective if the sum includes all *arithmetic* primes $\mathfrak{p} = pR$ with $p \in \mathbb{Z}$, or if it includes all the *geometric* primes $\mathfrak{p} = fR$ with $f \in \mathbb{Z}[T] - \mathbb{Z}$. Thus, a computation of SK_1R would depend on some sort of reciprocity law involving both the arithmetic and the geometric primes. Notice that $k(\mathfrak{p})$ is always a global field: either $\mathbb{F}_p(T)$ or $\mathbb{Q}[T]/f(T)$.

Consider the localization $R \to R_{\mathbb{Q}}$ which removes the arithmetic primes. Since $R_{\mathbb{Q}}$ is a Euclidean domain, $SK_1R_{\mathbb{Q}} = 0$, and the long exact sequence for the localization yields

$$K_2R_{\mathbb{Q}} \to \bigoplus_{\text{geom. } \mathfrak{p}} k(\mathfrak{p})^* \to SK_1R \to 0$$

Thus finding relations for SK_1R in this case amounts to finding generators for $K_2R_{\mathbb{Q}}$ and doesn't appear easy. A similar statement can be made about the localization which removes the geometric primes; call it $R \to R'$. Combining the two localizations with the ultimate localization to $\mathbb{Q}(T)$ yields, in a standard way, a Mayer–Vietoris sequence

$$K_2R_{\mathbb{Q}}\oplus K_2R'\to K_2\mathbb{Q}(T)\to SK_1R\to 0.$$

Now let us define $0<1<C<2<\infty$. The categories $\mathfrak{M}^i$ are all abelian, and if $i\le j$, then $\mathfrak{M}^j$ is a Serre subcategory of $\mathfrak{M}^i$. We let $\mathfrak{M}^{i/j}$ denote the corresponding quotient abelian category. We can identify $\mathfrak{M}^i$ with $\mathfrak{M}^{i/\infty}$ for any i, and can use the fact that $\mathfrak{M}^{i/j}=\mathfrak{M}^{i/k}/\mathfrak{M}^{j/k}$ for any $i\le j\le k$. Let $K_*^{i/j}$ denote $K_*(\mathfrak{M}^{i/j})$.

The localization theorem for abelian categories [9, Section 5, Theorem 5] yields a long exact sequence

$$\cdots\to K_q^{j/k}\to K_q^{i/k}\to K_q^{i/j}\to K_{q-1}^{j/k}\to\cdots$$

for $i\le j\le k$.

Proposition 5. $SSF=\ker(K_0^C\to K_0^{C/2})$.

Proof. In the notation of [4] we have $SSF=G=\ker(p:K_0(QI)\to\mathscr{P})$. Here $\mathscr{P}=R^*$, $QI\equiv\mathfrak{M}^C$, and p is the map which takes the characteristic polynomial of "multiplication by T" on $M\in\mathfrak{M}^C$. (Any M in $\mathfrak{M}^C$ is a finitely generated abelian group because the cyclotomic primes are generated by monic polynomials.)

Let $\mathfrak{p}_n=\varphi_nA$. Notice that $k(\mathfrak{p}_n)=\mathbb{Q}(\zeta_n)$, where ζ_n is a primitive nth root of 1. Using the fact that every object of $\mathfrak{M}^{C/2}$ has finite length we see [9, Section 5, Corollary 1; also Section 7, Theorem 5.4] that $K_0^{C/2}=\bigoplus_{n\ge 0}K_0(k(\mathfrak{p}_n))=\bigoplus_{n\ge 0}\mathbb{Z}$ and the map $K_0^C\to\bigoplus_n\mathbb{Z}$ is given by $M\mapsto l=\text{length}\,M_{\mathfrak{p}_n}$. We must see that φ_n occurs with multiplicity l in the characteristic polynomial over $\mathbb{Z}$ for T on M. Notice first that M has a filtration by modules of the form

(i) $\mathbb{Z}[T]/\varphi_n$,
(ii) $\mathbb{Z}[T]/\varphi_m$, $m\ne n$,
(iii) finite module

(standard theory for modules over a noetherian ring). The assertion about l is obvious in each of these cases, ($l=1, 0, 0$ resp.) and follows in general by additivity.

Proposition 6. $SSF=\operatorname{ckr}(K_1^{C/2}\to K_0^2)$.

Proof. Use the long exact sequence

$$K_1^{C/2}\to K_0^2\to K_0^C\to C_0^{C/2}\to 0$$

and the previous proposition.

Now $K_1^{C/2}=\bigoplus_{n\ge 0}\mathbb{Q}(\zeta_n)^*$,

$$K_0^2=\bigoplus_{\substack{\text{max ideals}\\ \text{of } \mathbb{Z}[T]}}\mathbb{Z}=\Bigg(\bigoplus_{n\ge 0}\ \bigoplus_{\substack{\text{max ideals}\\ \text{of } \mathbb{Z}[\zeta_n]}}\mathbb{Z}\Bigg)\Big/\sim$$

where $m_1 \sim m_2$ if m_1 and m_2 are maximal ideals of cyclotomic rings which correspond, via the maps $\mathbb{Z}[T] \to \mathbb{Z}[\zeta_{n_i}]$, to the same maximal ideal of $\mathbb{Z}[T]$.

In other words, $\sim$ takes into account the intersections of the cyclotomic curves. These intersections are described explicitly in [2, Theorem 5.3]. Taking into account the fact that $K_1^{C/2} \to K_0^2$ is "take the divisor" [9, Section 7, Theorem 5.14 and Lemma 5.16] we recover Bass' presentation [2, Theorem 6.1]

$$SSF = \bigoplus \mathrm{Pic}(\mathbb{Z}[\zeta_n])/\sim .$$

Proposition 7. $SSF = \mathrm{ckr}(K_2^{0/1} \to K_1^{1/C})$.

Proof. To do this we show the sequence

$$K_2^{0/1} \to K_1^{1/C} \to K_0^C \to K_0^{C/2} \to 0$$

is exact. Exactness at K_0^C involves a diagram chase in the following diagram.

$$\begin{array}{ccccccc}
K_1^{1/2} & \longrightarrow & K_1^{1/C} & \longrightarrow & K_0^{C/2} & \xrightarrow{\alpha} & K_0^{1/2} \\
\downarrow & & \downarrow & & \| & & \\
K_0^2 & \longrightarrow & K_0^C & \longrightarrow & K_0^{C/2} & & \\
\downarrow{\scriptstyle\beta} & & & & & & \\
K_0^1 & & & & & &
\end{array}$$

The rows and columns are exact because they are portions of localization exact sequences. The map α is injective because $K_0^{C/2}$ (resp. $K_0^{1/2}$) is $\bigoplus \mathbb{Z}$, where the sum runs over cyclotomic primes (resp. all height one primes). The map β is zero because K_0^2 is generated by residue fields $k(m)$ where m is a maximal ideal of $\mathbb{Z}[T]$. If $p\mathbb{Z} = m \cap \mathbb{Z}$, then m comes from a principal ideal of $(\mathbb{Z}/p)[T] = B$; the exact sequence $0 \to B \to B \to k(m) \to 0$ of $\mathfrak{M}^1$ shows $[k(m)] = 0$ in K_0^1.

Exactness at $K_1^{1/C}$ involves the following diagram with exact row and column

$$\begin{array}{ccccccc}
K_2^{0/1} & \longrightarrow & K_1^1 & \longrightarrow & K_1^0 & \xrightarrow{\gamma} & K_1^{0/1} \\
 & \searrow & \downarrow & & & & \\
 & & K_1^{1/C} & & & & \\
 & & \downarrow & & & & \\
 & & K_0^C & & & &
\end{array}$$

The map γ is injective because $K_1^0 = K_1\mathbb{Z}[T] = K_1\mathbb{Z} = \mathbb{Z}^*$, and $K_1^{0/1} = K_1\mathbb{Q}(T) = \mathbb{Q}(T)^*$. The equality $K_1^{0/1} = K_1\mathbb{Q}(T)$ is true because every object of $\mathfrak{M}^{0/1}$ has finite length; the only residue field in this case is that of the height 0 prime, namely, the fraction field.

Proposition 8. $SSF = \ker(K_1^{0/C} \to K_1^{0/1}) = SK_1R$.

Proof. The first equality follows from Proposition 7 and the long exact top row of the following diagram

$$\begin{array}{ccccccccc} K_2^{0/1} & \longrightarrow & K_1^{1/C} & \longrightarrow & K_1^{0/C} & \longrightarrow & K_1^{0/1} & \longrightarrow & K_0^{1/C} \\ \downarrow \wr & & \downarrow \wr & & \downarrow & & \downarrow \wr & & \downarrow \wr \\ K_2^{0/1}R & \longrightarrow & K_1^1R & \longrightarrow & K_1R & \longrightarrow & K_1^{0/1}R & \longrightarrow & K_0^1R. \end{array}$$

The bottom row is constructed from the K-theory of R, and the vertical maps come from the localization $\mathbb{Z}[T] \to R$. The indicated maps are isomorphisms because the groups involved are direct sums of K-groups of the same residue fields. By the five lemma, the center map is an isomorphism, also.

It follows that $SSF = \ker(K_1R \to K_1^{0/1}R = \mathbb{Q}(T)^*) = SK_1R$.

It is possible to use Proposition 8 to recover the Mennicke symbol presentation of SK_1R given in Corollary 3. The additional information required is that

(i) $K_2^{0/1} = K_2\mathbb{Q}(T)$, so is generated by Steinberg symbols $\{u, v\}$,

(ii) $K_1^{1/C} = \bigoplus k(\mathfrak{p})^*$, where $\mathfrak{p}$ runs over height 1 non-cyclotomic primes of $\mathbb{Z}[T]$,

(iii) the boundary map $K_2^{0/1} \to K_1^{1/C}$ is the tame symbol.

One proceeds by matching up $\left[{u \atop v}\right]$ with the residue class of u in $k(\mathfrak{p})^*$, where v is irreducible and $\mathfrak{p} = vA$.

Now we can use Proposition 7 to see that $\left[{u \atop v}\right]$ corresponds to the class of the module $A/(u, v)$ in $\ker(K_0^C \to K_0^{C/2})$. Indeed, the boundary map $K_1^{1/C} \to K_0^C$ is "take the divisor". To see this in the case where v is irreducible we use the commutative square

$$\begin{array}{ccc} k(\mathfrak{p})^* = K_1^{0/1}(A/\mathfrak{p}) & \hookrightarrow & K_1^{1/C} \\ \Big\downarrow \text{div} & & \Big\downarrow \\ K_0^1(A/\mathfrak{p}) & \to & K_0^C \end{array}$$

Here $\mathfrak{p} = vA$. This also explains the relationship between the Mennicke symbol presentation and Bass' presentation of SSF.

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