Projections, Cycles, and Algebraic K-Theory

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The most elegant application of higher algebraic K-theory to algebraic geometry is Bloch's formula

$$H^n(X, \mathcal{K}_n) = A^n(X).$$

The right-hand side of the formula denotes the group of codimension $n$ cycles on a non-singular variety $X$ modulo rational equivalence, while the left-hand side denotes sheaf cohomology with coefficients in the $K$-theory of the structure sheaf of rings on $X$.

The main step in Quillen's proof of the formula is the establishment of Gersten's conjecture, which is done by modifying Noether's normalization lemma to yield a projection with added smoothness conditions. Over an infinite field there is a simpler proof of Noether normalization which uses only linear projections. The purpose of this paper is to present the analogous simplification of Quillen's form of Noether normalization. The final result is no weaker because a trick of Roberts allows us to deduce Gersten's conjecture over a finite field once we know it over all infinite fields.

The contents of this paper are contained in the author's 1976 MIT thesis. I am indebted to Steve Kleiman for many good ideas.

1. Projections

We collect here some facts about projections which we will need. The situation is this: $X \subset \mathbb{A}^N$ is an affine variety over a field $k$, and $\bar{X} \subset \mathbb{P}^N$ is the closure in projective space. Let $T_1, ..., T_N$ be coordinates on $\mathbb{A}^N$, and $(T_0, ..., T_N)$ homogeneous coordinates on $\mathbb{P}^N$.

An affine projection $p: \mathbb{A}^N \to \mathbb{A}^n$ is determined by a choice of linearly independent linear forms $H_i = \sum_{j=1}^{N} a_{ij} T_j, 1 \leq i \leq n$. The corresponding projective projection $\bar{p}: \mathbb{P}^N \to \mathbb{P}^n$ is given by the $n+1$ linear forms $(T_0, H_1, ..., H_n)$, and the

0025-5831/78/0234/0069/S01.00
center of projection $C$ is the linear subspace of dimension $N - n - 1$ at infinity in $\mathbb{P}^N$ defined by the vanishing of these forms.

These projections $p$ are in $1 - 1$ correspondence with the rational points of a certain open subset of the affine space $\mathbb{A}^{nN} = \text{Spec}(k[A_{ij}])$. We say that a condition on a projection $p$ is open if the set of rational points of $\mathbb{A}^{nN}$ corresponding to projections which satisfy the condition is the set of rational points of an open subscheme of $\mathbb{A}^{nN}$.

**Proposition 1.1.** A projection $p$ yields a finite map $X \to \mathbb{A}^n$ if $C \cap \mathbb{X} = \emptyset$. If $\dim X \leq n$ and $k$ is infinite, then for suitable $p$, $C \cap \mathbb{X} = \emptyset$. The condition that $C \cap \mathbb{X} = \emptyset$ is an open condition.

**Proof.** The first two statements are proved in [5, I, §5, Theorem 7 and §6.2, Corollary 4 to Theorem 4] for an algebraically closed field $k$; the same proofs apply here. To see that the condition is open, let $\overline{C} \subset \mathbb{A}^{nN} \times \mathbb{P}^N$ be the closed set defined by $T_0 = \sum A_{ij} T_j = \ldots = \sum A_{nj} T_j = 0$, and let $q : \mathbb{A}^{nN} \times \mathbb{P}^N \to \mathbb{A}^{nN}$ be the projection. One can check that $q(\overline{C} \cap (\mathbb{A}^{nN} \times \mathbb{X}))$ is the complement of the points of $\mathbb{A}^{nN}$ corresponding to those $C$ which miss $\mathbb{X}$; this set is closed because $q$ is proper. Q.E.D.

**Proposition 1.2.** If $x \in X$, $X$ is smooth over $k$ at $x$, and $\dim X \geq n$, then $p$ is smooth at $x$ if and only if $dH_1, \ldots, dH_n$ are linearly independent in the $(k)$-vector space $\Omega^1_{X/k} \otimes k(x)$; moreover, the latter condition is satisfied by some $p$, and is an open condition.

**Proof.** The first statement is contained in [1, 17.11.1]. The existence of such a $p$ is easy to verify, because $\Omega^1_{X/k} \otimes k(x)$ is spanned by the differentials $dT_1, \ldots, dT_n$, and has dimension $= \dim X \geq n$; we may choose the $H_t$ from among the $T_j$. Choose a basis $v_1, \ldots, v_r$ of $\Omega^1_{X/k} \otimes k(x)$, $r = \dim X$, and choose elements $b^i_j$ of $k(x)$ so that $dT_j = \sum b^i_j v_k$. Then the condition that $dH_1, \ldots, dH_n$ be linearly independent is equivalent to the statement that some $n \times n$-minor of the $n \times r$-matrix $(\sum a_{ij} b^i_j)$ not vanish, i.e. there are certain polynomials with coefficients in $k(x)$, $P_x(a_{ij})$, such that $p$ is smooth at $x$ iff for some $x$, $P_x(a_{ij}) + 0$. If we choose a basis $e_\beta$ for $k(x)$ over $k$, and let $P_x = \sum Q_{x\beta} e_\beta$, where $Q_{x\beta}$ has coefficients in $k$, then smoothness of $p$ is determined by the non-vanishing of one of the $Q_{x\beta}$'s, so is an open condition. Q.E.D.

**Proposition 1.3.** Suppose $x_1, \ldots, x_r$ are smooth points of $X$, $r = \dim X$, $Z \subset X$ is a closed subscheme, $\dim Z < r$, and $k$ is infinite. Then there is a projection $p : X \to \mathbb{A}^{r-1}$ which is smooth at each $x_i$ and whose restriction to $Z$ is a finite map.

**Proof.** An intersection $U$ of non-empty open subsets of $\mathbb{A}^{nN}$ is non-empty and if $k$ is infinite, $U$ contains a rational point. Thus the statement follows from (1.2) and (1.3). Q.E.D.

2. Gersten's Conjecture

This section gives Quillen's proof of Gersten's conjecture, except that we are able to use (1.3) instead of Quillen's [3, Section 7, Lemma 5.12] by considering only
infinite fields. The details of the equivalence between Gersten’s conjecture and Bloch’s Formula are spelled out well in [3, Section 7]; we do not repeat them here.

**Definition 2.1.** $\mathcal{M}^p(X)$ is the abelian category of coherent sheaves on $X$ of codimension $\geq p$.

**Notation.** If $x_1, \ldots, x_i$ are points of $X$, let $X_x$ denote the semi-localization at these points.

**Proposition 2.2 [Gersten’s Conjecture].** Suppose $X/k$ is smooth and affine, $x_1, \ldots, x_i$ are points of $X$, $X$ has pure dimension $r$, and $k$ is an infinite field. Then the inclusion $\mathcal{M}^{p+1}(X_x) \subseteq \mathcal{M}^p(X_x)$ induces the zero map on K-theory: $K_q(\mathcal{M}^{p+1}(X_x)) \to K_q(\mathcal{M}^p(X_x))$.

**Note.** Since every regular variety is locally a filtering limit of smooth ones (for details see [3]), 2.2 holds more generally when $X$ is regular.

**Proof.** We know $K_q(\mathcal{M}^{p+1}(X_x)) = \lim_{\to} K_q(\mathcal{M}^{p+1}(U))$, where the limit ranges over the filtering system of open affine neighborhoods $U \subseteq X$ of $x$. Thus we need only show each $K_q(\mathcal{M}^{p+1}(U)) \to K_q(\mathcal{M}^p(X_x))$ is zero, and for this purpose we may assume $U = X$. Now $K_q(\mathcal{M}^{p+1}(X_x)) = \lim_{\to} K_q(\mathcal{M}^{p}(Z))$ where $Z$ runs over divisors $Z \subseteq X$, so it is enough to show $K_q(\mathcal{M}^{p}(Z)) \to K_q(\mathcal{M}^p(X_x))$ is zero. By (1.3) there is a projection $p : X \to \mathbb{A}^{r-1}$ which is smooth at each $x_i$ and whose restriction to $Z$ is a finite map. Let $W = X \times_{\mathbb{A}^{r-1}} Z$, as in

![Diagram](image)

We define the closed immersion $j$ so that $v_j = i$, $q_j = 1$. Let $Q = v^{-1}(\{x_1, \ldots, x_i\})$; $q$ is smooth of relative dimension 1 at each point of $Q$. Since $j$ is a closed immersion of schemes smooth over $Z$ at points of $Q \cap j(Z)$, we know that $j(Z) \subseteq W$ is a divisor at each point of $Q \cap j(Z)$ [2, II, Theorem 4.15]; it is trivially a divisor at the other points of $Q$. Since $Q$ is finite, by the Chinese Remainder Theorem, $j(X) \cap W$ is defined by a single non-zero-divisor in some neighborhood of $Q$; thus there is an exact sequence of $\mathcal{O}_W$-$\mathcal{O}$-modules:

$$0 \to \mathcal{O}_{W,Q} \to \mathcal{O}_{W,q} \to \mathcal{O}_{Z,x} \to 0.$$ Here $\mathcal{O}_{Z,x}$ is the semi-local ring on $Z$ at the points $\{x_1, \ldots, x_i\} \cap Z$. Notice that each of these modules is flat over $Z$ and finite over $X_x$. Thus the functor $\mathcal{M}^p(Z) \to \mathcal{M}^p(X_x)$ given by $M \mapsto M \otimes_Z \mathcal{O}_{W,Q}$ is exact; because $\text{Tor}_1^Z(M, \mathcal{O}_{Z,x}) = 0$, the sequence

$$0 \to M \otimes \mathcal{O}_{W,Q} \to M \otimes \mathcal{O}_{W,q} \to M_x \to 0$$

is an exact sequence of functors. By additivity of K-theory, the functor $M \mapsto M_x$ induces zero on $K_q$. 

3. The Case of a Finite Field

In this section we prove Gersten’s conjecture over a finite field using an idea of Roberts [4, p. 95].

**Proposition 3.1.** Suppose $X$ is a variety over $k$, and $k'$ is a field extension of $k$. Let $X'=X \otimes_k k'$. Then there is an exact functor $\mathcal{M}^p(X) \rightarrow \mathcal{M}^p(X')$ defined by $M \mapsto M \otimes_k k'$. If $k'/k$ is finite, then restriction of scalars defines an exact functor $\mathcal{M}^p(X') \rightarrow \mathcal{M}^p(X)$; moreover, the composite map $K_q(\mathcal{M}^p(X)) \mapsto K_q(\mathcal{M}^p(X')) \mapsto K_q(\mathcal{M}^p(X))$ is simply multiplication by $n=[k':k]$.

**Proof.** The only statement that may not be entirely clear is the last one. If we choose a basis of $k'/k$ then we can set up a functorial isomorphism $M \otimes_k k' \cong M \otimes^n$ of coherent sheaves on $X$, and then we can apply additivity of $K$-theory. Q.E.D.

**Proposition 3.2 [Gersten’s Conjecture for a finite field].** Suppose $X/k$ is smooth and affine, $x_1, \ldots, x_r$ are points of $X$. $X$ has pure dimension $r$, and $k$ is finite. Then $K_q(\mathcal{M}^{p+1}(X_x)) \mapsto K_q(\mathcal{M}^p(X_x))$ is the zero map.

**Proof.** Suppose $r$ is a number $\geq 2$. A finite field has a unique extension of any degree, so let $k^{(i)}$ denote the extension of degree $i$. Then $k=k^{(0)} \subset k^{(1)} \subset k^{(2)} \subset \cdots$; let $k'=\cup k^{(i)}$. Notice that $k'$ is an infinite field. Let $X'=X \otimes_k k'$, $X^{(i)}=X \otimes_k k^{(i)}$; let $X^{(i)}_x=X^{(i)} \times_k X_x$ and $X_x'=X' \times_k X_x$. The fibers of $X' \rightarrow X$ are finite; for let $L$ be the residue field of a point in $X$, and let $s_1, \ldots, s_m$ be a transcendence basis of $L/k$: for each $i$, $k(s) \otimes_k k^{(i)}$ is a field because it is a domain and is finite over $k(s)$, so $k(s) \otimes_k k'$ $=\cup k(s) \otimes_k k^{(i)}$ is a field, and thus $L \otimes_k k'$ is artinian. The map $X'_x \rightarrow X_x$ is integral, and thus closed; therefore $X'_x$ has only a finite number of maximal ideals, and must be the semilocalization of $X'$ at the finite set of points lying above $\{x_1, \ldots, x_r\}$. Applying (2.2) we see that $K_q(\mathcal{M}^{p+1}(X'_x)) \mapsto K_q(\mathcal{M}^p(X'_x))$ is zero. Combining this with the fact that $K_q$ commutes with direct limits, we find that any element in the image of $K_q(\mathcal{M}^{p+1}(X'_x)) \mapsto K_q(\mathcal{M}^p(X'_x))$ dies in $K_q(\mathcal{M}^p(X_x))$ for some $i$, and thus, by (3.1), is killed by multiplication by $r^i$ in $K_q(\mathcal{M}^p(X_x))$.

Since $r$ was arbitrary, we see that $K_q(\mathcal{M}^{p+1}(X'_x)) \mapsto K_q(\mathcal{M}^p(X'_x))$ is zero. Q.E.D.

**References**


Received June 28, 1977