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Localization for Flat Modules in Algebraic K-Theory

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There are two localization theorems in algebraic K-theory which yield long exact sequences for the K-groups of certain triples of exact categories: One handles the quotient of an Abelian category [9], and the other relates a ring and its localization by nonzero divisors.

We present here a localization theorem for certain categories of R-modules flat over a base ring A (Theorem 4.8). The problem encountered in [6] was the lack of suitable non-Abelian quotient categories for the filtration by relative codimension of support: We find a suitable quotient category and use techniques of Quillen to obtain the long exact sequence. The proof is longer than the proof in [9] of the localization theorem for Abelian categories, and does not retain the symmetry between monics and epis. This asymmetry leads to the introduction of relatively torsion-free modules, for they tend to map into or onto enough other modules. Useful tools include a generalization of Waldhausen's cofinality theorem and a modification of Quillen's resolution theorem.

The quotient category obtained is not very computable; a slightly larger exact category is more so. The K-groups are the same if cofinality can be established, but we can only do this for codimension $\leqslant 1$ (Theorem 5.2). This is good enough to show that the first two sheaves in the resolution of $\mathscr{K}_n(X)$ constructed in [6] are flasque (Proposition 6.4).

If X is smooth over S of relative dimension 2, and S is finite, then we can see that $A^2(X) = H^2(X, \mathscr{K}_2)$ is generated by classes of coherent $\mathcal{O}_X$-modules finite and free over $\mathcal{O}_S$, and the relations can be described (Section 9) with no reference to higher K-theory.

Recall that when S is a field, $A^2(X)$ is the group of zero cycles on X modulo rational equivalence. When S is not a field, $A^2(X)$ may yield information about the infinitesimal structure of the cycle class group, as in [2]; moreover $A^2(X)$ is part of a graded ring $A^{\cdot}(X)$ which might harbor Chern classes for vector bundles on the singular variety X.

The relations for $A^2(X)$ can be found by computing boundary maps from higher K-theory and using the fact that K_2 of a local ring is generated by symbols. They are those given by exact sequences and those of the form $c(y, f, g) = 0$,

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where y is a point in X of codimension 1 in X and $f, g \in k(X)^*$. The symbol c is defined in Section 9. If y is the only component of $g = 0$ and y is not a component of f (assuming now f and g are regular functions on X) then $c(y, f, g) = [\mathcal{O}_X/(f, g)]$, and is the divisor of f as a function on the curve $g = 0$.

While computing boundary maps we found that to two exact sequences

$$E: 0 \to L \to M \to N \to 0,$$
$$F: 0 \to N \to M \to L \to 0$$

in an exact category $\mathcal{M}$ is associated an element $T(E, F)$ in $K_1\mathcal{M}$ (Proposition 7.7). We use it to show that the boundary map $K_2 \to K_1$ is the tame symbol when we expect it to be (Corollary 7.13).

1. Cofinality

Suppose $\mathcal{M}$ is a full exact subcategory of an exact category $\mathcal{P}$ and is closed under extensions in $\mathcal{P}$. The inclusion $\mathcal{M} \subset \mathcal{P}$ is called cofinal if for any $P \in \mathcal{P}$ there is some $Q \in \mathcal{P}$ so that $P \oplus Q \in \mathcal{M}$. In this case, Waldhausen has shown that the map $K_i\mathcal{M} \to K_i\mathcal{P}$ is an isomorphism for $i > 0$ and injective for $i = 0$, provided every exact sequence in $\mathcal{P}$ splits (see [3, Propositions 1.1, 1.3]), but the result holds in general.

THEOREM 1.1. *If $\mathcal{M} \subset \mathcal{P}$ is cofinal, then $K_i\mathcal{M} \to K_i\mathcal{P}$ is an isomorphism for $i > 0$ and injective for $i = 0$.*

Proof. Let $\pi = ckr(K_0\mathcal{M} \to K_0\mathcal{P})$, and let $[P]$ denote the class of P in π. Call objects $P_1, P_2 \in \mathcal{P}$ equivalent (modulo $\mathcal{M}$) if $P_1 \oplus M_1 \simeq P_2 \oplus M_2$ for some $M_1, M_2 \in \mathcal{M}$. Equivalence is an equivalence relation, and we denote the set of equivalence classes $\langle P \rangle$ by G. Define $\langle P \rangle + \langle Q \rangle = \langle P \oplus Q \rangle$ (this definition respects equivalence classes) and then G is a group (cofinality ensures that inverses exist).

Suppose $0 \to P' \to P \to P'' \to 0$ is an exact sequence in $\mathcal{P}$. Choosing P_1, P_2 so that $P' \oplus P_1$, $P'' \oplus P_2 \in \mathcal{M}$ yields an exact sequence

$$0 \to P' \oplus P_1 \to P \oplus P_1 \oplus P_2 \to P'' \oplus P_2 \to 0$$

which shows that $P \oplus P_1 \oplus P_2 \in \mathcal{M}$ by closure under extensions. Thus $\langle P \rangle = -\langle P_1 \oplus P_2 \rangle = -\langle P_1 \rangle - \langle P_2 \rangle = \langle P' \rangle + \langle P'' \rangle$ holds in G.

It is now clear that $G \simeq \pi$. According to [3, Proposition 1.1] we must show that given $P_1, ..., P_n \in \mathcal{P}$ such that $\langle P_1 \rangle = \cdots = \langle P_n \rangle \in G$, there exists a Q in $\mathcal{P}$ so that $P_1 \oplus Q, ..., P_n \oplus Q \in \mathcal{M}$. In any case, there is a Q so that $P_1 \oplus Q \in \mathcal{M}$, and thus $\langle P_1 \oplus Q \rangle = \cdots = \langle P_n \oplus Q \rangle = 0$, so we may assume $\langle P_1 \rangle = \cdots = \langle P_n \rangle = 0$. Then each P_i is equivalent to zero, so $P_i \oplus M_i \in \mathcal{M}$ for suitable $M_i \in \mathcal{M}$. Now take $Q = M_1 \oplus \cdots \oplus M_n$ and we are done. Q.E.D.

2. Resolution

The following theorem differs from Quillen's resolution theorem in that left exactness is not required, but there must be enough projectives.

Theorem 2.1. *Suppose $\mathscr{M}$ is an exact category, $\mathscr{P}$ is a full exact subcategory closed under extensions, and for every object M of $\mathscr{M}$ there is an exact sequence $0 \to P \to Q \to M \to 0$ with $P, Q \in \mathscr{P}$ and Q projective. Then $K_i\mathscr{P} \simeq K_i\mathscr{M}$ is an isomorphism for all $i \geqslant 0$.*

Proof. Let $\mathscr{F}$ be the full subcategory of $\mathscr{M}$ whose objects are all $F \in \mathscr{M}$ such that $F \oplus P \in \mathscr{P}$ for some $P \in \mathscr{P}$. To see that $\mathscr{F}$ is closed under extensions in $\mathscr{M}$, let $0 \to F' \to F \to F'' \to 0$ be an exact sequence with $F', F'' \in \mathscr{F}$. If $F' \oplus P', F'' \oplus P'', P', P'' \in \mathscr{P}$ then the exact sequence

$$0 \to F' \oplus P' \to F \oplus P' \oplus P'' \to F'' \oplus P'' \to 0$$

shows $F \oplus P' \oplus P'' \in \mathscr{P}$ and $F \in \mathscr{F}$.

Thus $\mathscr{F}$ is an exact category, and the cofinality of $\mathscr{P} \subset \mathscr{F}$ shows that $K_i\mathscr{P} \simeq K_i\mathscr{F}$ is an isomorphism all $i \geqslant 0$. (Surjectivity for $i = 0$ is clear.)

The resolution theorem of Quillen [9, Sect. 4, Theorem 3] will show that $K_i\mathscr{F} = K_i\mathscr{M}$ once we verify

(i) given any $M \in \mathscr{M}$, there is a resolution $0 \to F' \to F \to M \to 0$ with $F', F \in \mathscr{F}$;

(ii) given any $0 \to G \to F \to M \to 0$ in $\mathscr{M}$ with $F \in \mathscr{F}$, then $G \in \mathscr{F}$.

To check (ii) let $0 \to P \to Q \to M \to 0$ be a resolution of M with Q projective and $P, Q \in \mathscr{P}$. Construct the pullback diagram

$$\begin{tikzcd}
 & & 0 & 0 & \\
0 \arrow[r] & G \arrow[r] & F \arrow[r] \arrow[u] & M \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & G \arrow[r] \arrow[u, equal] & T \arrow[r] \arrow[u] & Q \arrow[r] \arrow[u] & 0 \\
 & & P \arrow[r, equal] \arrow[u] & P \arrow[u] & \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Then $P, F \in \mathscr{F}$ implies $T \in \mathscr{F}$, but $T = G \oplus Q$ and $Q \in \mathscr{P}$ implies $G \in \mathscr{F}$.

Now (i) follows from (ii) and the hypothesis. Q.E.D.

3. Relatively Torsion-Free Modules and Localization

Let $A \to R$ be a flat map of commutative Noetherian rings. We record first the form of the local criterion of flatness we find most useful:

Lemma 3.1. *Suppose $M \to N$ is a map of finite R-modules, and N is flat over A. Then the following conditions are equivalent:*

(i) *$M \to N$ is injective and its cokernel is flat over A.*

(ii) *$M \otimes_A k(\mathfrak{p}) \to N \otimes_A k(\mathfrak{p})$ is injective for each prime $\mathfrak{p} \subset A$ of the form $\mathfrak{p} = q \cap A$ where q is a maximal ideal of R.*

(iii) *$M \to N$ is universally injective relative to A.*

The lemma follows easily from [1, VII 4.1, p. 142].

Let $\mathscr{M}_{R/A}$ denote the exact category of finitely generated R-modules flat over A. If $s \in R$ and $M \in \mathscr{M}_{R/A}$ then we call s a relative nonzero divisor on M if s is a nonzero divisor on M and M/sM is flat over A, or equivalently, s is a nonzero divisor on $M \otimes_A k(\mathfrak{p})$ for each $\mathfrak{p}$. We call s a relative nonzero divisor if s is a relative nonzero divisor on $M = R$. Now suppose we are given a multiplicative subset S of R consisting of relative nonzero divisors.

We say $L \in \mathscr{M}_{R/A}$ is relatively torsion-free if every $s \in S$ is a relative nonzero divisor on L. Let $\mathscr{L}$ denote the full subcategory of $\mathscr{M}_{R/A}$ whose objects are the relatively torsion-free modules. Since $\mathscr{L}$ is closed under extensions in $\mathscr{M}_{R/A}$, it is an exact category.

Lemma 3.2. *If $0 \to M \to L \to N \to 0$ is an exact sequence in $\mathscr{M}_{R/A}$ with $L \in \mathscr{L}$, then $M \in \mathscr{L}$.*

Proof. For each $s \in S$ and each $\mathfrak{p} \in \operatorname{Spec}(A)$, we know s is a nonzero divisor on $M \otimes_A k(\mathfrak{p})$ because the latter is a submodule of $L \otimes_A k(\mathfrak{p})$, on which s is a nonzero divisor. Q.E.D.

Lemma 3.3. *Suppose $f: L \to M$ is a map in $\mathscr{M}_{R/A}$ with $L \in \mathscr{L}$ such that $S^{-1}f$ is an admissible monomorphism in $\mathscr{M}_{S^{-1}R/A}$ (i.e., is a monomorphism with flat cokernel). Then f is an admissible monomorphism in $\mathscr{M}_{R/A}$.*

Proof. The hypotheses yield the diagram

$$\begin{array}{ccc} S^{-1}L \otimes_A k(\mathfrak{p}) & \subset & S^{-1}M \otimes_A k(\mathfrak{p}) \\ \cup & & \uparrow \\ L \otimes_A k(\mathfrak{p}) & \xrightarrow{f \otimes 1} & M \otimes_A k(\mathfrak{p}) \end{array}$$

for any $\mathfrak{p} \in \operatorname{Spec} A$, which shows that $f \otimes 1$ is injective. Now apply Lemma 3.1. Q.E.D.

Lemmas 3.2 and 3.3 are the properties of $\mathscr{L}$ which are most important for us.

PROPOSITION 3.4. *The inclusion $\mathscr{L} \subset \mathscr{M}_{R/A}$ induces isomorphisms*

$$K_i\mathscr{L} \xrightarrow{\simeq} K_i\mathscr{M}_{R/A}$$

on K-theory, i.e., the map

$$Q\mathscr{L} \to Q\mathscr{M}_{R/A}$$

is a homotopy equivalence.

Proof. Since any free R-module is in $\mathscr{L}$, any $N \in \mathscr{M}_{R/A}$ has a resolution of length 1 by objects of $\mathscr{L}$ (Lemma 3.2). In addition, if $0 \to M \to L \to N \to 0$ is exact with $L \in \mathscr{L}$, then $M \in \mathscr{L}$. Thus the resolution theorem [9, Sect. 4] applies. Q.E.D.

We now define an exact category $\mathscr{V}$ which plays the role of the "image" of $\mathscr{L}$ in $\mathscr{M}_{S^{-1}R/A}$.

DEFINITION 3.5. $\mathscr{V}$ is the full subcategory of $\mathscr{M}_{S^{-1}R/A}$ whose objects are those V isomorphic to $S^{-1}L$ for some $L \in \mathscr{L}$.

LEMMA 3.6. *Suppose $L \in \mathscr{L}$, $M \in \mathscr{M}_R$, and $S^{-1}L \subset S^{-1}M$. Then for some $s \in S$ there is a monomorphism p: $sL \to M$ which makes*

$$\begin{tikzcd} sL \arrow[r, hook, "p"'] \arrow[d, phantom, "\cap"] & M \arrow[d, "j"] \\ S^{-1}L \arrow[r, phantom, "\subset"] & S^{-1}M \end{tikzcd}$$

commute.

Proof. By shrinking L (replacing L by sL with $s \in S$) we may assume $L \subset \operatorname{im} j$. Let $R^m \xrightarrow{g} R^n \to L \to 0$ be a presentation of L. Choose f so that

$$\begin{tikzcd} R^m \arrow[r, "g"] \arrow[d, "f"'] & R^n \arrow[r] \arrow[dl, dashed, "h"'] & L \arrow[r] \arrow[d, hook, "i"] & 0 \\ M \arrow[rr, "j"'] & & S^{-1}M & \end{tikzcd}$$

commutes. Since $S^{-1}(fg) = 0$, shrinking L ensures that $fg = 0$, yielding the mono h. Q.E.D.

LEMMA 3.6.1. *If $0 \to V' \to W \to V'' \to 0$ is an exact sequence of $\mathcal{M}_{S^{-1}R/A}$ with $V', V'' \in \mathcal{V}$, when $W \in \mathcal{V}$. Thus $\mathcal{V}$ is an exact category. Moreover, the exact sequence comes from an exact sequence in $\mathcal{L}$.*

Proof. Let $V' = S^{-1}L'$, $V'' = S^{-1}L''$, with $L', L'' \in \mathcal{L}$, and let $W = S^{-1}M$ with $M \in \mathcal{M}_R$ (M may not be flat over A). By Lemma 3.6 we may assume $L' \subset M$; if $N = M/L'$ then $V'' = S^{-1}N$. Shrinking L'' to arrange $L'' \subset N$ and pulling back yields an exact sequence $0 \to L' \to L_0 \to L'' \to 0$ with $S^{-1}L_0 \simeq W$ and $L_0 \in \mathcal{L}$. Q.E.D.

DEFINITION 3.7. $\mathcal{C}$ is the full subcategory of $Q\mathcal{M}_{R/A}$ whose objects are the objects of $\mathcal{L}$.

The arrows of $\mathcal{C}$ may be represented by diagrams $L \twoheadleftarrow L_1 \rightarrowtail L'$, where by Lemma 3.2, $L_1 \in \mathcal{L}$. Thus an arrow $L \to L'$ of $\mathcal{C}$ is determined by a layer $L_0 \subset L_1 \subset L'$ admissible in $\mathcal{M}_{R/A}$ and an isomorphism $L \simeq L_1/L_0$.

DEFINITION 3.8. $\mathcal{D}$ is the subcategory of $Q\mathcal{M}_{S^{-1}R/A}$ whose objects are the objects of $\mathcal{V}$, but whose arrows are isomorphism classes of diagrams $V \overset{p}{\twoheadleftarrow} V_1 \rightarrowtail V'$ where $V_1 \in \mathcal{V}$ and $\ker p \in \mathcal{V}$.

DEFINITION 3.8.1. $\mathcal{W} = \mathcal{W}_{R/A}$ is the full subcategory of $\mathcal{M}_{S^{-1}R/A}$ consisting of all W isomorphic to $S^{-1}M$ for some $M \in \mathcal{M}_{R/A}$.

LEMMA 3.8.2. *$\mathcal{W}$ is closed under extension in $\mathcal{M}_{S^{-1}R/A}$ and thus is an exact category. Moreover, the* map *$Q\mathcal{V} \to Q\mathcal{W}$ is a homotopy equivalence.*

Proof. Suppose E: $(0 \to S^{-1}M' \to W \to S^{-1}M'' \to 0)$ is an exact sequence and $M', M'' \in \mathcal{M}_{R/A}$. Using finite presentation of W, we can find an exact sequence D: $(0 \to N' \to N \to N'' \to 0)$ of R-modules so that $S^{-1}D \simeq E$. There are maps $f'\colon N' \to M'$ and $f''\colon M'' \to N''$ so that $S^{-1}f'$ and $S^{-1}f''$ are multiplication by some $s', s'' \in S$. Pulling D back along f'' and then pushing it out along f' yields an exact sequence C: $(0 \to M' \to M \to M'' \to 0)$ so that $S^{-1}C \simeq E$ and $M \in \mathcal{M}_{R/A}$.

Suppose $W = S^{-1}M$ and $M \in \mathcal{M}_{R/A}$. We may find a resolution B: $(0 \to L \to F \to M \to 0)$ with F free over R and $L \in \mathcal{L}$ by Lemma 3.2. Then $S^{-1}B$ is a resolution of the form required to apply Theorem 2.1, and thus $Q\mathcal{V} \to Q\mathcal{W}$ is a homotopy equivalence. Q.E.D.

Since any epimorphism in $\mathscr{L}$ is admissible, $L \to S^{-1}L$ is a functor $a\colon \mathscr{C} \to \mathscr{D}$ which fits in the following diagram:

$$\begin{tikzcd}
Q\mathscr{L} \arrow[r, hook, "\sim"] \arrow[d] & \mathscr{C} \arrow[r, hook, "\sim"] \arrow[d, "a"] & Q\mathscr{M}_{R\backslash A} \arrow[d, "b"] & \\
Q\mathscr{V} \arrow[r, hook, "\sim"] & \mathscr{D} \arrow[r, hook, "\sim"] & Q\mathscr{W} \arrow[r, hook] & Q\mathscr{M}_{S^{-1}R/A}
\end{tikzcd} \tag{1}$$

The upper maps are shown to be homotopy equivalences in the proof of the resolution theorem [9, Sect. 4] used in Proposition 3.4; half of that proof applies equally well to show $Q\mathscr{V} \to \mathscr{D}$ is a homotopy equivalence. That $\mathscr{D} \to Q\mathscr{W}$ is a homotopy equivalence follows from Lemma 3.8.2.

Definition 3.9. $\mathscr{N}$ is the full exact subcategory of $\mathscr{M}_{R/A}$ consisting of all N such that $S^{-1}N = 0$.

Definition 3.10. $\mathscr{H}$ is the category whose objects are all surjective arrows $(N \twoheadleftarrow L')$ from $Q\mathscr{M}_{R/A}$ with $N \in \mathscr{N}$ and $L' \in \mathscr{L}$, and is a full subcategory of the category of arrows in $Q\mathscr{M}_{R/A}$. Thus an arrow in $\mathscr{H}$ from $(N \twoheadleftarrow L')$ to $(N_0 \twoheadleftarrow L_0')$ is an isomorphism class of diagrams:

$$\begin{tikzcd}
N & L' \arrow[l, two heads] \\
N_1 \arrow[u, two heads] \arrow[d, tail] \arrow[dr, phantom, "\square"] & L_1' \arrow[l, two heads] \arrow[u, two heads] \arrow[d, tail] \\
N_0 & L_0', \arrow[l, two heads]
\end{tikzcd}$$

where the top square is commutative and the bottom square is Cartesian.

We let c denote the functor $\mathscr{H} \to Q\mathscr{N}$ given by $c(N \twoheadleftarrow L') = N$; note that c is fibered.

Lemma 3.11. *$c\colon \mathscr{H} \to Q\mathscr{N}$ is a homotopy equivalence.*

Proof. By Theorem A [9], it is enough to show that each fiber $c^{-1}(N)$ is contractible. Given $N \in \mathscr{N}$, there is a free R-module L_0' and a surjection $L_0' \twoheadrightarrow N$, so $c^{-1}(N)$ is nonempty. The category $c^{-1}(N)$ has as objects all surjections $(N \twoheadleftarrow L')$ with $L' \in \mathscr{L}$, and as arrows all commutative diagrams

$$\begin{tikzcd}
N & L' \arrow[l, two heads] \\
N \arrow[u, equal] & L''. \arrow[l, two heads] \arrow[u, two heads]
\end{tikzcd}$$

The natural transformations $(N \twoheadleftarrow L') \to (N \twoheadleftarrow L' \oplus L'_0) \leftarrow (N \twoheadleftarrow L'_0)$ show that $c^{-1}(N)$ is contractible. Q.E.D.

Recall that the right fiber $0/a$ has as objects all pairs $(0 \xrightarrow{i} S^{-1}L', L')$ where i is an arrow of $\mathscr{D}$ and $L' \in \mathscr{L}$, and arrows are commutative diagrams:

$$\begin{tikzcd}
& S^{-1}L' \arrow[dd, "S^{-1}f"] & L' \arrow[dd, "f"] \\
0 \arrow[ur] \arrow[dr] & & \\
& S^{-1}L'' & L''
\end{tikzcd}$$

where f is an arrow of $\mathscr{C}$.

We let $d_0\colon \mathscr{H} \to 0/a$ denote the functor $d_0(N \twoheadleftarrow L') = (0 \twoheadleftarrow S^{-1}L', L')$. Let $e_0\colon \mathscr{H} \to \mathscr{C}$ denote the functor $e_0(N \twoheadleftarrow L') = L'$, so that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{H} \arrow[r, "d_0"] \arrow[dr, "e_0"'] & 0/a \arrow[d] \\
& \mathscr{C}
\end{tikzcd}$$

Consider the following diagram:

$$\begin{tikzcd}
& & Q\eta \arrow[r] \arrow[dd] & Q\mathfrak{M}_{R/A} \arrow[dd, "b"] \\
\mathscr{H} \arrow[urr, "c", "\sim"'] \arrow[r, "e_0"'] \arrow[dd] & \mathscr{C} \arrow[urr, "\sim"] \arrow[dd, "a"] & & \\
& & 0 \arrow[r] & Q\mathcal{W} \\
0/1_{\mathfrak{D}} \arrow[urr, "\sim"] \arrow[r] & \mathfrak{D} \arrow[urr, "\sim"'] & &
\end{tikzcd} \tag{2}$$

It is immediate from Definition 3.10 that the top square commutes up to homotopy; so does the bottom square, and the others actually commute. (To avoid problems with base point, we assume all our categories have only one zero object.)

THEOREM 3.12. *The front square in* (2) *is homotopy Cartesian, and thus the back square is too.*

COROLLARY 3.13. *There is a long exact sequence*

$$\cdots \to K_i\mathcal{N} \to K_i\mathcal{M}_{R/A} \to K_i\mathcal{W} \to K_{i-1}\mathcal{N} \to \cdots$$

Note 3.13.1. Given a cube of pointed CW-complexes such as (2):

$$\begin{tikzcd}
 & F' \arrow[rr, "j'"] \arrow[dd, "i'" near start] & & G' \arrow[dd, "q'"] \\
F \arrow[ur, "\phi"] \arrow[rr, "j" near end] \arrow[dd, "i"] & & G \arrow[ur, "\gamma"] \arrow[dd, "q" near start] & \\
 & E' \arrow[rr, "p'" near start] & & B' \\
E \arrow[ur, "\varepsilon"] \arrow[rr, "p"] & & B \arrow[ur, "\beta"] &
\end{tikzcd} \tag{2'}$$

such that

(i) there are homotopies $\zeta\colon j'\varphi \simeq \gamma j$ and $\eta\colon p'\epsilon \simeq \beta p$, and
(ii) the other four faces of the cube commute,

we consider the square

$$\begin{tikzcd}
F \arrow[r, "\varphi"] \arrow[d, "\lambda"'] & F' \arrow[d, "\lambda'"] \\
\Omega \arrow[r, "\omega"] & \Omega'
\end{tikzcd} \tag{*}$$

where $\Omega = E \times_B B^I \times_B G$ is the homotopy-theoretic fiber product (resp. $\Omega' = \cdots$) and λ is the map

$$\lambda\colon F \to \Omega,$$
$$x \mapsto (ix, pix = qjx, jx)$$

(resp. λ' is ...). Here ω is the map

$$\omega\colon \Omega \to \Omega',$$
$$(e, c, g) \mapsto (\epsilon e, p'\epsilon e \xrightarrow{\eta e} \beta pe \xrightarrow{\beta c} \beta qg = q'\gamma g, \gamma g).$$

If further

(iii) $q'\zeta = \eta i$,

then we can construct a homotopy $\omega\lambda \simeq \lambda'\varphi$ as illustrated

$$\begin{tikzcd}
\varepsilon ix \arrow[d, equal] & p'\varepsilon ix \arrow[r, "\eta ix"] \arrow[d, equal] & \beta pix = \beta qjx = q'\gamma jx & \gamma jx \\
i'\phi x & p'i'\phi x \arrow[r, equal] & q'j'\phi x \arrow[u, "q'\zeta x"'] & j'\phi x \arrow[u, "\zeta x"']
\end{tikzcd}$$

If in addition

(iv) ϵ, β, γ are homotopy equivalences,

then ω is a homotopy equivalence. To see this we use the following lemma.

LEMMA 3.13.2. *Suppose $f: X \to Y$ and $g: Y \to X$ are homotopy inverse maps, and $\alpha: fg \simeq 1$ is a homotopy. There is a homotopy $\delta: gf \simeq 1$ so that the two homotopies αf and $f\delta: fgf \simeq f$ are homotopic (through homotopies of fgf with f).*

Choose homotopy inverses $\bar{\epsilon}, \bar{\beta}, \bar{\gamma}$ and homotopies for $\beta, \bar{\beta}$ with this property. The diagram

$$\begin{tikzcd}
E' \arrow[r, "p'"] \arrow[d, "\bar{\epsilon}"'] & B' & G' \arrow[l, "q'"'] \arrow[d, "\bar{\gamma}"] \\
E \arrow[r, "p"] & B \arrow[r, "q"] \arrow[u, "\beta"] & G
\end{tikzcd}$$

commutes up to homotopy (e.g., $\bar{\beta}p' \simeq \bar{\beta}p'\epsilon\bar{\epsilon} \simeq \bar{\beta}\beta p\bar{\epsilon} \simeq p\bar{\epsilon}$) so a slight extension of the procedure by which we defined ω allows us to define

$$\bar{\omega}: \Omega' \to \Omega$$

$$(e', p'e' \overset{c'}{—} q'g', g') \mapsto (\bar{\epsilon}e', p\bar{\epsilon}e' — \bar{\beta}\beta p\bar{\epsilon}e' — \bar{\beta}p'\epsilon\bar{\epsilon}e' — \bar{\beta}p'e' \overset{\bar{\beta}c'}{—} \cdots$$

We only draw the left half, for the right half is obtained by symmetry. The following diagram shows that $\omega\bar{\omega} \simeq 1$.

$$\begin{tikzcd}
(\varepsilon\bar{\varepsilon}e', \arrow[d, dash] & p'\varepsilon\bar{\varepsilon}e' \arrow[r, dash] \arrow[d, dash] & \beta p\bar{\varepsilon}e' \arrow[r, dash] & \beta\bar{\beta}\beta p\bar{\varepsilon}e' \arrow[r, dash] & \beta\bar{\beta}p'\varepsilon\bar{\varepsilon}e' \arrow[r, dash] & \beta\bar{\beta}p'e \arrow[r, dash, "\beta\bar{\beta}c'"] & \cdots \\
(e', & p'e' \arrow[rrrrr, dash, "c'"'] & & & & & \cdots
\end{tikzcd}$$

$*$

The * marks the spot where we use the property from the lemma for β. The same reasoning that has just shown that ω has a right inverse in the homotopy category applies to show that $\bar{\omega}$ has a right inverse. It follows immediately that ω is a homotopy equivalence.

Proof of 3.13.2. Let η: $gf \simeq 1$ be any homotopy, and define $\delta = (gf\eta)^{-1} \circ (g\alpha f) \circ \eta$. The following diagram illustrates the homotopy we want.

$$\begin{tikzcd}
fgf \arrow[rrr, bend left=30, "f\delta"] \arrow[d, "\alpha f"'] & fgfgf \arrow[l, "fgf\eta"'] \arrow[r, "fg\alpha f"'] \arrow[d, "\alpha fgf"'] & fgf \arrow[r, "f\eta"'] \arrow[d, "\alpha f"'] \arrow[dl, bend left=70, no head, "\mathrm{id}"] & f \arrow[dlll, bend left=60, no head, "\mathrm{id}"] \\
f & fgf \arrow[l, "f\eta"'] \arrow[r, "\alpha f"'] & f &
\end{tikzcd}$$

Q.E.D.

If we further assume

(v) φ is a homotopy equivalence

then, since we have shown that (*) commutes up to homotopy, we may conclude that λ is a homotopy equivalence if and only if λ' is, i.e., the front square of (2′) is homotopy Cartesian if and only if the back square is.

In the case at hand, (2), we have

$$\zeta(N \twoheadleftarrow L') = (N \twoheadleftarrow L'),$$
$$\eta(0 \twoheadleftarrow S^{-1}L') = (0 \twoheadleftarrow S^{-1}L')$$

so that (iii) is clearly satisfied. This justifies the second part of the statement of Theorem 3.12. I thank Richard Swan for pointing out to me that one must be careful here.

The following diagram can serve as a guide to the proof of Theorem 3.12, which will occupy the rest of the section.

$$\begin{tikzcd}
 & & V/a \\
 & \mathscr{H} \arrow[r, "pk"'] \arrow[ur, "d_L"] \arrow[dl, "c"'] \arrow[d, "qk"] & \mathscr{G}_V \arrow[u, "t_V"', "\wr"] \arrow[d, "s_V", "\wr"'] \\
Q\mathscr{N} & \mathscr{E}_L \arrow[l, "k_L"', "\sim"] \arrow[r, "\sim", "r_L"'] & b^{-1}(V)
\end{tikzcd} \tag{3}$$

Definition 3.14. For $V \in \mathscr{V}$ we define the category $b^{-1}(V)$. Its objects are all pairs $(V \simeq S^{-1}M, M)$ with $M \in \mathscr{M}_{R/A}$. An arrow is an arrow $M \to M'$ in $Q\mathscr{M}_{R/A}$ which makes the following triangle commute:

$$\begin{tikzcd}
 & S^{-1}M \arrow[dd, "\wr"'] \\
V \arrow[ur] \arrow[dr] & \\
 & S^{-1}M'
\end{tikzcd}$$

(This category is $\underline{\underline{F}}_V$ in the notation of [9, Sect. 5, Theorem 5].)

Definition 3.15. For $V \in \mathscr{V}$, let $\mathscr{I}_V$ denote the ordered set whose objects are R-submodules $L \subset V$ such that $L \in \mathscr{L}$ and $S^{-1}L \cong V$. For each $L \in \mathscr{I}_V$ let $\mathscr{E}_L$ be the category whose objects are (admissible) monomorphisms $(L \subset M)$ in $\mathscr{M}_{R/A}$ such that $V \simeq S^{-1}M$, and whose arrows are given by arrows $(M \rightarrowtail M_0 \twoheadleftarrow M')$ in $Q\mathscr{M}_{R/A}$ which make the following diagram commute in $\mathscr{M}_{R/A}$:

$$\begin{tikzcd}
L \subset M \arrow[d, equal] \arrow[d, shift left=6ex, tail] \\
L \subset M_0 \\
L \subset M' \arrow[u, equal] \arrow[u, shift left=6ex, two heads]
\end{tikzcd}$$

(This category is $\underline{\underline{E}}'_L$ in the notation of [9].) When $L' \subset L$ are in $\mathscr{I}_V$, there is an obvious functor $\mathscr{E}_L \to \mathscr{E}_{L'}$. We define the functor r_L: $\mathscr{E}_L \to b^{-1}(V)$ by $r_L(L \subset M) = (V \simeq S^{-1}M, M)$.

Lemma 3.16. *For any* $V \in \mathscr{V}$, *the map*

$$r\colon \varinjlim_{L\in(\mathscr{I}_V)^0} \mathscr{E}_L \to b^{-1}(V)$$

is an isomorphism of categories (*note that* $(\mathscr{I}_V)^0$ *is a filtering ordered set and the* L*'s get smaller*).

Proof. First, we check surjectivity on objects: given $(V \xrightarrow{\sim} S^{-1}M, M) \in b^{-1}(V)$ we use Lemma 3.6 to find $i\colon L \subset M$ with $S^{-1}L = V$; by Lemma 3.3 i is an admissible mono of $\mathscr{M}_{R/A}$ and thus an object of $\mathscr{E}_L$.

Second, injectivity on objects: given $L \in \mathscr{I}_V$ and $(i_1\colon L \subset M_1), (i_2\colon L \subset M_2) \in \mathscr{E}_L$ suppose $(V \cong S^{-1}M_1, M_1) = (V \cong S^{-1}M_2, M_2)$. Then $M_1 = M_2$ and $S^{-1}i_1 = S^{-1}i_2$. For some $s \in S$, we know $s(i_1 - i_2) = 0$, so by replacing L by sL we see that $(sL \subset M_1) = (sL \subset M_2) \in \mathscr{E}_{sL}$.

Third, surjectivity on arrows: Suppose $L \in \mathscr{I}_V$, and $(i\colon L \subset M), (i'\colon L \subset M') \in \mathscr{E}_L$, and we are given an arrow from $(V \cong S^{-1}M, M)$ to $(V \cong S^{-1}M', M')$ in $b^{-1}(V)$, i.e., an arrow $f\colon M \to M'$ in $Q\mathscr{M}_{R/A}$ which makes

$$\begin{tikzcd}
 & S^{-1}M \arrow[dd, "S^{-1}f", "\wr"'] \\
V \arrow[ur, "S^{-1}i"] \arrow[dr, "S^{-1}i'"'] & \\
 & S^{-1}M'
\end{tikzcd}$$

commute. We may assume that f is either injective or surjective; say it's injective. Then $S^{-1}(fi - i') = 0$ so for some $s \in S$ we know $sfi = si'$. Thus f gives an arrow in $\mathscr{E}_{sL}$:

$$\begin{tikzcd}
sL \arrow[d, equal] & \subset & M \arrow[d, tail, "f"] \\
sL & \subset & M'
\end{tikzcd}$$

which does the trick.

Finally, injectivity on arrows is clear. Q.E.D.

Definition 3.17. Given $V \in \mathscr{V}$ and $L \in \mathscr{I}_V$, we define a functor $k_L\colon \mathscr{E}_L \to Q\mathscr{N}$ to be $k_L(L \subset M) = M/L$ on objects, and to act on arrows as diagrammed:

$$\left(\begin{tikzcd}[row sep=small] L \subset M \arrow[d, equal, shift left=-3ex] \arrow[d, tail, shift left=3ex] \\ L \subset M_1 \\ L \subset M' \arrow[u, equal, shift left=-3ex] \arrow[u, two heads, shift left=3ex] \end{tikzcd}\right) \mapsto \begin{tikzcd}[row sep=small] M/L \arrow[d, tail] \\ M_1/L \\ M'/L \arrow[u, two heads] \end{tikzcd}$$

Checking that k_L is a functor amounts to showing that if

$$\begin{tikzcd} M \arrow[r, tail] & M_1 \\ M_0 \arrow[r, tail] \arrow[u, two heads] \arrow[ur, phantom, "\square"] & M' \arrow[u, two heads] \end{tikzcd}$$

is a Cartesian square, each of whose corners contains L, then

$$\begin{tikzcd} M/L \arrow[r, tail] & M_1/L \\ M_0/L \arrow[r, tail] \arrow[u, two heads] & M'/L \arrow[u, two heads] \end{tikzcd}$$

is also Cartesian, which a diagram chase shows easily.

LEMMA 3.18. *Give $V \in \mathscr{V}$ and $L \in \mathscr{I}_V$, the map k_L: $\mathscr{E}_L \to Q\mathscr{N}$ is a homotopy equivalence.*

The proof is the dual of the proof of [9, Sect. 5, Lemma 2].

LEMMA 3.19. *The functor r_L: $\mathscr{E}_L \to b^{-1}(V)$ is a homotopy equivalence.*

Proof. Given $L' \subset L$ in $\mathscr{I}_V$, the diagram

$$\begin{tikzcd} \mathscr{E}_L \arrow[rr] \arrow[dr, "k_L"'] & & \mathscr{E}_{L'} \arrow[dl, "k_{L'}"] \\ & Q\mathscr{N} & \end{tikzcd}$$

commutes up to the homotopy given by the natural transformation $M/L \twoheadleftarrow M/L'$. Thus, by Lemma 3.18, $\mathscr{E}_L \to \mathscr{E}_{L'}$ is a homotopy equivalence, so by Lemma 3.16, so is r_L. Q.E.D.

DEFINITION 3.20. Given $V \in \mathscr{V}$, let $\mathscr{G}_V$ denote the category whose objects are pairs ($V \cong S^{-1}M$, u: $M \to L'$) where u is an arrow in $Q\mathscr{M}_{R/A}$ and $L' \in \mathscr{L}$, and whose arrows from ($V \cong S^{-1}M_0$, $M_0 \to L_0'$) to ($V \cong S^{-1}M_1$, $M_1 \to L_1'$)

are commutative squares in $Q\mathcal{M}_{R/A}$ compatible with the isomorphisms in $Q\mathcal{M}_{S^{-1}R/A}$:

$$\begin{array}{ccccc}(V \neq S^{-1}M_0\,, & M_0 & \xrightarrow{u_0} & L_0') \\ \| \quad {\scriptstyle S^{-1}f}\big\downarrow\wr & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle g} \\ (V \neq S^{-1}M_1\,, & M_1 & \xrightarrow[u_1]{} & L_1').\end{array}$$

Let $s_V\colon \mathcal{G}_V \to b^{-1}(V)$ be the functor $s_V(V \cong S^{-1}M, M \to L') = (V \cong S^{-1}M, M)$, and let $t_V\colon \mathcal{G}_V \to V/a$ be the functor $t_V(V \cong S^{-1}M, M \to^h L') = (V \to S^{-1}L', L')$. We must verify that $S^{-1}f\colon S^{-1}M \to S^{-1}L'$ is an arrow of $\mathcal{D}$; if f is injective this is clear, so suppose f is surjective, $f\colon M \twoheadleftarrow L'$. By Lemma 3.2, $\ker(f) \in \mathcal{L}$, so $\ker(S^{-1}f) \in \mathcal{V}$ and thus $S^{-1}f$ is an arrow in $\mathcal{D}$.

Lemma 3.21. *The functor $s_V\colon \mathcal{G}_V \to b^{-1}(V)$ is a homotopy equivalence.*

Proof. Since s_V is fibered, it is enough to show that each fiber $s_V^{-1}(V \cong S^{-1}M, M)$ is contractible; that fiber is simply the category $\mathcal{B}$ of maps in $Q\mathcal{M}_{R/A}$ from M to an object of $\mathcal{L}$. Let $\mathcal{A}$ be the full subcategory of $\mathcal{B}$ whose objects are the surjective arrows $(M \twoheadleftarrow L_1)$ with $L_1 \in \mathcal{L}$. If $X = (M \twoheadleftarrow L_1 \rightarrowtail L)$ is an object of $\mathcal{B}$, then $L_1 \in \mathcal{L}$ by Lemma 3.2. If we let $\bar{X} = (M \twoheadleftarrow L_1) \in \mathcal{A}$, then the arrow $\bar{X} \to X$ given by the diagram

$$\begin{array}{c}\bar{X} \\ \big\downarrow \\ X\end{array} \qquad \left(\begin{array}{ccccc} M & \twoheadleftarrow & L_1 & \text{---} & L_1 \\ \| & & \| & & \big\downarrow\!\!\vee \\ M & \twoheadleftarrow & L_1 & \rightarrowtail & L \end{array}\right)$$

is a universal arrow to X from an object of $\mathcal{A}$. Thus the functor $X \to \bar{X}$ is an adjoint to the inclusion $\mathcal{A} \subset \mathcal{B}$. Now we can see that $\mathcal{A}$ is contractible just as in the proof of Lemma 3.11. Q.E.D.

Lemma 3.22. *The functor $t_V\colon \mathcal{G}_V \to V/a$ is a homotopy equivalence.*

Proof. Since t_V is cofibered, it is enough to show that each fiber $t_V^{-1}(V \to S^{-1}L', L')$ is contractible. Now the arrow $V \to S^{-1}L'$ in $\mathcal{D}$ is represented by a layer $V_0 \subset V_1 \subset S^{-1}L'$ and an isomorphism $V \cong V_1/V_0$, such that $V_0 \subset V_1$ is an admissible monomorphism of $\mathcal{V}$. The fiber mentioned above is equivalent to the ordered set of layers $L_0 \subset L_1 \subset L'$ (admissible in $\mathcal{M}_{R/A}$ with $L_0, L_1 \in \mathcal{L}$) such that $S^{-1}L_0 \cong V_0$ and $S^{-1}L_1 \cong V_1$, By Lemma 3.3 this set is nonempty. Moreover, it is filtering, because for any two such layers $L_0 \subset L_1$, $L_0' \subset L_1'$ of L', there is an $s \in S$ such that

$$\begin{array}{ccccc} & & L_0' \subset L_1' & & \\ & \subset & & \supset & \\ sL_0 & & & & s^{-1}L_1 \\ & \supset & & \subset & \\ & & L_0 \subset L_1 & & \end{array}$$

Q.E.D.

DEFINITION 3.23. Given $V \in \mathscr{V}$ and $L \in \mathscr{I}_V$, the functor q_L: $\mathscr{H} \to \mathscr{E}_L$ is given by $q_L(N \to L') = (L \subset L \oplus N)$. The functor p_L: $\mathscr{H} \to \mathscr{G}_V$ is given by $p_L(N \to L') = (V \cong S^{-1}(L \oplus N), L \oplus N \to L \oplus L')$. The functor d_L: $\mathscr{H} \to V/a$ is given by $d_L(N \to L') = (V \cong S^{-1}(L \oplus N) \to S^{-1}(L \oplus L'), L \oplus L')$. This agrees with the earlier definition of d_0.

We claim that diagram (3) commutes: $d_L = t_V \circ p_L$ is clear. Also, $s_V p_L(N \to L') = s_V(V \cong S^{-1}(L \oplus N), L \oplus N \to L \oplus L') = (V \cong S^{-1}(L \oplus N), L \oplus N) = r_L(L \subset L \oplus N) = r_L q_L(N \to L')$, so $s_V p_L = r_L q_L$. Furthermore, $k_L q_L = c_0$ is clear.

It follows from Lemmas 3.11, 3.18, 3.19, 3.21, and 3.22 that every map in (3) is a homotopy equivalence.

Proof of Theorem 3.12. We know that d_0: $\mathscr{H} \to 0/a$ is a homotopy equivalence; it only remains to show that all the base-change maps $V/a \to V'/a$ are homotopy equivalences, and for this purpose it is enough to treat base-change by i: $0 \rightarrowtail V$ and j: $0 \twoheadleftarrow V$.

Consider the following diagram, where $L \in \mathscr{I}_V$:

$$\begin{tikzcd}
Q\mathscr{N} \arrow[d, equal] & & \mathscr{H} \arrow[ll, "c"', "\sim"] \arrow[rr, "d_L", "\sim"'] \arrow[dl, "q_0"'] \arrow[dr, shift left, "\bar{\jmath}"] \arrow[dr, shift right, "\bar{\imath}"'] & & V/a \arrow[d, shift right, "i^*"'] \arrow[d, shift left, "j^*"] \\
Q\mathscr{N} & \mathscr{E}_0 \arrow[l, "\sim"'] \arrow[r, "\sim", "r_0"'] & b^{-1}(0) & \mathscr{G}_0 \arrow[l, "\sim"', "s_0"] \arrow[r, "\sim", "t_0"'] & 0/a
\end{tikzcd} \tag{4}$$

We define functors $\bar{\imath}, \bar{\jmath}$: $\mathscr{H}_0 \to \mathscr{G}_0$ by $\bar{\imath}(N \to^f L') = (0 \cong S^{-1}N, N \rightarrowtail L \oplus N \to_{1 \oplus f} L \oplus L')$ and $\bar{\jmath}(N \to^f L') = (0 \cong S^{-1}N, N \twoheadleftarrow L \oplus N \to_{1 \oplus f} L \oplus L')$. It is easy to check that $t_0 \bar{\imath} = i^* d_L$, $t_0 \bar{\jmath} = j^* d_L$, and $r_0 q_0 = s_0 \bar{\imath} = s_0 \bar{\jmath}$. Thus i^* and j^* are homotopy equivalences. Q.E.D.

4. LOCALIZATION IN CODIMENSION p

Let $A \to R$ be a flat map of noetherian rings, and let $\mathscr{M}^p_{R/A}$ denote the exact category of finitely generated R-modules of codimension $\geqslant p$ flat over A. In order to apply the results of Section 3 to determine the third term in the exact sequence

$$\cdots \to K_i \mathscr{M}^{p+1}_{R/A} \to K_i \mathscr{M}^p_{R/A} \to \ ? \to K_{i-1} \mathscr{M}^{p+1}_{R/A} \to \cdots$$

we must impose some constraints on R. For instance, when $p = 0$ there must be relative nonzero divisors annihilating each $M \in \mathscr{M}^1_{R/A}$. For $p > 0$ we must find sufficiently many quotient rings R' which themselves have enough relative nonzero divisors.

DEFINITION 4.1. If A is local with residue field k, call R pure (over A) if the closure in Spec(R) of any associated point of a fiber $R \otimes_A k(\mathfrak{p})$ contains an associated point of the closed fiber $R \otimes_A k$.

This definition is taken from [10]. For the rest of the section A is local and has residue field k.

DEFINITION 4.2. Call R superpure over A if any nonzero quotient R' of R defined by a regular sequence of length $p \geqslant 0$ is pure over A.

LEMMA 4.3. (i) *R is pure over A if and only if every $f \in R$ which is a nonzero divisor on $R \otimes_A k$ is also a relative nonzero divisor in R.*

(ii) *If R is local and the map $A \to R$ is local then R is superpure over A.*

(iii) *If R is smooth over A with connected fibers, then R is pure over A.*

(iv) *If A is Artin, then R is superpure over A.*

Proof. (i) follows from Lemma 3.1, as does (ii). As for (iii), the points of Spec R which are associated points in their fibers are of the form pR where $p \in$ Spec A. Statement (iv) holds because there is only one fiber. Q.E.D.

LEMMA 4.4. *Suppose R is superpure over A and $R \otimes_A k$ is Cohen–Macaulay. Then*

(i) *for any $M \in \mathscr{M}^1_{R/A}$; there is a relative nonzero divisor $f \in R$ with $fM = 0$,*

(ii) *if $f \in R$ is a relative nonzero divisor, then $R' = R/fR$ has Cohen–Macaulay closed fiber,*

(iii) *for any nonzero $M \in \mathscr{M}^p_{R/A}$ there is a relative regular sequence $(f_1, ..., f_p)$ in R so that $f_1M = \cdots = f_pM = 0$, and*

(iv) *$\mathscr{M}^{p+q}_{R/A} = \lim \mathscr{M}^q_{R'/A}$ is a filtering inductive limit of exact categories, where the limit runs over all quotients R' of R by a regular sequence of length p.*

Proof. (i) Since $R \otimes_A k$ is Cohen–Macauley, its associated primes are minimal, and since $\operatorname{supp}(M) \cap \operatorname{Spec}(R \otimes_A k) = \operatorname{supp}(M \otimes_A k)$ has codimension 1 in Spec($R \otimes_A k$), we may choose $f \in R$ vanishing on M and not on the minimal points of $R \otimes_A k$.

(ii) Clear.

(iii) Follows from (i) and (ii) by induction.

(iv) The system is filtering, for if R_1' and R_2' are two such quotients of R, we may find a third dominating both of them by applying (iii) to $M = R_1' \oplus R_2'$.

COROLLARY 4.5. *There is a long exact sequence*

$$\cdots \to K_i\mathscr{M}^{p+1}_{R/A} \to K_i\mathscr{M}^{p}_{R/A} \to \varinjlim K_i(\mathscr{W}_{R'/A}) \to \cdots$$

where the limit runs over all R' as in (iv).

Proof. Use Corollary 3.13. Notice that the transition maps for the system $K_i\mathscr{W}_{R'/A}$ are defined on the level of topological spaces by the universal property of homotopy fibers. Q.E.D.

For the remainder of Section 4 we assume $A \to R$ is superpure with Cohen–Macaulay closed fiber.

In actuality, the transition maps are realized on the level of exact categories as follows: Suppose R_1' and R_2' are quotients of R defined by relative regular sequences of length p. Let S_1 and S_2 denote the sets of relative nonzero divisors in R_1' and R_2'. Suppose R_2' is smaller than R_1', and p: $R_1' \twoheadrightarrow R_2'$ is the canonical surjection.

LEMMA 4.5. $p(S_1) = S_2$.

Proof. By Lemma 4.3(i), we may assume $A = k$. The associated primes of R_2' are minimal, and thus are also associated primes in R_1', because the two rings are Cohen–Macaulay and have the same codimension. Thus $p(S_1) \subset S_2$.

Suppose $f \in R_1'$ and $p(f) \in S_2$, and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be those minimal primes of R_1' not containing $I = \ker p$. Now since $\mathfrak{p}_i$ is prime, $\mathfrak{p}_i \not\supset \bigcap_{j\neq i} \mathfrak{p}_j \cap I$, so choose $f_i \notin \mathfrak{p}_i$, and $f_i \in \bigcap_{j\neq i} \mathfrak{p}_j \cap I$. The only thing that could prevent f from being relative nonzero divisor is the possibility that $f \in \mathfrak{p}_i$ for some i. Let $f' = f + \sum f_i$ where the sum runs over i with $f \in \mathfrak{p}_i$. Then $f - f' \in I$ and $f' \notin \bigcup \mathfrak{p}_i$. Thus $f' \in S_1$ and $p(f') = p(f)$. Q.E.D.

COROLLARY 4.6. *$S_1^{-1}R_1' \to S_2^{-1}R_2'$ is a well-defined surjective ring homomorphism, and restriction of scalars defines an exact functor $\mathscr{W}_{R_2'/A} \to \mathscr{W}_{R_1'/A}$ which in turn yields the transition map in* (4.4).

Since $S_1^{-1}R_2' = S_2^{-1}R_2'$, it pays to adopt notation to deemphasize the role S_2 plays.

DEFINITION. $k(R'/A)$ denotes $S^{-1}R'$, where S is the set of all relative nonzero divisors in R'.

DEFINITION 4.7. Let $\mathscr{W}^{p/p+1}_{R/A}$ denote the category of R-modules W with the property that for some quotient $R' = R/(f_1, \ldots, f_p)R$ of R by a relative regular sequence of length p with $(f_1, \ldots, f_p)W = 0$, W is a $k(R'/A)$-module and is of the form $M \otimes_{R'} k(R'/A)$ for some $M \in \mathscr{M}_{R'/A}$.

THEOREM 4.8. (i) *$\mathscr{W}^{p/p+1}_{R/A} = \varinjlim \mathscr{W}_{R'/A}$, where the limit runs over R' as above, and is an exact category,*

(ii) *there is a long exact sequence*

$$\cdots \to K_i\mathscr{M}^{p+1}_{R/A} \to K_i\mathscr{M}^{p}_{R/A} \to K_i\mathscr{W}^{p/p+1}_{R/A} \to \cdots .$$

Proof. The point is that $\mathscr{W}_{R'/A}$ is actually a full subcategory of the category of all R-modules. Q.E.D.

The map $K_i\mathscr{M}^{p}_{R/A} \to K_i\mathscr{W}^{p/p+1}_{R/A}$ comes from an exact functor b: $\mathscr{M}^{p}_{R/A} \to \mathscr{W}^{p/p+1}_{R/A}$ described as follows: Given $M \in \mathscr{M}^{p}_{R/A}$, find an R' which supports it, and take $M \otimes_{R'} k(R'/A)$. The resulting R-module is independent of the choice of R'.

It is likely that the functor $\mathscr{M}^{p}_{R/A} \to \mathscr{W}^{p/p+1}_{R/A}$ is the universal exact functor killing $\mathscr{M}^{p+1}_{R/A}$, but that doesn't seem worth pursuing.

If A is a field, then $\mathscr{W}^{p/p+1}_{R/A}$ provides an explicit construction for the quotient Abelian category $\mathscr{M}_R{}^p/\mathscr{M}^{p+1}_R$, at least when R is Cohen–Macaulay. Compare this with the quotient of an abelian category by a localizing Serre subcategory [11, p. 40].

5. Cofinality in Codimension 1

Let $A \to R$ be as in Section 4, that is, superpure, with Cohen–Macaulay closed fiber.

DEFINITION 5.1. Let $\mathscr{M}^{p/p+1}_{R/A}$ be the category of R-modules X with the property that for some quotient $R' = R/(f_1, \ldots, f_p)R$ of R by a relative regular sequence of length p with $(f_1, \ldots, f_p)X = 0$, X is a finitely generated $k(R'/A)$-module flat over A.

Note that $\mathscr{W}^{p/p+1}_{R/A} \subset \mathscr{M}^{p/p+1}_{R/A}$.

THEOREM 5.2. *Suppose $A \to R$ is essentially smooth and A is Artin. Then $\mathscr{W}^{p/p+1}_{R/A} \subseteq \mathscr{M}^{p/p+1}_{R/A}$ is cofinal for $p \leqslant 1$.*

Proof. The case $p = 0$ was implicitly handled in [6, Theorem 3.8]: The closed fiber of $k(R/A)$ over A is a field, so flatness of X over A implies freedom of X as $k(R/A)$-module, and thus X comes from a free R-module.

Suppose $p = 1$, and $X \in \mathscr{M}^{1/2}_{R/A}$. Choose $R' = R/f'R$ where f' is a relative nonzero divisor and X is a finitely generated $k(R'/A)$-module. Let T' be the semilocalization of R at the minimal primes of $R' \otimes_A k$: this amounts to inverting all elements of R which map to relative nonzero divisors in R'; moreover, the elements inverted are relative nonzero divisors of R, as well. We see easily that:

(1) $T' \otimes_A k$ is a regular semilocal ring of dimension 1,

(2) $k(T'/A) = k(R/A)$,

(3) $T'/f'T' = k(R'/A)$.

By (3), X is finitely generated as T'-module, so we may find a resolution

$$E\colon (0 \to U \to F \to X \to 0),$$

where F is a free T'-module of finite rank. By (1) and flatness of X, $U \otimes_A k$ is a free $T' \otimes_A k$-module, so by Nakayama's lemma, U is a free T'-module. Since T' is a localization of R by relative nonzero divisors, we may find an exact sequence

$$D\colon (0 \to R^n \to R^n \to M \to 0)$$

such that $D \otimes_R T' \cong E$, and by (3.3) we know $M \in \mathscr{M}_{R/A}$. The remaining problem is that perhaps $f'M \neq 0$. Since $f'X = 0$, then $0 = X \otimes_{T'} k(R/A) = M \otimes_R k(R/A)$. Choose a relative nonzero divisor $f'' \in R$ so $f''M = 0$, and assume f' divides f''. Let $R'' = R/f''R$; since M is supported on R'', we see that $M \otimes_{R''} k(R''/A) \in \mathscr{W}^{1/2}_{R/A}$. Let T'' be the semilocalization of R at minimal primes of $R'' \otimes_A k$; we list its properties:

(2′) $k(T''/A) = k(R/A)$,

(3′) $T''/f''T'' = k(R''/A)$.

Consider the diagram

$$\begin{tikzcd}
M \otimes_R T'' \arrow[r, two heads, "\sim"] \arrow[d] & M \otimes_R k(R''/A) = W \arrow[d, two heads] \\
X = M \otimes_R T' \arrow[r, two heads, "\sim"] & M \otimes_R k(R'/A).
\end{tikzcd}$$

The top map is an isomorphism because $f''M = 0$, the bottom map is an isomorphism because $f'X = 0$, and the right-hand map is surjective by Lemma 4.5.

So $W \to W \otimes_{T''} T' = X$ is surjective; since T' is a localization of T'' by elements which are relative nonzero divisors on R', there is some $h \in R$ which is a relative nonzero divisor on R' so that $hY = 0$, where $Y = \ker(W \twoheadrightarrow X)$. Since no prime of height 1 in $R'' \otimes_A k$ contains both h and f', no prime of T'' contains h and f' so $(h, f')T'' = 1$, and we can let $1 = ah + bf'$ in T''. Notice that $0 \to Y \to W \to X \to 0$ is an exact sequence of T''-modules.

Define $W \to Y$ as $w \quad w(bf')$. ($wbf' \in W$ because $f'X = 0$.) If $w \in Y$ then $wbf' = w(1 - ah) = w$. Therefore $Y \oplus X \cong W$. Q.E.D.

COROLLARY 5.3. *There is a long exact sequence*

$$\cdots \to K_i\mathcal{M}^2_{R/A} \to K_i\mathcal{M}^1_{R/A} \to K_i\mathcal{M}^{1/2}_{R/A} \to K_{i-1}\mathcal{M}^2_{R/A} \to \cdots$$

which ends at $K_0\mathcal{M}^{1/2}_{R/A}$.

Proof. Apply Theorems 1.1 and 4.8. Q.E.D.

6. FLASQUENESS

Let A be an Artin local ring with residue field k and $S = \mathrm{Spec}(A)$.

LEMMA 6.1. *If R is a Noetherian flat A-algebra and $R \otimes_A k$ is Macaulay then $k(R/A) = \prod k(R_y/A)$, where y runs over the minimal primes of R.*

Proof. Since $R \otimes_A k$ is Macaulay, the relative nonzero divisors of R are those elements f contained in no minimal prime x. Thus $k(R/A)$ is the semi-localization of R at its minimal primes, and is the product of its local rings. Q.E.D.

PROPOSITION 6.2. *If R is an essentially smooth A-algebra, then*

$$\mathcal{M}^{p/p+1}_{R/A} \cong \coprod_{y} \mathcal{M}^{p/p+1}_{R_y/A},$$

where y runs over all primes in R of height p.

Proof. We simply combine Definition 5.1 with Lemma 6.1 applied to $R' = R/(f_1, \ldots, f_p)R$. Q.E.D.

DEFINITION 6.3. If X is a flat S-scheme, let $K_q^{p/p+1} = \pi_{q+1}(BQ\mathcal{M}^p(X/S), BQ\mathcal{M}^{p+1}(X/S))$. Let $\mathcal{K}_q^{p/p+1}(X/S)$ be the sheaf on X associated to the presheaf $U \to K_q^{p/p+1}(U/S)$, and let $\mathcal{K}_q{}^p(X/S)$ be the sheaf associated to $U \to K_q\mathcal{M}^p(U/S)$. (See [6, Sect. 1].)

PROPOSITION 6.4. *If X is a smooth S-scheme, $q \geqslant 1$, and $p \leqslant 1$, then*

$$\mathcal{K}_q^{p/p+1}(X, S) = \bigoplus_{y} (j_y)_* K_q\mathcal{M}^{p/p+1}(\mathcal{O}_y/A),$$

where y runs over all points of X of codimension p, and thus is a flasque sheaf.

Proof. The case $p = 0$ is treated in [6, Theorem 3.8]. For $p = 1$ the result follows from Corollary 5.3 and Lemma 6.2. Q.E.D.

COROLLARY 6.5. *There is an exact sequence*

$$\bigoplus_y K_1^{1/2}(\mathcal{O}_{X,y}/A) \to H^0(X, \mathcal{K}_0^{2/3}(X, S)) \to H^2(X, \mathcal{K}_2(X)) \to 0.$$

Proof. This follows from the exact sequence

$$0 \to \mathcal{K}_2 \to \mathcal{K}_2^{0/1} \to \mathcal{K}_1^{1/2} \to \mathcal{K}_0^{2/3} \to 0$$

obtained in [6, Corollary 1.6].

7. TAME SYMBOLS

Let R be a ring, $T \subset R$ a multiplicative set of central nonzero divisors, $F = T^{-1}R$, and let $\mathcal{H}$ be the exact category of R-modules of projective dimension 1 killed by T. There is a long exact sequence

$$\cdots \to K_{i+1}F \xrightarrow{d} K_i\mathcal{H} \to K_iR \to \cdots$$

If $f, g \in T$, the symbol $\{f, g\}$ is an element of K_2F. We propose to calculate $d\{f, g\} \in K_1\mathcal{H}$.

The long exact sequence results from the homotopy Cartesian square [5]:

$$\begin{tikzcd} S^{-1}S(\mathcal{P}_F) & S^{-1}E' \arrow[l, "\sim"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S^{-1}E \arrow[r, "\sim"] \arrow[d] & pt \\ & Q\mathcal{H} \arrow[r] & Q\mathcal{P}_R^1. & \end{tikzcd}$$

We must find a representation of $\{f, g\}$ which is easy to lift to $S^{-1}E'$ from $S^{-1}S(\mathcal{P}_F)$.

Consider the commutative diagram in $S^{-1}S(\mathcal{P}_F)$ depicted in Fig. 7.1. Here s denotes the isomorphism of $F^2 = F \oplus F$ which interchanges the two factors. Since the boundary is (0, 0), we have described a certain element γ of $\pi_2 S^{-1}S(\mathcal{P}_F) = K_2F$. Notice also that the arrows in Fig. 7.1 have as first component either s or id, and these isomorphisms come from isomorphisms of R^2; this will make the lifting possible.

For computations in the Steinberg group, we will abuse notation and consider (f) and $\left(\begin{smallmatrix} f & 0 \\ 0 & 1 \end{smallmatrix}\right)$ to denote the same matrix. We now do some calculations along the lines of [7], and adopt the notation there.

The diagram in Fig. 7.2 represents the image of γ in $\pi_1 F' = G(F) \cong St(F) \rtimes Gl(F)$. It follows that

$$\begin{aligned} j(\gamma) &= (k(s)^{(id, f\oplus 1)}\, k(s)^{-1})^{(id, g\oplus 1)} (k(s)^{(id, f\oplus 1)}\, k(s)^{-1})^{-1} \\ &= k(s)^{(id, fg\oplus 1)} \cdot k(s)^{(id, g\oplus 1)} \cdot k(s) \cdot k(s)^{(id, f\oplus 1)} \\ &= j((fg \oplus 1)^{-1} \star s)\, k(s)\, j((g \oplus 1)^{-1} \star s)\, k(s)^2 j((f \oplus 1)^{-1} \star s)\, k(s) \\ &= j(((fg)^{-1} \star s) \cdot (g^{-1} \star s)^s \cdot (f^{-1} \star s)^s). \end{aligned}$$

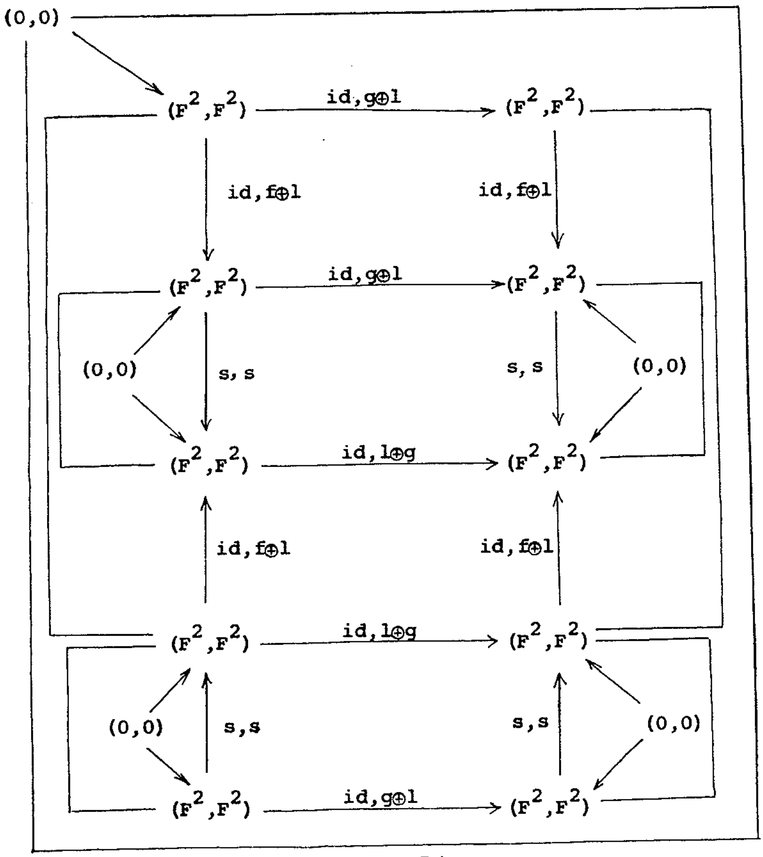


FIGURE 7.1

Here s also denotes the matrix $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. Let $w = \left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$. Then $g \star w = [h_{13}(g), w_{12}(1)] = (h_{13}(g)\, w_2(1))\, w_{12}(-1) = w_{12}(g)\, w_{12}(-1) = h_{12}(g)$.

Let

$$t = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

the image in $Gl(F)$ of $h_{14}(-1)$. Then

$$wt = \left(\begin{array}{cc|c|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & -1 \end{array}\right) = s \oplus 1 \oplus -1.$$

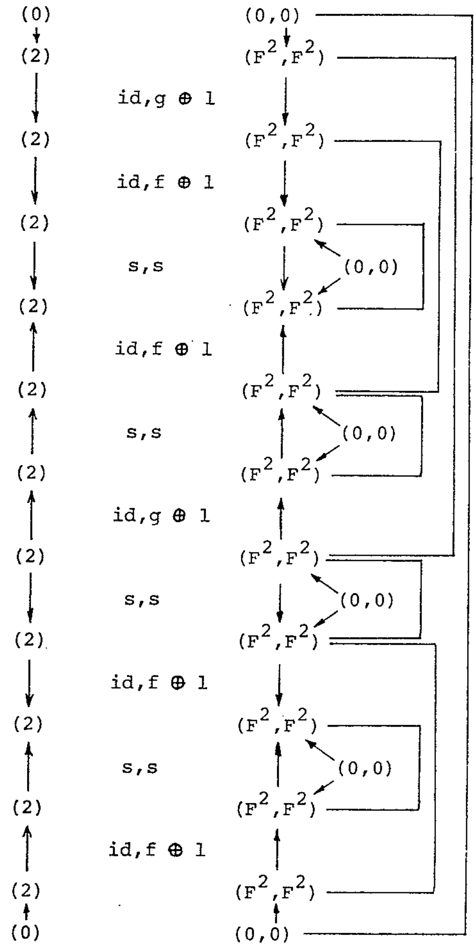


FIGURE 7.2

Thus

$$g \star s = g \star wt = (g \star w) \cdot ((g \star -1)^{w^{-1}})$$
$$= (g \star w)(\{g, -1\}^{w^{-1}}) = (g \star w)\{g, -1\}.$$

Also

$$1 = g \star 1 = g \star s^2 = g \star ((wt)^2)$$
$$= (g \star wt) \cdot (g \star wt)^{wt} = (g \star s) \cdot (g \star s)^s,$$

so

$$(g \star s)^s = (g \star s)^{-1}.$$

Thus

$$\begin{aligned}
\gamma &= ((fg)^{-1} \star s)(f^{-1} \star s)^{-1}(g^{-1} \star s)^{-1} \\
&= ((fg)^{-1} \star w)(f^{-1} \star w)^{-1}(g^{-1} \star w)^{-1}\{(fg)^{-1}, -1\} \\
&\qquad \cdot \{f^{-1}, -1\}^{-1} \cdot \{g^{-1}, -1\}^{-1} = h_{12}((fg)^{-1})\, h_{12}(f^{-1})^{-1}\, h_{12}(g^{-1})^{-1} \\
&= \{f^{-1}, g^{-1}\} = \{f, g\}.
\end{aligned}$$

We have shown (for any ring F):

PROPOSITION 7.3. *The element of K_2F represented in Fig.* 7.1 *is* $\{f, g\}$.

It is hard to see directly that Fig. 7.1 and Fig. 1 from [7] are homotopic.

Now we lift Fig. 7.1 to $S^{-1}E'$. We depict objects of $S^{-1}E'$ as pairs $(P, Q \twoheadrightarrow M)$ where $P, Q \in \mathscr{P}_R$ and $M \in \mathscr{H}$. Arrows are of the following three types:

$$(P, Q \twoheadrightarrow M) \to (P \oplus P', Q \oplus P' \twoheadrightarrow M)$$

$$\begin{tikzcd}
\cdot \arrow[d] & (P, \arrow[d, "\wr"'] & Q \arrow[r, two heads] \arrow[d, tail] & M) \arrow[d, tail] \\
\cdot & (P', & Q' \arrow[r, two heads] & M')
\end{tikzcd}
\quad \text{or} \quad
\begin{tikzcd}
\cdot \arrow[d] & (P, \arrow[d, "\wr"'] & Q \arrow[r, two heads] \arrow[d, equal] & M) \\
\cdot & (P', & Q \arrow[r, two heads] & M') \arrow[u, two heads]
\end{tikzcd}$$

In Fig. 7.4 is depicted an element of $\pi_2 S^{-1}E'$; it is a lifting of Fig. 7.1 and is uniformly four times as large. (The notation used in Fig. 7.4 is as follows. We have objects

$$\begin{aligned}
A &= (R, R \twoheadrightarrow 0), \\
B &= (R, R \twoheadrightarrow R/f), \\
C &= (R, R \twoheadrightarrow R/g), \\
D &= (R, R \twoheadrightarrow R/fg), \\
0 &= (0, 0 \twoheadrightarrow 0)
\end{aligned}$$

in $S^{-1}E'$. The arrow s is the switch isomorphism. Unmarked arrows are identities. The other arrows all involve the identity map on the S component, and involve arrows from E' as follows.

$$\begin{tikzcd}
A \arrow[d, "1"'] & R \arrow[r, two heads] \arrow[d, equal, "1"'] & 0 \\
B & R \arrow[r, two heads] & R/f \arrow[u, two heads]
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[d, "f"'] & R \arrow[r, two heads] \arrow[d, tail, "f"'] & 0 \arrow[d, tail] \\
B & R \arrow[r, two heads] & R/f
\end{tikzcd}$$

$$\begin{tikzcd}
C \arrow[d, "f"'] & R \arrow[r, two heads] \arrow[d, tail, "f"'] & R/g \arrow[d, tail] \\
D & R \arrow[r, two heads] & R/fg.
\end{tikzcd}$$

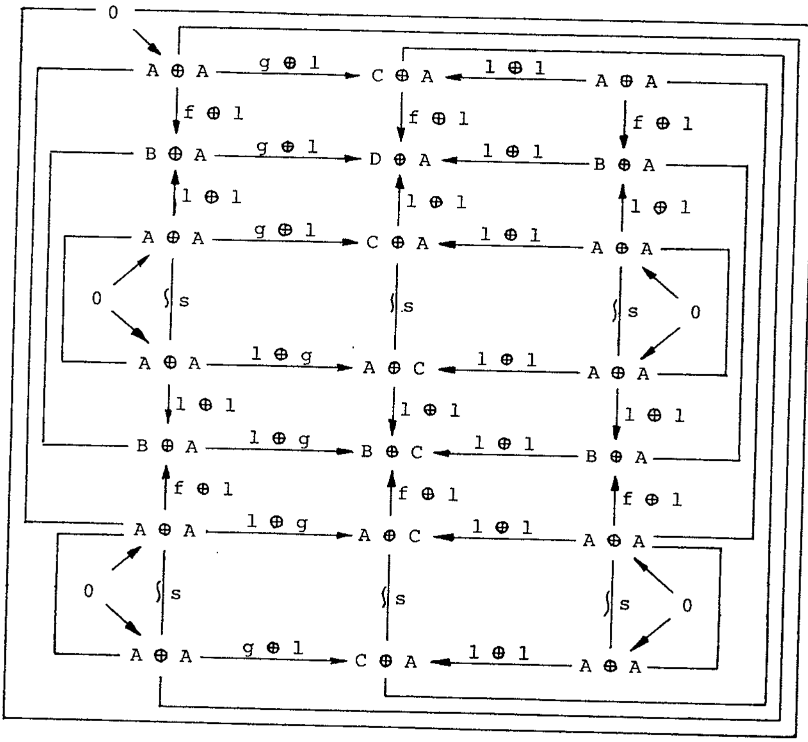


FIGURE 7.4

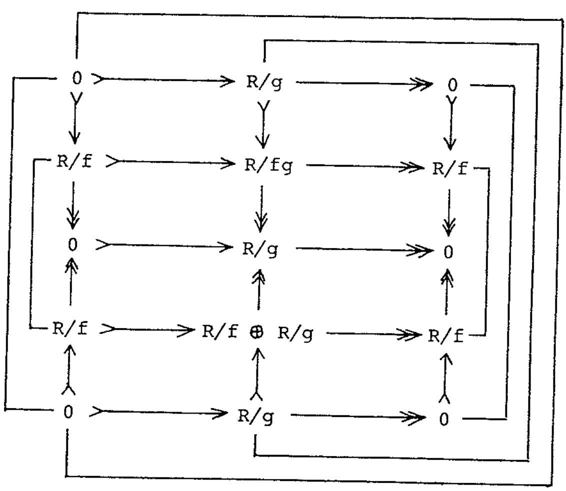


FIGURE 7.5

The arrows $1: A \to C$, $1: B \to D$, $1: C \to D$, $g: A \to C$, and $g: B \to D$ are analogous.) It is now easy to apply the forgetful functor $S^{-1}E' \to Q\mathscr{H}$ to Fig. 7.4, yielding Fig. 7.5.

If $\mathscr{M}$ is any exact category, then we can always get an element of $K_1\mathscr{M}$ as follows. Suppose

$$E: 0 \to L \to M \to N \to 0,$$
$$F: 0 \to N \to M \to L \to 0$$

are two exact sequences in $\mathscr{M}$. Let $T(E,F)$ denote the class in $K_1\mathscr{M} = \pi_2 Q\mathscr{M}$ of the commutative diagram in Fig. 7.6 in $Q\mathscr{M}$:

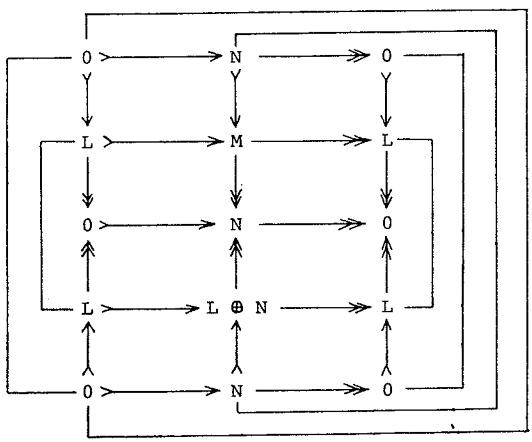


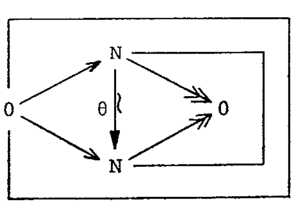
FIGURE 7.6

Note that if $M = L \oplus N$ and E and F are the usual exact sequences, then $T(E,F) = 0$.

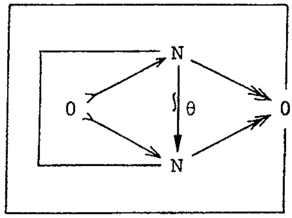
With this notation we have:

PROPOSITION 7.7. *Given $f, g \in T$, let $E = (0 \to R/f \to R/fg \to R/g \to 0)$ and let $F = (0 \to R/g \to R/fg \to R/f \to 0)$. Then $d\{f,g\} = T(E,F)$.*

Remark 7.8. If $L = 0$, it is easy to see that $T(E,F) = [N, \theta]$, where θ is the automorphism $N \to M \to N$, and $[N, \theta]$ is the element of K_1 defined by

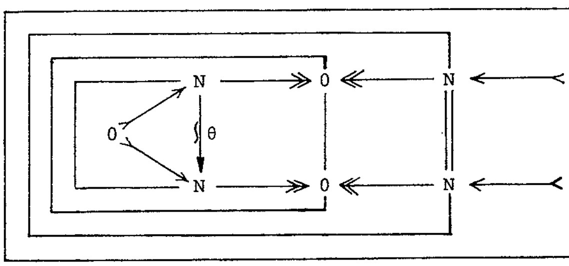


Note that



represents the same class in $\pi_2 Q\mathcal{M}$; this follows from the fact that $\pi_1 Q\mathcal{M}$ acts trivially on $\pi_2 Q\mathcal{M}$.

We can use the fibration $S^{-1}S \to S^{-1}E \to Q\,\mathcal{M}$ if $\mathcal{M}$ is semisimple [4] to see what the corresponding element of $\pi_1 S^{-1}S = K_1\mathcal{M}$ is. We may lift to $S^{-1}E$ and get the diagram in Fig. 7.9; we have actually lifted



which can be seen to represent the same class in $\pi_2 Q\mathcal{M}$ by using homotopies such as

$$\begin{tikzcd}
N \arrow[r, dash] \arrow[d, two heads] & N \arrow[d, two heads] & & & N \arrow[dl] \arrow[dd, dash] \arrow[dr, two heads] & \\
0 \arrow[r, dash] & 0 \arrow[r] & 0 & & & 0. \\
N \arrow[r, dash] \arrow[u, tail] & N \arrow[u, tail] & & & N \arrow[ul] \arrow[ur, tail] &
\end{tikzcd}$$

(The notation used in Fig. 7.9 is as follows. We have objects

$$A = (0, N \overset{1}{\twoheadrightarrow} N),$$

$$B = (N, N \twoheadrightarrow 0),$$

$$C = (0, N \twoheadrightarrow 0)$$

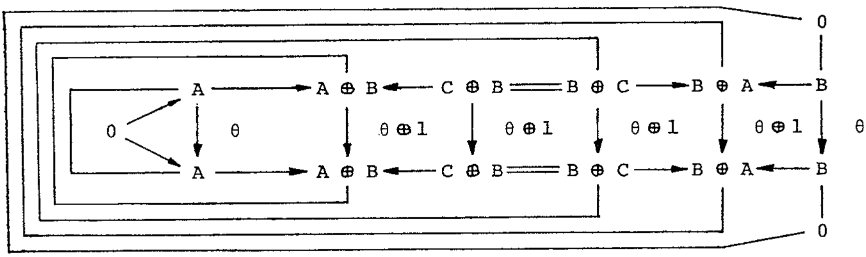


FIGURE 7.9

of $S^{-1}E$ and arrows

$$\begin{tikzcd} C \arrow[d] & 0 \arrow[d, equal] & N \arrow[r, two heads] \arrow[d, equal] & 0 \\ A & 0 & N \arrow[r, two heads] & N \arrow[u, two heads] \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[d] & 0 \arrow[d, equal] & 0 \arrow[r, two heads] \arrow[d, tail] & 0 \arrow[d, tail] \\ A & 0 & N \arrow[r, two heads] & N. \end{tikzcd}$$

The arrow $A \to A \oplus B$ arises from the action of S, and the equality $C \oplus B = B \oplus C$ is just that; it is not the switch map. We let θ denote the arrows which are 1 on the S component, and θ on N.)

The boundary of (7.9) yields the class in the following lemma.

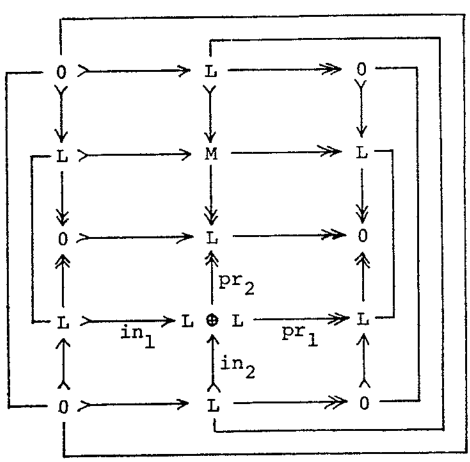


FIGURE 7.12

LEMMA 7.10. *If $\mathcal{M}$ is semisimple, then $[N, \theta]$, as defined in Remark 7.8, is given by the path*

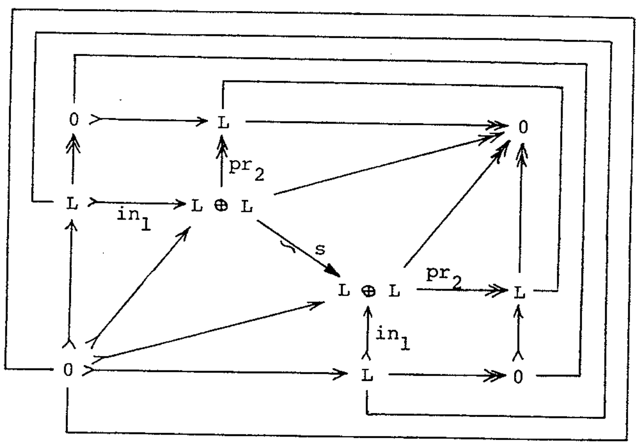
$$(0, 0) \to (N, N) \xrightarrow{1,\theta} (N, N) \to (0, 0)$$

in $S^{-1}S$.

We treat now the case where $E = F$.

LEMMA 7.11. *If $E = F$ then $T(E, F) = [L^2, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)]$.*

Proof. $E = F$ means $L = N$, so we can transform Fig. 7.6 into Fig. 7.12. The top half of Fig. 7.12 can be tucked away by folding on the diagonal, and the bottom half can be recast as



where $s = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Folding this along the diagonal yields the result. Q.E.D.

COROLLARY 7.13. *If R is a discrete valuation ring with valuation v and residue field k, and $T = R - 0$, then $d\{f, g\}$ is the usual tame symbol in $K_1 k = k^*$ given by*

$$T(f, g) = (-1)^{v(f)v(g)} \overline{f^{v(g)} g^{-v(f)}}.$$

Proof. Let π be a uniformizing parameter. Then Lemma 7.11 shows that $d\{\pi, \pi\} = [k^2, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)] = -1$. If $f, g \in R^*$, then $d\{f, g\} = 0$ because $\{f, g\}$ comes from $K_2 R$.

If $f \in R^*$, then $d\{f, \pi\} = [k, \bar{f}] = \bar{f}$ by Remark 7.8 and Lemma 7.10.

These three cases of the formula, together with additivity of both sides, yield the result. Q.E.D.

8. Divisors

We return to the notation of Section 3.

In this section we will compute the effect of the map

$$d: K_1\mathscr{V} \to K_0\mathscr{N}$$

from Lemma 3.13 on the elements $T(E, F)$ defined in Fig. 7.6. It will be clear that the computations will apply equally well to the boundary map in the localization theorem for projective modules. We anticipate the calculation with the following definitions.

Call $L \subset V$ a lattice if $L \in \mathscr{L}$ and $S^{-1}L = V$. If L_1, L_2 are lattices in V, then there is a lattice L with $L \subset L_1$ and $L \subset L_2$ (e.g., $L = sL_2$ for suitable $s \in S$); let $\chi(L_1, L_2) = [L_1/L] - [L_2/L] \in K_0\mathscr{N}$. It is easy to see that $\chi(L_1, L_2)$ does not depend on L.

Suppose $E_1 = (0 \to V' \to V \to V'' \to 0)$ and $E_2 = (0 \to V'' \to V \to V' \to 0)$ are exact sequences in $\mathscr{V}$. There are exact sequences $F_1 = (0 \to L_1' \to L_1 \to$

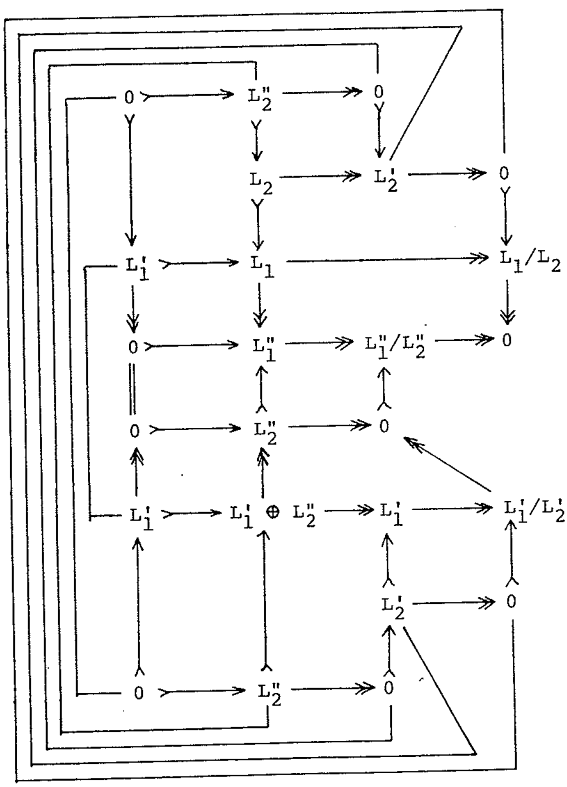


Figure 8.2

$L_1' \to 0)$ and $F_2 = (0 \to L_2'' \to L_2 \to L_2' \to 0)$ in $\mathscr{L}$ so that $S^{-1}F_1 = E_1$ and $S^{-1}F_2 = E_2$, by Lemma 3.6.1. Let $\chi(E_1, E_2)$ denote $\chi(L_1, L_2) - \chi(L_1', L_2') - \chi(L_1'', L_2'') \in K_0\mathscr{N}$. If $F_1 \subset \tilde{F}_1$ are two exact sequences of lattices in E_1, then $\chi(\tilde{L}_1, L_2) = [\tilde{L}_1/L_1] + \chi(L_1, L_2)$, etc., so $\chi(\tilde{E}_1, E_2) = \chi(E_1, E_2) + [\tilde{L}_1/L_1] - [\tilde{L}_1'/L_1'] - [\tilde{L}_1''/L_1''] = \chi(E_1, E_2)$. Since the set of suitable F_1 is filtering, $\chi(E_1, E_2)$ is independent of the choice of F_1; similarly for F_2.

PROPOSITION 8.1. $d(T(E_1, E_2)) = \chi(E_1, E_2)$.

Proof. We choose F_1, F_2 as above. We may assume $L_1 \supset L_2$, $L_1' \supset L_2'$, and $L_1'' \supset L_2''$ by replacing F_2 by sF_2 for large $s \in S$. We lift as much as we can of Fig. 7.6 (in which L, M, and N have been replaced by V', V, and V'') along the map $Q\mathscr{M}_{R/A} \to Q\mathscr{W}$ and obtain Fig. 8.2. Its boundary is in $Q\mathscr{N}$, and represents $[L_1/L_2] - [L_1''/L_2''] - [L_1'/L_2'] = \chi(E_1, E_2)$. Q.E.D.

COROLLARY 8.3. *If $V \in \mathscr{V}$ and θ: $V \to V$ is an automorphism, then $d[V, \theta] = \chi(L, \theta L)$ where $L \subset \dot{V}$ is a lattice.*

Proof. Let $E_1 = (0 \to 0 \to V \xrightarrow{\theta} V \to 0)$ and $E_2 = (0 \to V \xrightarrow{1} V \to 0 \to 0)$ so that $[V, \theta] = T(E_1, E_2)$; let $F_1 = (0 \to 0 \to L \to \theta L \to 0)$ and $F_2 = (0 \to L \to L \to 0 \to 0)$. Then $\chi(E_1, E_2) = \chi(L, L) - \chi(\theta L, L) = \chi(L, \theta L)$. Q.E.D.

9. DESCRIPTION OF $H^2(X, \mathscr{K}_2)$

We return to the notation of Section 6, where $S = \mathrm{Spec}(A)$, X is smooth over S, and A is an Artin local ring with residue field k.

Suppose that X is a family of surfaces; then there are no modules of codimension 3 in X, so

$$H^0(X, \mathscr{K}_0^{2/3}(X/S)) = \bigoplus_x K_0{}^2(\mathcal{O}_{X,x}/A),$$

where x runs of points of codimension 2 in X, i.e., closed points in X. (Notice that the modules in $\mathscr{M}^2(\mathcal{O}_{X,x}/A)$ are finitely generated free A-modules.)

Let $A^2(X)$ denote $H^2(X, \mathscr{K}_2(X))$; from Corollary 6.5 we have the exact sequence

$$\bigoplus_y K_1^{1/2}(\mathcal{O}_y/A) \xrightarrow{d_0} \bigoplus_x K_0{}^2(\mathcal{O}_x/A) \to A^2(X) \to 0.$$

We must compute the image of d_0 in order to get relations for $A^2(X)$.

Notice that $\mathcal{O}_y$ has dimension 1, so

$$K_1^{1/2}(\mathcal{O}_y/A) = K_1{}^1(\mathcal{O}_y/A).$$

It follows from [6, Theorem 1.3] that $K_1^1(\mathcal{O}_y/A) \to K_1^0(\mathcal{O}_y/A)$ is zero, and thus $d_{1,y}\colon K_2^{0/1}(\mathcal{O}_y/A) \to K_1^1(\mathcal{O}_y/A)$ is surjective. Suppose X is irreducible. From [6, 3.8] we see that $K_2^{0/1}(\mathcal{O}_y/A) = K_2k(X/S)$. Since $k(X/S)$ is a local ring, $K_2k(X/S)$ is generated by symbols $\{f, g\}$ where f, g are (relative) nonzero divisors in $\mathcal{O}_y$. Thus sufficient relations for $A^2(X)$ are given by elements $d_0(d_{1,y}\{f, g\})$. Suppose $x \in X$ is a closed point. Since everything above is contravariant in X for flat maps, we may replace X by $X_x = \operatorname{Spec}(\mathcal{O}_x)$ in order to compute $d_{0,x}(d_{1,y}\{f, g\}) \in K_0^2(\mathcal{O}_x/A)$. By additivity we may assume $f, g \in \mathcal{O}_x$. Let $R = \mathcal{O}_x$,

$$E_y = (0 \to (R/f)_y \to (R/fg)_y \to (R/g)_y \to 0),$$

and

$$F_y = (0 \to (R/g)_y \to (R/fg)_y \to (R/f)_y \to 0).$$

Then $d_{1,y}\{f, g\} = T(E_y, F_y) \in K_1^1(\mathcal{O}_y/A)$ by (7.7). Now by (6.4), $K_2^{1/2}(\mathcal{O}_x/A) = \bigoplus_z K_1^1(\mathcal{O}_z/A)$ where z runs over points in $\mathcal{O}_x$ of codimension 1. For those $z \neq y$ let

$$E_z = (0 \to (R/f)_z \to (R/f \oplus R/g)_z \to (R/g)_z \to 0),$$

and

$$F_z = (0 \to (R/g)_z \to (R/f \oplus R/g)_z \to (R/f)_z \to 0).$$

(Note that all but a finite number of these are zero.) Let $E_x = \bigoplus E_z$, $F_x = \bigoplus F_z$. Since the functor $\mathscr{M}^1(\mathcal{O}_x/A) \to \mathscr{M}^{1/2}(\mathcal{O}_x/A)$ carries R/f (resp. R/g) to $\bigoplus(R/f)_z$ (resp. $\bigoplus(R/g)_z$) we see that E and F have end terms actually in $\mathscr{W}^{1/2}(\mathcal{O}_x/A)$. From Lemma 3.8.2 and Definition 4.7 we see that E and F are in $\mathscr{W}^{1/2}(\mathcal{O}_x/A)$. For $z \neq y$, we know $T(E_z, F_z) = 0$ in $K_1^1(\mathcal{O}_z/A)$, so $T(E_x, F_x) = \sum T(E_z, F_z) = T(E_y, F_y)$. Thus $d_{0,x}(d_{1,y}\{f, g\}) = d_{0,x}(T(E_x, F_x)) = \chi(E_x, F_x)$ by Proposition 8.1.

The above furnishes a complete description of the relations for $A^2(X)$ with no reference to higher K-theory; still, it is very complicated.

To sum up, for each triple (y, f, g) with $y \in X$ a point of codimension 1, $f, g \in k(X/S)^*$, there is a family of elements $c_x = c_x(y, f, g) \in K_0^2(\mathcal{O}_x/A)$ for a finite number of closed points x in the closure of y. If $f, g \in \mathcal{O}_x$, then $c_x = \chi(E_x, F_x)$, where E_x, F_x are constructed from f and g and χ is described in Section 8. Finally, $A^2(X)$ is the quotient of $\bigoplus K_0^2(\mathcal{O}_x/A)$ by the subgroup generated by the elements $c(y, f, g) = \sum_x c_x(y, f, g)$.

Observe that if $f, g \in \mathcal{O}_y^*$, then $c_x = \chi(E_y, F_y) = 0$. Moreover, if y is the only component of f or g which passes through x, then $\chi(E_x, F_x) = 0$ (because $T(E_x, F_x)$ comes from $K_1^1(\mathcal{O}_x/A)$).

If $f \in \mathcal{O}_y^*$, then $T(E_y, F_y) = [(\mathcal{O}_x/g)_y, f]$; if moreover y is the only component of g, then $\chi(E_y, F_y) = \chi((R/g), f \cdot (R/g))$ by Corollary 8.3, so $c_x = [\mathcal{O}_x/(f, g)] \in K_0^2(\mathcal{O}_x/A)$. The latter element is the one we would expect to be the divisor of f considered as a function on the support of g.

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