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K-THEORY AND LOCALIZATION OF NONCOMMUTATIVE RINGS

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Suppose R is a ring with 1 and S is a multiplicative set of nonzerodivisors. We suppose S satisfies the left Ore condition, namely, given $s \in S$ and $r \in R$ there exist $s_1 \in S$ and $r_1 \in R$ so that $r_1 s = s_1 r$. If, in addition, S satisfies the right Ore condition, then the ring of left quotients $S^{-1}R = \{s^{-1}r\}$ is also a ring of right quotients. We will say simply that S satisfies the two-sided Ore condition.

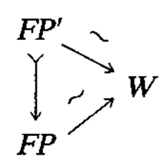
The purpose of this note is to point out that there is a long exact sequence of K-groups

$$(*) \qquad \cdots K_1\mathcal{H} \to K_iR \to K_iS^{-1}R \to K_{i-1}\mathcal{H} \cdots$$

ending at $K_0S^{-1}R$ provided S satisfies the two-sided Ore condition. The usual proofs [2, 3] of the localization theorem for projective modules in K-theory do the job.

The interest in this situation arises from some work of Justin Smith [5] which was pointed out to me by Andrew Ranicki. He deals with the case where $G \to H$ is a surjective homomorphism of groups, H is a finite extension of a polycyclic group, $\ker(G \to H)$ is finitely generated nilpotent, $R = \mathbf{Z}G$, $I = \ker \mathbf{Z}G \to \mathbf{Z}H$ and $S = 1 + I$. He shows S satisfies the Ore conditions by showing I satisfies the Artin–Rees property. Thus the map $\mathbf{Z}G \to \mathbf{Z}H$ is a composite of a nice localization $\mathbf{Z}G \to S^{-1}\mathbf{Z}G$ followed by a surjection $S^{-1}\mathbf{Z}G \to \mathbf{Z}H$ with kernel in the radical.

The barest ingredients needed for a localization theorem seem to be the following. We are given an exact functor $F: \mathcal{P}' \to \mathcal{W}$ of exact categories. For any $W \in \mathcal{W}$ we define the category $\mathcal{L}_W$ to have for objects all pairs $(P, g: FP \xrightarrow{\sim} W)$ where P is a projective object of $\mathcal{P}'$ and g is an isomorphism in $\mathcal{W}$, and to have for arrows all admissible monomorphisms $P' \rightarrowtail P$ in $\mathcal{P}'$ which make



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commute. We define $\ker F$ to be the full exact subcategory of $\mathscr{P}'$ whose objects are all M with $FM \simeq 0$.

Theorem 1. *Suppose that*

(i) *$\mathscr{W}$ is semisimple (i.e. every object is projective),*

(ii) *$\mathscr{P}'$ is hereditary (i.e. every object has projective dimension $\leqslant 1$ inside $\mathscr{P}'$), and*

(iii) *for each $W \in \mathscr{W}$ the category $\mathscr{L}_W$ is contractible (and thus, is not empty).*

Then there is an exact sequence

$$\cdots K_i(\ker F) \to K_i\mathscr{P}' \to K_i\mathscr{W} \to K_{i-1}(\ker F) \cdots \to \cdots K_0\mathscr{P}' \to K_0\mathscr{W} \to 0.$$

The proof is the same as the proof of the localization theorem in [3] without the appeal to the cofinality theorem.

Now suppose $f: R \to T$ is an injective ring homomorphism. Let $\mathscr{W}$ be the category of finitely generated free left T-modules, and let $\mathscr{P}'$ be the category of finitely presented left R-modules M with projective dimension $\leqslant 1$ such that $T \otimes_R M$ is free. If T is right flat, then we may define an exact functor $F: \mathscr{P}' \to \mathscr{W}$ by $FM = T \otimes_R M$. The category $\mathscr{L}_w$ is equivalent to the ordered set of lattices in W; we call $P \subset W$ a lattice if P is a finitely generated projective left R- submodule of W and $T \otimes_R P \simeq W$. Since a filtering ordered set is contractible, we have

Theorem 2. *Suppose*

(a) *$f: R \to T$ is injective and T is flat as right R-module, and*

(b) *T is a filtering union of lattices in T relative to R. Then the conditions of Theorem 1 are satisfied for $F: \mathscr{P}' \to \mathscr{W}$.*

The objects of $\mathscr{H} = \ker F$ are easy to describe; they are all the finitely generated left R-modules M with $T \otimes_R M = 0$ and which have projective resolutions

$$0 \to P \to Q \to M \to 0,$$

for such M also have resolutions where Q is free, and this implies $T \otimes_R P$ and $T \otimes_R Q$ are free, so that $P, Q \in \mathscr{P}'$.

Now the inclusion $\mathscr{W} \subset \mathscr{P}_T$ (where $\mathscr{P}_T$ denotes the category of finitely-generated projective left T-modules) is cofinal (i.e. $X \in \mathscr{P}_T \Rightarrow \exists X' \in \mathscr{P}_T$ such that $X \oplus X' \in \mathscr{W}$) so [1, Proposition 1.3] we see that

$$K_i\mathscr{W} \to K_iT$$

is an isomorphism for $i > 0$ and is injective for $i = 0$.

Let $\mathscr{F}$ denote the category of projective objects in $\mathscr{P}'$. Since the kernel of a surjective map of free modules need not be free we must use the slight generalization of the resolution theorem [4, Theorem 2.1] to obtain the isomorphism $K_i\mathscr{F} \xrightarrow{\sim} K_i\mathscr{P}'$. Now the cofinality theorem (as above) applies to $K_i\mathscr{F} \to K_iR$.

Thus we may obtain the long exact sequence (∗) provided we check exactness of

$$K_0\mathscr{H} \to K_0R \to K_0T,$$

which we proceed to do. Suppose we are given $\alpha = [P]-[Q] \in \ker K_0R \to K_0T$ with $P, Q \in \mathscr{P}_R$. Since $T\otimes P$ and $T\otimes Q$ are stably isomorphic, adding a suitable projective module allows us to assume $T\otimes P$ and $T\otimes Q$ are isomorphic and free. By hypothesis (b) of Theorem 2 we see there is a lattice L in $T\otimes P$ containing both P and Q. Thus $\alpha = [L/Q]-[L/P] \in \operatorname{im}(K_0\mathscr{H} \to K_0R)$.

We have shown

Corollary 3. *Under the hypotheses of Theorem 2 there is a long exact sequence* (∗).

Now suppose $S \subset R$ satisfies the two-sided Ore condition. We will show that $f: R \to S^{-1}R = T$ satisfies the hypotheses of Theorem 2.

Two elements $s^{-1}r$ and $t^{-1}p$ in $S^{-1}R$ are equal if and only if there are $s_1, t_1 \in S$ so that $s_1s = t_1t$ and $s_1r = t_1p$. This equivalence relation is generated by the requirement that $s^{-1}r = (s_1s)^{-1}(s_1r)$.

Given $s, t \in S$, we have $s^{-1}R \subset (s_1s)^{-1}R = (t_1t)^{-1}R \supset t^{-1}R$ when s_1 and t_1 are chosen to satisfy $s_1s = t_1t$. Thus $S^{-1}R = \cup s^{-1}R$ is a filtering union of free right R-submodules, and therefore is right flat. Similarly, we see that $S^{-1}R = \cup Rs^{-1}$ is a filtering union of left lattices, so the conditions of Theorem 2 are fulfilled.

I don't know whether there are useful rings satisfying the conditions of Theorem 2 which are not classical rings of quotients.

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