Reduction theory using semistability, II

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In this paper we extend the result of [G] to include the case of an arithmetic subgroup $\Gamma$ of a semisimple algebraic group $G$. We represent the symmetric space $X = G/K$ (where $K$ is a maximal compact subgroup of $G$) as a certain space of inner products $H$ on the Lie algebra $\mathfrak{g}$ of $G$, namely, those inner products that come from the Killing form via a Cartan involution.

If we assume that $\Gamma$ is the stabilizer of a lattice $\mathfrak{g}_X \subset \mathfrak{g}$, then we may use the notions of semistability and of Harder–Narasimhan canonical filtration to study the action of $\Gamma$ on $X$. Our main result is the explicit construction of a closed submanifold (with boundary) in $X$ (of codimension zero) which is contractible, compact modulo $\Gamma$, and has boundary $\Gamma$-homotopy equivalent to the Tits building of $G$. Thus this subspace of $X$ provides an alternate route toward the theorems of Borel–Serre [BS] about the homological properties of $\Gamma$.

The proof in [G] of contractibility for the intersections of the neighborhoods of the cusps $[G, 7.18(c), 7.7]$ had the side effect of proving something already known, namely contractibility of the space $X$. In section 4 we use a simpler technique that makes use of the contractibility of $X$.

We make essential use of the assumption that $G$ is semisimple; it would have been nice to avoid it altogether, the way Borel–Serre do.

The idea of formulating the results of reduction theory in terms of functions which measure the distance to the cusps, here and in [G], is due to Harder [Ha1, Ha2]. Indeed, his function $\pi(P, \mathcal{O})$ of [Ha1, 2.4, p. 54] is presumably closely related to our vol $(L \cap W^1)$, where $W^1$ is the nilpotent radical of a parabolic subalgebra of $\mathfrak{g}$. In a sense, we answer the question in the last sentence of [Ha1], hopefully by a method as direct and transparent as the one anticipated there by Harder, thereby completing the program he laid out.

Motivation for doing reduction theory via semistability of Lie algebra bundles comes from the paper of Atiyah and Bott [AB], where the canonical filtration for a principal $G$-bundle on a Riemann surface is introduced.

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661
I thank Ranga Rao, Ofer Gabber, Hyman Bass, and David Kazhdan for valuable assistance.

1. Inner products on a Lie algebra

Let G be a connected semisimple algebraic group defined over \( \mathbb{Q} \). Since we will be dealing with the adjoint representation of G, we will replace G by \( G' = G/Z(G) \), where Z = “center of.” Since G is semisimple, \( Z(G) \) is finite, and \( G' \) is also a semisimple linear algebraic group; its adjoint representation is now faithful, so \( Z(G') = 1 \) [W, 3.50]. Any arithmetic subgroup \( \Gamma \subset G(\mathbb{R}) \) has a congruence subgroup \( \Gamma' \subset \Gamma \) of finite index which maps injectively to \( G'(\mathbb{R}) \); its image in \( G' \) is an arithmetic subgroup of \( G' \) [B, 7.13(2)]. Since all theorems we intend to prove about \( \Gamma \) are (equivalent to a theorem) of the form “\( \Gamma \) has a subgroup \( \Gamma' \) of finite index such that \( \Gamma' \) satisfies . . . ,” we may as well replace \( \Gamma \) by \( \Gamma' \).

Let \( \mathfrak{g} \) be the Lie algebra of the group of real points \( G(\mathbb{R}) \) of G; it is a semisimple Lie algebra, so the Cartan–Killing form \( B \) (induced from the canonical bilinear form \( (X, Y) \mapsto \text{tr}(XY) \) on \( \text{End}_{\mathbb{R}}(\mathfrak{g}) \) via \( \text{ad} : \mathfrak{g} \to \text{End}_{\mathbb{R}}(\mathfrak{g}) \)) is nondegenerate. The form \( B \) is characteristic (i.e. for all \( \theta \in \text{Aut}_{\text{Lie}_\text{alg}}(\mathfrak{g}) \) we have \( B(X, Y) = B(\theta X, \theta Y) \)) and is invariant (i.e. \( B([X, Y], Z) = B(X, [Y, Z]) \)).

Let \( H \) be an inner product on \( \mathfrak{g} \), i.e. a positive definite real symmetric bilinear form. If we think of a bilinear form also as a map \( \mathfrak{g} \to \mathfrak{g}^* \) (or as a matrix formed with respect to a basis of \( \mathfrak{g} \) and its dual basis of \( \mathfrak{g}^* \)), then an expression like \( H^{-1}B \) makes sense as an \( \mathbb{R} \)-linear endomorphism of \( \mathfrak{g} \) (and is independent of the choice of basis). We say that \( H \) is compatible (with the Lie bracket) if \( -H^{-1}B \) is a Lie algebra automorphism of \( \mathfrak{g} \). One consequence of compatibility is that \( \theta : = -H^{-1}B \) is a Cartan involution of \( \mathfrak{g} \), as we see now. The fact that \( \theta^2 = 1 \) follows from \( \theta B \theta = B \) (\( B \) is characteristic) and \( \theta B = -BH^{-1}B = B\theta \). Letting \( \kappa = (+1) \)-eigenspace of \( \theta \) and \( \rho = (-1) \)-eigenspace, we see from \( B\theta = -H < 0 \) that \( B \) is negative definite on \( \kappa \) and positive definite on \( \rho \). It follows from \( \theta B\theta = B \) that \( \kappa \) and \( \rho \) are orthogonal with respect to \( B \), and thus that \( \theta \) is a Cartan involution.

Another consequence of compatibility is that \( H \) is compatible with \( B \) in the sense of [G, Definition 7.3]. For the equation \( \theta^2 = 1 \) may be rewritten as \( \theta BH^{-1}B = H \), which says that \( B \) is an isometry \( \mathfrak{g} \to \mathfrak{g}^* \) with respect to the inner products \( H \) on \( \mathfrak{g} \) and \( H^{-1} \) on \( \mathfrak{g}^* \).

Now let \( X \) be the space of all compatible inner products on \( \mathfrak{g} \); it is a closed subspace of the space of all inner products on \( \mathfrak{g} \). Suppose \( F \) is any Lie algebra automorphism of \( \mathfrak{g} \); then \( FB = B \) because \( B \) is characteristic. For any \( H \in X \) we
find that $H' := FH F \in X$, too. See that by computing $-H'^{-1} B = F^{-1} \theta F$. Thus we may let $G$ act on $X$ via the adjoint representation, i.e. $(H, g) \mapsto (\text{Ad } g) H (\text{Ad } g)$, $g \in g$, $H \in X$. In terms of $\theta$, the action of $G$ on $X$ is expressed by $\theta \mapsto (\text{Ad } g)^{-1} \theta (\text{Ad } g)$.

Now choose $\Theta : G(\mathbb{R}) \rightarrow G(\mathbb{R})$ a Cartan involution with $\theta = d \Theta$. We require $\Theta$ to extend to an algebraic map $G \rightarrow G$ as in [BS, 1.6]; this makes $\Theta$ unique even if $G(\mathbb{R})$ is not connected, because $G$ is. It is known that $K := \{ g \in G(\mathbb{R}) \mid \Theta(g) = g \}$ is a maximal compact subgroup of $G(\mathbb{R})$. We now prove that $K$ is the stabilizer of $\theta$ (and thus is also the stabilizer of $H$). Let $C_g : G \rightarrow G$ denote conjugation by $g$, and let $( )^0$ denote "connected component of the identity in." Then $\theta = (\text{Ad } g)^{-1} \theta (\text{Ad } g)$ iff $d(C_g^{-1} \Theta C_g) = d \Theta$ iff for all $h \in G(\mathbb{R})$, $\Theta(h) = g^{-1} \Theta(g) \Theta(h) \Theta(g^{-1})_g$ iff $g^{-1} \Theta(g) \in Z(G(\mathbb{R})^0)$ = 1 iff $g \in K$. (We know $Z(G(\mathbb{R})^0) = 1$ because $Z(G(\mathbb{C})) = 1$ and $G(\mathbb{R})^0$ is Zariski dense in $G$). Since any two Cartan involutions are conjugate under $G(\mathbb{R})$ [[V, Propositions 2 & 9, p. 193–195, Part II, Section 1]], we have a diffeomorphism $X \simeq G(\mathbb{R})/K$. In particular, we now know that $X$ is contractible [BHC, Lemma 1.7], [B, 9.10].

Since $G$ is defined over $\mathbb{Q}$, its Lie algebra has a rational structure $g_\mathbb{Q} \subset g$. Moreover, the adjoint representation $G \rightarrow \text{Aut } (g_\mathbb{Q})$ is rational (regular map with rational number coefficients). Let $G \subset G$ be any arithmetic subgroup. Then by [B, 7.13(1)] we may find an integral structure $g_\mathbb{Z} \subset g_\mathbb{Q}$ such that $(\text{Ad } G) g_\mathbb{Z} \subset g_\mathbb{Z}$. Since $g_\mathbb{Q}$ is a Lie algebra, by clearing denominators (i.e. replacing $g_\mathbb{Z}$ by $N g_\mathbb{Z}$ for some large integer $N$) we can achieve $[g_\mathbb{Z}, g_\mathbb{Z}] \subset g_\mathbb{Z}$. Having done this, we see that the form $B$ is integral (meaning $B(g_\mathbb{Z}, g_\mathbb{Z}) \subset \mathbb{Z}$). For any $H \in X$, the pair $L = (g_\mathbb{Z}, H)$ is an example of what we call a lattice in $[G]$. Henceforth $g_\mathbb{Z}$ remains fixed.

By a $\mathbb{Q}$-subspace $W$ of $g$ we mean an $\mathbb{R}$-subspace of the form $W = W_\mathbb{Q} \otimes \mathbb{R}$, with $W_\mathbb{Q} \subset g_\mathbb{Q}$. For any such $W$ and any inner product $H$ on $g$ we defined a number $d_W(H) = d(W, H) > 0$ in [G, 2.1] with the property that $d_W(H) > 1$ iff $W$ is in the canonical filtration of $L$.

For $t \geq 1$ we define $X_w(t) := X(W, t) := \{ x \in X \mid d_W(H) > t \}$; this is an open subset of $X$ because $d_W$ is a continuous function of $H$, see [G, 3.1]. We will be interested mainly in the case where $W$ is a parabolic $\mathbb{Q}$-subalgebra of $g$, because the other $W$'s will correspond to cusps for the larger group $\text{GL}(g)$ which are "uninhabited" by $X$ (i.e. $X$ does not approach the ends of these other cusps without also approaching the cusp corresponding to a $\mathbb{Q}$-parabolic). We make this precise as follows. Define $X_w(t) = X - \bigcup_w X(W, t)$, where $W$ runs over all parabolic $\mathbb{Q}$-subalgebras of $g$; this is a closed subspace of $X$.

If $g \in \Gamma$, then $\text{Ad } g$ is a Lie algebra automorphism of $g_\mathbb{Z}$, and so we have $g \cdot X(W, t) = X((\text{Ad } g) \cdot W, t)$ as in [G, 2.2(b)]. Thus $X_w(t)$ is stable under $\Gamma$.

THEOREM 1.1 $X_w(t)$ is compact modulo $\Gamma$. 
COROLLARY 2.3. If $L$ and $M$ are lattices, then there is a constant $c$ depending only on the ranks of $L$ and $M$ so that

$$\min L \otimes M \geq \min M + \min L - c,$$

and

$$\max L \otimes M \leq \max M + \max L + c.$$

Proof. Apply 2.2 to $\Hom(L^*, M) = L \otimes M$ and use $\min L = -\max L^*$.

COROLLARY 2.4. Suppose $L$, $M$, and $N$ are lattices, and $f: L \otimes M \to N$ is a nonzero map. There is a constant $c$ (depending only on $\|f\|$ and the ranks of $L$, $M$, and $N$) such that any sublattices $L' \subset L$, $M' \subset M$, and $N' \subset N$ with $\min N/N' > \max L' + \max M' + c$ also satisfy $f(L' \otimes M') \subset N'$.

Proof. Let $N'' = N^*/N'$, and consider the map $f': L' \otimes M' \to N''$ induced by $f$. For suitable orthonormal bases, we see that the matrix of $f'$ occurs as part of the matrix of $f$, so $\|f\| \geq \|f'\|$. Applying 2.2 and 2.3 we find constants with

$$\min \Hom(L' \otimes M', N'') \geq \min N'' - \max L' \otimes M' - c_1$$

$$\geq \min N'' - \max L' - \max M' - c_2.$$

If $f' \neq 0$, then we know that

$$\log \|f'\| \geq \min \Hom(L' \otimes M', N'').$$

Letting $c = \log \|f\| + c_2$ we find

$$\min N/N' \leq \max L' + \max M' + c.$$

QED.

3. The search for parabolic subalgebras

We need the following convenient characterization of parabolic subalgebras, which was used in [AB].

PROPOSITION 3.1 [AB]. Suppose $\mathfrak{g}$ is semisimple, $W \subset \mathfrak{g}$ is a subalgebra, and the orthogonal $W^\vee$ (formed with respect to the Killing form $B$) is contained in $W$ and is $\ad_W$-nilpotent. Then $W$ is parabolic. The converse is also true.
Proof [Ranga Rao]. We tensor everything with $C$, so we are dealing with complex Lie algebras, and are in the case where Borel subalgebras exist. Since $W^\gamma$ is solvable, it is contained in a Borel subalgebra $\mathfrak{b}$, Let $\mathfrak{h} = \{\mathfrak{h}, \mathfrak{n}\}$, and write $\mathfrak{h} = \mathfrak{h} \oplus \mathfrak{n}$, where $\mathfrak{h}$ is a Cartan subalgebra.

We claim that $W^\gamma \subset \mathfrak{n}$. Choose $y \in W^\gamma$ and write $y = h + n$, $h \in \mathfrak{h}$, $n \in \mathfrak{n}$. We have an ordering on the roots $\alpha$ so that $\mathfrak{n} = \sum_{\alpha > 0} \mathfrak{g}_\alpha$, and $\mathfrak{h} = \sum_{\alpha = 0} \mathfrak{g}_\alpha$. If we choose a basis of $\mathfrak{h}$ which contains bases for the root spaces, and is totally ordered in a way that extends the partial ordering of the roots, then the matrix of $\text{ad}_h n$ is strictly lower triangular (because $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha + \beta}$), and the matrix of $\text{ad}_h h$ is diagonal. But $\text{ad}_h (h + n) = \text{ad}_h y$ is nilpotent by hypothesis, so $\text{ad}_h h = 0$. Thus $\alpha(h) = \text{ad}_{\mathfrak{h}}(h) = 0$ for $\alpha > 0$. Now the simple roots are a basis of $\mathfrak{h}^*$, and so $h = 0$, and thus $W^\gamma \subset \mathfrak{n}$.

Next we claim that $\mathfrak{h}^* = \mathfrak{h}$. This follows from $B(H, \mathfrak{g}_n) = 0$ (\(\alpha \neq 0\)), $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ (\(\alpha + \beta \neq 0\)), and $\dim \mathfrak{n} = \text{codim} \mathfrak{h}$.

Finally, we have $\mathfrak{h} = \mathfrak{n}^\vee \subset W^\vee = W$, so $W$ contains a Borel subalgebra, and is parabolic (by definition). QED.

Proposition 3.1 describes parabolic subalgebras completely in terms of trilinear and bilinear data, namely $[\cdot, \cdot]: \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ and $B: \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$. We now see how to apply the facts from section 2. Recall the notation $L = (\mathfrak{g}_Z, H)$ for inner products $H \in X$.

**Lemma 3.2.** Let $b = [\cdot, \cdot] \in \text{Hom}(L \otimes L, L)$. The norm $\|b\|$ is independent of the choice of $H \in X$.

Proof [D. Kazhdan]. Suppose $H' \in X$ also, and $L' = (\mathfrak{g}_Z, H')$. Then for some $g \in G(\mathbb{R})$ we have $H' = (\text{Ad} g) H (\text{Ad} g)$. Since $\text{Ad} g$ is a Lie algebra automorphism of $\mathfrak{g}$ we get a commutative diagram

$$
\begin{array}{ccc}
V \otimes V & \cong & V \\
\text{Ad} g \otimes \text{Ad} g & \downarrow & \text{Ad} g \\
V' \otimes V' & \cong & V'
\end{array}
$$

where $V = L \otimes \mathbb{R}$, $V' = L' \otimes \mathbb{R}$, and the vertical maps are isometries. The vertical maps do not preserve the integral structure; nevertheless, the diagram shows that $\|b\| = \|b'\|$. QED.

For $H \in X$ and $L = (\mathfrak{g}_Z, H)$, we let $0 = L_0 \subset L_1 \subset \cdots \subset L_{t-1} \subset L_t = L$ be the canonical filtration of $L$, $\sigma_i = \text{slope} (L_i/L_{i-1})$, and $L_i^* = \text{the dual with respect to}$
B. We introduce the notation \( \text{err}(c) \) to denote any (undetermined) number of absolute value \( \leq c \), i.e. \( x = y + \text{err}(c) \) will mean \( |x - y| \leq c \).

**PROPOSITION 3.3.** There is a constant \( c > 0 \) so that for all \( H \in X \) we have

\[
\begin{align*}
(P1) & \quad \forall i, j, k \quad \sigma_i + \sigma_j + c \leq \sigma_{k+1} \Rightarrow [L_i, L_j] \subseteq L_k \\
(P2) & \quad \forall i \quad \sigma_{i+1} \geq \sigma_i + c \quad \Rightarrow \exists! j \quad L_j = L_i^\gamma, \quad \sigma_j = -\sigma_{i+1} + \text{err}(c), \\
& \quad \text{and } \sigma_{i+1} = -\sigma_i + \text{err}(c).
\end{align*}
\]

**Proof.** P1 follows from (3.2) and (2.4) because \( \sigma_1 = \max L_i \) and \( \sigma_{k+1} = \min L/L_k \).

P2 follows from [G, 7.14] and the proof of [G, 7.13]. QED.

**PROPOSITION 3.4.** With \( c \) as in 3.3, assume \( i \) satisfies \( 1 \leq i \leq s - 1 \) and

\[
\begin{align*}
(A) & \quad 2c \leq \sigma_{i+1}, \\
(B) & \quad \sigma_i + c \leq \sigma_{i+1}, \text{ and} \\
(C) & \quad 2\sigma_i + c \leq \sigma_{i+1}.
\end{align*}
\]

Then \( L_i \) is a parabolic subalgebra of \( \mathfrak{g} \).

**Proof.** By (B) and (P2) we know \( L_i^\gamma = L_j \) for some \( j \). In addition, \( \sigma_j = -\sigma_{j+1} + \text{err}(c) \leq -2c + \text{err}(c) \leq -c \). Were it true that \( j \geq i + 1 \) we would have \( \sigma_j \geq \sigma_{j+1} \geq 2c \) as well, so we must have \( j \leq i \) and thus \( L_j \subseteq L_i \).

By (P1) and (C) we know \( [L_i, L_j] \subseteq L_i \), so \( L_i \) is a subalgebra of \( \mathfrak{g} \).

For all \( k \) we have \( \sigma_j + \sigma_k + c \leq -c + \sigma_k + c = \sigma_{(k-1)+1} \), so by (P1) we have \( [L_j, L_k] \subseteq L_{k-1} \), which shows that \( L_i^\gamma = L_j \) is ad-\( \mathfrak{g} \)-nilpotent. Now by (3.1) we know \( L_i \) is parabolic. QED.

Now we have enough to prove Theorem 1.2.

**Proof.** of Theorem 1.2. Suppose we have an \( H \in X_\sigma(t) \). Then for all \( j \) (\( 1 \leq j \leq s \)) we have either \( j = 1 \) (in which case \( \sigma_j \leq \text{slope } L \)); or \( L_{j-1} \) is parabolic, in which case \( \sigma_j \leq \sigma_{j-1} + \log t \); or, finally, \( L_{j-1} \) is not parabolic, in which case 3.4 shows that either \( \sigma_j < 2c \), or \( \sigma_j < \sigma_{j-1} + c \), or \( \sigma_j < 2\sigma_{j-1} + c \). We apply descending induction to prove that for each \( i < s \) there are a finite number of polynomials \( P_\sigma(T) \) with positive coefficients such that \( \sigma_i < \max_\sigma P_\sigma(T) \). We find that \( \max L = \sigma_s < \max_\sigma P_\sigma(T) \leq \max_\sigma P_\sigma(slope L) \); the latter number is independent of \( L \) because slope \( L \) is. The dependence of that number on \( s \) can be removed because \( 1 \leq s \leq \dim \mathfrak{g} \), yielding an upper bound for max \( L \) which is independent of \( H \). Clearly an upper bound on max \( L \) provides also a lower bound on min \( L \),
again using that slope $L$ is independent of $H$. But $\min L \leq \log \|v\|$ for any nonzero $v \in L$. Thus we have proved the contrapositive of 1.2. QED.

4. Topology

To prove results about $X_r$, we introduce flows on $X$ for each parabolic subalgebra $W \subset \mathfrak{g}$, namely, the geodesic action of $[B, S]$. 

**Lemma 4.1.** Suppose $W \subset \mathfrak{g}$ is a parabolic subalgebra. Then for some $s$, $\mathfrak{g}$ has a filtration by subspaces $0 = W^{-s} \subset \cdots \subset W^{-1} \subset W^0 \subset \cdots \subset W^s = \mathfrak{g}$ such that $W^0 = W$, $W^i = (W^{-i+1})^\vee$ all $i$, and $[W^i, W^j] \subset W^{i+j}$ all $i, j$ (using the obvious convention when $i + j$ falls outside the range $[s + 1, -s]$). Moreover, if $W$ is a $\mathbb{Q}$-subspace, so is each $W^i$.

**Proof.** Define $W^0 = W$, $W^1 = W^\vee$ (= orthogonal with respect to Killing form $B$); for $i > 1$ define $W^i$ inductively by $W^i = [W^{i-1}, W^1]$; for $i < 0$ define $W^i = (W^{-i+1})^\vee$. This is clearly a filtration, and $W^{1+i} = 0$ for some $s$ because $W^1$ is nilpotent (3.1). To prove that $[W^i, W^j] \subset W^{i+j}$ we may first tensor with $\mathbb{C}$ and make use of roots relative to a Cartan subalgebra $\mathfrak{h} \subset W^0$. [This approach was explained to me by Ranga Rao; I was also able to make a complicated combinatorial proof, using only the Jacobi identity and the invariance of $B$.] Let $S$ be a set of simple roots chosen so $\alpha \in S \to \varrho_\alpha \subset W$. Then by [H, Exercise 6, p. 87] find a subset $S' \subset S$ so $W$ is generated by the subspaces $\varrho_\alpha$ for $\alpha \in S$, $\varrho_{-\alpha}$ for $\alpha \in S'$, and $\mathfrak{h}$. The root spaces $\varrho_\alpha$ are 1-dimensional, and if $\alpha, \beta, \alpha + \beta$ are all roots, then $[\varrho_\alpha, \varrho_\beta] = \varrho_{\alpha + \beta}$ [H, Prop. 8.4(d)]; if $\alpha$ is a root, then $[\varrho_\alpha, \varrho_{-\alpha}] \in \mathfrak{h}$ [H, Prop. 8.3(d)]; and $[\mathfrak{h}, \varrho_\alpha] \subset \varrho_\alpha$. We also know that $B(\varrho_\alpha, \varrho_{\beta}) = 0$; and $B(\varrho_\alpha, \varrho_{\beta}) = 0$ if $\alpha + \beta \neq 0$. Introduce the notation $g(R) = \bigoplus_{\alpha \in R \cap \mathfrak{h}} \varrho_\alpha$, where $\Delta$ is the roots, and $R$ is any subset of $\mathbb{Z} \Delta$ (we include $\alpha = 0$, and let $\varrho_0 = \mathfrak{h}$). Then we see that $W^0 = g(\mathbb{N}S + \mathbb{Z}S')$, $W^1 = g(\mathbb{Z} \Delta - (\mathbb{N}S + \mathbb{Z}S')) = g(T_1)$, where $T_1 = \{ \sum \alpha \in \mathbb{N}n_\alpha \alpha \in \mathbb{Z} \Delta : \sum \alpha \in S - S' n_\alpha \geq i \}$ and """ denotes "complement of."" Using the fact that if $\sum n_\alpha \alpha$ is a root, then all $n_\alpha$ have the same sign, we deduce that $W^i = g(T_i)$ for $i \geq 1$, and then that $W^{-i} = (W^{i+1})^\vee = g(\mathbb{Z} \Delta - T_{i+1}) = g(T_{-i})$. The conclusion then follows easily. QED.

**Construction 4.2.** Fix $W$ and $W^i$ as in 4.1. Fix an inner product $H \in X$ and its corresponding Cartan involution $\theta$. Use $(\cdot)^\perp$ to denote the orthogonal complement with respect to $H$. Define $V^i = W^i \cap (W^{i+1})^\perp$, so $\mathfrak{g} = V^1 \oplus \cdots \oplus V^1 \oplus V^0 \oplus V^{-1} \oplus \cdots \oplus V^i$ is a decomposition orthogonal with respect to $H$.

By compatibility of $H$ and $B$, and using [G, proof of 7.5] deduce that
\((W^{i+1})^\perp = (W^{-i})^\perp = \theta W^{-i}\) and \(V' = W^I \cap \theta W^{-i}\). Since \(\theta\) preserves the bracket, we see that \([V', V'] \subset V^{i+j}\) (all \(i, j\)) and thus we have a Lie algebra grading of \(\mathfrak{g}\).

[In terms of roots (after tensoring with \(\mathbb{C}\)) we see that \(V' = \mathfrak{g}(U_i)\) where \(U_i = \{\sum_{a \in S} n_a \alpha_a : \sum_{a \in S} n_a \alpha_a = i\}\), provided \(\mathcal{A}\) is chosen to be \(\mathcal{A} \cap \theta \mathcal{A}\), where \(\mathcal{A}\) is a Borel subalgebra contained in \(W\).] Given \(r \in \mathbb{R}\), \(r > 0\), the graded map \(F_r : \mathfrak{g} \to \mathfrak{g}\), defined as multiplication by \(r^{-1}\) on \(V_i\), is a Lie algebra automorphism of \(\mathfrak{g}\). Define \(H' = F_r \cdot H \cdot F_r \in X\). The corresponding lattice \(L' = (\mathbb{Q}z, H')\) will be denoted by \(L(W; r)\); the same notation was used in [G, 7.16] for a related concept in a different context.

These operations commute to some extent. If \(W' \subset W\) are \(\mathbb{Q}\)-parabolic subalgebras then

\[L(W'; r') \{W; r\} = L(W; r) \{W'; r\}. \tag{4.2.1}\]

This is best seen by examining the root spaces, as in the previous paragraph, the point being that \(F_r\) is multiplication by a scalar on each root space \(\mathfrak{g}_\alpha\). The hypothesis that \(W' \subset W\) ensures that we may assume \(\mathcal{A} \subset W'\) and \(\mathcal{A} \subset W\), so the corresponding map \(F_r\) is diagonal for the same root space decomposition that \(F_r\) is diagonal for, and thus \(F_r\) and \(F_r'\) commute.

**LEMMA 4.3.** If \(W' \subset W\) are \(\mathbb{Q}\)-parabolic subalgebras of \(\mathfrak{g}\), \(L \in X\), and \(r > 1\), then

(a) \(d(W, L(W; r)) \leq r \cdot d(W, L)\)

(a') \(d(W, L(w, r^{-1})) \leq r^{-1} \cdot d(W, L)\)

(b) \(d(W', L(W; r)) \leq d(W', L)\)

(b') \(d(W', L(W; r^{-1})) \leq d(W', L)\).

**Remark.** It would have been nice to have (b) for the case \(W \subset W'\), but that is not true.

**Proof.** Let \(V = \mathfrak{g}\). Let \(0 \subset W^{i} \subset \cdots \subset W^1 \subset W^0 = W \subset W^{-1} \subset \cdots \) and \(0 \subset W^{i'} \subset \cdots \subset W^1 \subset W^0 = W' \subset W^{-1} \subset \cdots \) be chosen as in 4.2. It is not always true that the union of these two filtrations is a filtration, but we can manage anyway. Notice that (a) \(\Rightarrow (a')\) and (b) \(\Rightarrow (b')\).

We prove (b) assuming \(W' \neq W\). In the notation of [G, 2.4] we have \(L(W; r) = L[r^{-1}][W^i; r] \cdots [W^1; r][W^0; r] \cdots [W^1; r^{-1}; r]\) and \(W' \subset \cdots \subset W^1 \subset W^0 \subset \cdots \subset W^{-1}\). Since \(d(W', L[r^{-1}]) = d(W', L)\) [see G, 2.1], all we need is the following lemma (from which (a) also follows). QED.

**LEMMA 4.4.** If \(L\) is a lattice, \(W' \subset W\) are \(\mathbb{Q}\)-subspaces of \(V = L \otimes \mathbb{R}\) and
\[ r \geq 1, \text{ then} \]
(a) \( d(W, L[W; r]) = r \cdot d(W, L) \)
(b) \( d(W, L[W'; r]) \geq d(W, L) \)
(c) \( d(W', L[W; r]) \geq d(W', L) \)

**Proof.** Part (a) was proved in [G, 2.4]. By duality [G, 7.1.] (b) will follow from (c). To prove (c) we replace \( L \) by \( L/W' \cap L \); then we must show that \( \min (L[W; r]) \geq \min (L) \) (notation from [G, 1.23]). So for each \( \mathcal{Q} \)-subspace \( U \subset V \) we must show that \( \text{vol} (L[W; r] \cap U) \geq \text{vol} (L \cap U) \). Let \( m = \dim U \), choose an orthonormal basis for \( W \) which contains an orthonormal basis for \( W \), and write \( \Lambda^m(U \cap L) = \mathbb{Z}_u \), with \( u = \sum a_i e_i \), where \( i = (i_1, \ldots, i_m) \in \mathbb{N}^m \) and \( e_j = e_i \wedge \cdots \wedge e_j \). Then \( \text{vol} (L \cap U) = \| u \| = (\sum a_i^2)^{1/2} \), but \( \text{vol} (L[W; r] \cap U) = (\sum (a_i r^{1/2})^2)^{1/2} \), where \( \langle I \rangle = (i : e_i \notin W) \). Since \( r \geq 1 \), the desired inequality is clear. \( \square \)

Our next goal is to identify the homotopy types of \( X_{ss} \), its boundary, and its complement. In order to do these three tasks simultaneously, we abstract the information available to us. We let \( \mathcal{W} \) be the partially ordered set of parabolic \( \mathcal{Q} \)-subalgebras of \( g \); for \( W \in \mathcal{W} \) let \( h_w : X \to \mathbb{R} \) be the continuous map defined by \( h_w(x) = \log (d(W, x)/r) \), where \( t \) is the fixed number referred to in 1.1. Let \( \varphi_w : X \times \mathbb{R} \to X \) be defined by \( \varphi_w(L, r) = L \cdot e^r \). Let \( \mathcal{W}_i = \{ W \in \mathcal{W} : \dim W = i \} \). Let \( T = \dim g \). These data satisfy the following axioms.

(A1) \( \mathcal{W} \) is a partially ordered subset which is the disjoint union of subsets \( \mathcal{W}_i \), \( (i = 1, \ldots, T) \), such that \( W \in \mathcal{W}_i, U \in \mathcal{W}_j \) with \( W \cap U \) implies \( i < j \).

(A2) \( X \) is a nonempty topological space.

(A3) \( \forall W \in \mathcal{W} h_w : X \to \mathbb{R} \) is a continuous map.

(A4) \( \forall W \in \mathcal{W} \varphi_w : X \times \mathbb{R} \to X \) is a continuous action of the topological group \( \mathbb{R} \) on \( X \), i.e. \( \varphi_w(x, 0) = x \) and \( \varphi_w(\varphi_w(x, r), s) = \varphi_w(x, r + s) \).

(A5) \( \forall W \in \mathcal{W} \forall x \in X \) the map \( \mathbb{R} \to \mathbb{R} \) defined by \( r \mapsto h_w(\varphi_w(x, r)) \) is an increasing bijection.

(A6) \( \forall x \in X \forall W \subseteq U \in \mathcal{W} [r = 0 \Rightarrow h_w(\varphi_w(x, r)) = h_w(x)] \) and \( [r \leq 0 \Rightarrow h_w(x) \geq h_w(\varphi_w(x, r))] \).

(A7) The sets \( \bar{X}_w = \{ x \in X : h_w(x) \geq 0 \} \) form a locally finite family, and moreover, any \( x \in X \) has a neighborhood \( \mathcal{U} \) such that \( \{ W \in \mathcal{W} : \mathcal{U} \cap \bar{X}_w \neq \phi \} \) is a chain in \( \mathcal{W} \).

We also define \( X_{ss} = \{ x \in X : \forall W h_w(x) \leq 0 \} \), and \( X^+ = \bigcup_{W \in \mathcal{W}} \bar{X}_w \). The following proofs will be based solely upon the axioms.

**THEOREM 4.5.** (a) **The boundary** \( \partial X_{ss} \) is the set \( \{ x \in X_{ss} : \exists W h_w(x) = 0 \} \).
(b) There is a deformation retraction of $X$ onto $X_{ss}$ which restricts to a deformation retraction of $X^+$ onto $\partial X_{ss}$.

(c) Given $W_1 < \cdots < W_m \in W$, there is a deformation retraction of $X$ onto $\dot{X}_w \cap \cdots \cap \dot{X}_{w_m}$.

First, we need some lemmas.

LEMMA 4.6. Given $W \in W$, define $\psi_w : X \times \mathbb{R} \to \mathbb{R}$ by decreeing that $\psi_w(x, r) = t$ iff $h_w(\varphi_w(x, t)) = r$ (such a $t$ exists by (A5)). The map $\psi_w$ is continuous.

Proof. Let $\varphi = \varphi_w$, $h = h_w$, and $\psi = \psi_w$. Suppose $\psi(x, r) = t$ and $\varepsilon > 0$ is given. We know $h(\varphi(x, t)) = r$, so by (A5) we may choose $\delta > 0$ so $h(\varphi(x, t - \varepsilon)) < r - \delta < r + \delta < h(\varphi(x, t + \varepsilon))$. Choose a neighborhood of $x$ so $\forall y \in U h(\varphi(y, t - \varepsilon)) < r - \delta$ and $r + \delta < h(\varphi(y, t + \varepsilon))$. If $y \in U$ and $s \in (r - \delta, r + \delta)$, then $h(\varphi(y, t - \varepsilon)) < s = h(\varphi(y, x, s)) < h(\varphi(y, t + \varepsilon))$, and thus $t < \varepsilon < \psi(y, s) < t + \varepsilon$. QED.

LEMMA 4.7 [Rectification of $\varphi_w$]. Define $\varphi'_w : X \times \mathbb{R} \to X$ by the formula $\varphi'_w(x, r) = \varphi_w(x, \psi_w(x, r + h_w(x)))$. The maps $\{\varphi'_w\}$ satisfy the axioms (A1)–(A7) as well as

(A8) $\forall x \in X \forall r h_w(\varphi_w(x, r)) = h_w(x) + r$ [equivariance of $h_w$].

Proof. Axiom (A8) for $\varphi'_w$ follows from the definition.

Check (A4). Continuity follows from the previous lemma. Check $h_w(\varphi'_w(\varphi'_w(x, r), s)) = h_w(\varphi'_w(x, r)) + s = h_w(x) + r + s = h_w(\varphi'_w(x, r + s))$; thus $\varphi'_w(\varphi'_w(x, r), s) = \varphi'_w(x, r + s)$, as both sides are in the orbit of $x$ for the $\varphi_w$ action.

It is immediate that (A8) implies (A5).

To check (A6) notice that $\psi_w(x, r + h_w(x)) \geq 0$ iff $r \geq 0$, which itself follows from (A6) for $\varphi_w$. QED.

Proof of 4.5a. For any $\varepsilon > 0$, $W \in W$, and $x \in X_{ss}$ with $h_w(x) = 0$, we have $h_w(\varphi_w(x, \varepsilon)) > 0$, and thus $\varphi_w(x, \varepsilon) \notin X_{ss}$. Since $\varphi_w(x, \varepsilon)$ is continuous in $\varepsilon$, we see $x \in \partial X_{ss}$.

Proof of 4.5b. We define the homotopy $H : X \times [0, T] \to X$ as a composite of
homotopies $H_1, \ldots, H_T : X \times [0, 1] \to X$, i.e. for $s \in [i - 1, i]$ and $x \in X$ we have

$$H(x, s) = H_i(H_{i-1}(H_{i-2}(\ldots H_1(x, 1) \ldots, 1), s - i + 1).$$

Each $H_i$ will satisfy $H_i(x, 0) = x$ (all $x \in X$) so $H$ will be continuous.

Since 4.5b makes no reference to the maps $\varphi_w$, we may use 4.7 to rectify the $\varphi_w$'s, and thus may assume (A8) is satisfied. Now set

$$H_i(x, s) = \begin{cases} 
\varphi_w(x, -sh_w(x)) & \text{if } x \in X_w \text{ and } W \in \mathcal{W}_i \\
\varphi_w(x) & \text{if } h_w(x) \leq 0 \text{ all } W \in \mathcal{W}_i
\end{cases}$$

Axioms (A1, 7) ensure that the sets $X_w (W \in \mathcal{W}_i)$ are disjoint. Since $\varphi_w(x, 0) = x$ we see that when the two cases in the definition of $H_i$ overlap, they agree; moreover the closed sets involved form a locally finite family, so $H_i$ is continuous.

Now we claim that

$$\forall j \leq i \quad \forall U \in \mathcal{W}_j \quad \forall x \in X \quad h_U(H(x, i)) \leq 0$$

(\#)

We prove this by induction on $i$, the case $i = 0$ being vacuous. Let $z = H(x, i)$ and $y = H(x, i - 1)$, so $z = H_i(y, 1)$. By the inductive hypothesis, we know that $h_w(y) \leq 0$ for $U \in \mathcal{W}_i$, $j > i$. Look at the definition of $H_i$: if $h_w(y) \leq 0$ all $W \in \mathcal{W}_i$, then $z = y$ and (\#) is clear. On the other hand, if $W \in \mathcal{W}_i$ and $x \in X_w$, then $z = \varphi_w(y, -h_w(u))$. Pick $j \leq i$ and $U \in \mathcal{W}_j$ and establish (\#): there are three cases. If $U = W$ then $h_U(z) = h_w(z) = h_w(y) - h_w(y) = 0$. If $U < W$, then (A6) yields $h_U(z) \leq h_U(y) \leq 0$. If $\{U, W\}$ is not a chain, then $h_w(z) = 0$ forces $h_U(z) < 0$ by (A7).

Now we check that $H$ provides the deformation retraction we need; this means we must check the following properties:

(i) $H(x, 0) = X$

(ii) $H(x, T) \in X_{sr}$

(iii) $x \in X_{sr} \Rightarrow H(x, s) = x$, all $s$

(iv) $x \in X^+ \Rightarrow H(x, s) \in X^+$, all $s$

(v) $x \in X^+ \Rightarrow H(x, T) \in \partial X_{sr}$

Property (i) is immediate, and (v) follows from the others because $\partial X_{sr} = X^+ \cap X_{sr}$. Property (ii) is (\#) for $i = T$. Property (iii) is immediate from the definition of $H_i$. Property (iv) follows from the analogous statements for each $H_i$, which are immediate. QED.

Proof of 4.5c. We replace $h_w$ by $h_w' = -h_w$, and $\varphi_w$ by $\varphi_w'$ defined as $\varphi_w'(x, r) = \varphi_w(x, -r)$. The axioms (A1–6) are preserved. We replace $\mathcal{W}$ by
\( \mathcal{W} = \{\alpha_1, \ldots, \alpha_m\} \), so now (A7) is satisfied as well, because \( \mathcal{W} \) is itself a chain. Now 4.5b yields what we want. QED.

In order to prove that various subspaces of \( X \) are manifolds, we introduce some new axioms, which we show later are satisfied in our situation. These are necessary because, for example, the assertion \( M \times \mathbb{R} \) is a manifold does not imply that \( M \) is a manifold.

(A9) \( X \) is a \( C^\infty \)-manifold

(A10) \( \phi_W \) is a \( C^\infty \)-map

(A11) If \( W < U \in \mathcal{W} \), then \( \phi_W \) and \( \phi_U \) commute, i.e. \( \phi_W(\phi_U(x, r), s) = \phi_W(\phi_U(x, s), r) \).

(A12) Given \( W_1 < \cdots < W_k \in \mathcal{W} \), there are numbers \( n_1, \ldots, n_k > 0 \) so that if we define \( \phi(x, r) = \phi_{W_k}(\phi_{W_{k-1}}(\cdots \phi_{W_1}(x, n_1 r) \cdots, n_k r), n_1 r) \), then \( \forall \epsilon \exists r \geq 0 \forall x \ h_{W_k}(\phi(x, r)) \geq r + h_{W_1}(x) \). Moreover, there is a \( C^\infty \)-map \( h : X \to \mathbb{R} \) so \( h(\phi(x, r)) \geq h(x) + r \), all \( x \in X, r \geq 0 \).


(a) \( X_{ss} \) is a manifold with boundary, and the boundary is \( \partial X_{ss} \).

(b) Given \( W_1 < \cdots < W_k \in \mathcal{W} \), the space \( X_{W_1} \cap \cdots \cap X_{W_k} \) is a manifold with boundary.

Proof. Using "inversion" as in the proof of 4.5c, we see that (b) follows from (a).

The idea for proving (a) comes from [G, proof of 3.4]. We see that the differential \( dh \) is nonzero everywhere (look at the composite \( r \mapsto h(\phi(x, r)) \), so the level set \( h^{-1}(0) \) is a submanifold of \( X \) of codimension 1, and is the boundary \( \partial Y \) of the manifold \( Y^h^{-1}((-\infty, 0]) \).

Now suppose we choose a point \( x \in X \) and try to show that \( X_{ss} \) is a manifold near \( x \), assuming \( x \in \partial X_{ss} \). We use (A7) to choose a neighborhood \( U \) of \( x \) and let \( \{W_1 < \cdots < W_k\} = \{W \in \mathcal{W} : U \cap X_{W} \neq \phi\} \). Define \( h'(x) = \sup_{x \in X} h_W(x) \) for \( x \in X \), so letting \( Y' = h'^{-1}((-\infty, 0]) \), we have \( \mathcal{U} \cap X_{ss} = \mathcal{U} \cap Y' \), \( \partial Y' = h'^{-1}(0) \), and \( \mathcal{U} \cap \partial X_{ss} = \mathcal{U} \cap \partial Y' \); thus it will suffice to prove that \( Y' \) is a manifold with boundary \( \partial Y' \). From (A12) we see that

(A13) \( h'(\phi(x, r)) \geq h'(x) + r \), all \( x \in X, r \geq 0 \).

Then using 4.6 for \( h \) and \( h' \), it is not hard to set up a homeomorphism \( Y = Y' \) which restricts to a homeomorphism \( \partial Y = \partial Y' \).

PROPOSITION 4.9. (A9–12) are actually satisfied in our situation.
Proof. (A9, 10) are clear; (A11) was proved in 4.2.1.

Prove (A12). We define \( h(x) = \text{slope} (x/x \cap W_i) - \text{slope} (x \cap W_i) \); it is a \( C^* \) function*. We let \( N = \dim \mathfrak{g} \), and define \( n_i = N \cdot i \). It is easy to check the required properties in terms of root spaces. Each \( W_i \) and its orthogonal complement for the inner product \( H \) is a sum of root spaces, and each \( \varphi_{W_i} \) comes from a map \( \varphi \) which is diagonal with respect to the root space decomposition. The numbers \( n_i \) have been chosen to make it clear that the composite of all these maps \( \varphi \) has the following property: for each \( W_i \), the scalars occurring in \( W_i \) are strictly smaller than the scalars occurring in its orthogonal complement. The result follows then from 4.4. QED.

5. Conclusion

THEOREM 5.1. Suppose \( t > 1 \). The space \( X_G(t) \) is a manifold with boundary, is contractible, and is compact modulo \( \Gamma \). The boundary is homotopy equivalent to the Tits building by a homotopy equivalence which respects the action of \( \Gamma \).

Proof. We combine 4.8b, 4.5b, and 1.1 to get the assertion about \( X_W \).

Now consider the covering \( X^+ = \bigcup_{W \in W} \tilde{X}_W \) by closed subsets; we will use [BS, 8.1, 8.2] to identify its homotopy type.

Suppose \( W_1, \ldots, W_i \in W \). If they do not form a chain, then \( \tilde{X}_{W_1} \cap \cdots \cap \tilde{X}_{W_i} = \varphi \), by (A7). If they do form a chain, then \( \tilde{X}_{W_1} \cap \cdots \cap \tilde{X}_{W_i} \) is a contractible manifold with boundary, according to 4.5c, contractibility of \( X \), and 4.8b, and thus is an absolute retract [BS, 8.1].

Now we apply [BS, 8.2.1] to conclude that \( X^+ \) (and thus \( \partial X_W \) also, by 4.5b) is \( \Gamma \)-homotopy equivalent to the simplicial complex associated to the poset \( A \) of \( \mathbb{Q} \)-parabolic subalgebras of \( \mathfrak{g} \), and thus to the Tits building of \( G \). QED.

Thus we have proved for our space \( X_W \) the same qualitative properties which Borel and Serre prove for their manifold with corners \( \tilde{X} \), namely [BS, 8.4.2 and 9.3]. The cohomological properties for \( \Gamma \) deduced in [BS, Section 11, except for 11.3] can be derived using \( X_W \), except that we must appeal to [KS, p. 123] for the fact that \( X_W / \Gamma \) is equivalent to a finite simplicial complex, and appeal to [B, 9.10] for the fact that \( \Gamma \) contains only finitely many conjugacy classes of torsion elements (which implies that \( \Gamma \) contains a torsion-free subgroup of finite index).

* One may choose a basis so that each of these slopes is a constant times the log of the determinant of a principal minor of the matrix (or its inverse) of the inner product \( x \).
Ultimately, I hope that both of these latter facts can be proved directly by these methods.

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Received February 24, 1986