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On the K-theory of Fields

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Abstract. We give a survey of recent results on the K-theory of fields, with particular attention to the Merkuriev-Suslin theorem and the K-theory of algebraically closed fields.

1. The algebraic K-groups of fields.

We give a brief survey of recent results on the K-theory of fields, and for some theorems, we give brief sketches of proofs. I have found **[50]** a useful source of information.

Let F be a field. Few theorems of a general nature are known about the algebraic K-groups $K_n(F)$, due to the fact that their definition by Quillen in **[36]** (the Q-construction) presents them as the homotopy groups $K_n(F) = \pi_n K(F)$ of a certain space $K(F)$ which can be constructed easily and explicitly, but whose homotopy groups are mysterious. The homology groups of the connected component of the basepoint of $K(F)$ are the same as those of the infinite general linear group $G\ell(F)$, and Quillen's direct construction of this component from $G\ell(F)$ is denoted by $BG\ell(F)^+$. Concerning the homotopy groups, we do know that

$$\begin{aligned} K_0F &= \mathbb{Z} \\ K_1F &= F^\times \\ K_2F &= (F^\times \otimes_{\mathbb{Z}} F^\times)/\langle a \otimes (1-a) \mid a \in F - \{0,1\}\rangle \end{aligned} \tag{1.1}$$

The standard notation for the image of $a \otimes b$ in K_2F is $\{a, b\}$, and the relation $\{a, 1-a\} = 1$ is called the Steinberg relation. From the Steinberg relation one can deduce that $\{a, b\} = \{b, a\}^{-1}$ **[31**, p.95**]**.

There is an operation

$$K_mF \otimes K_nF \longrightarrow K_{m+n}F$$

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1980 *Mathematics subject classifications.* Primary 18F25; Secondary 19C30, 19D99 which endows $\bigoplus_{n=0}^{\infty} K_nF$ with the structure of a skew-commutative graded ring. In the case $m = n = 1$ this product agrees with the Steinberg symbol (at least up to a possible sign).

One defines the Milnor ring of the field F to be the quotient of the tensor algebra of $F^\times$ by the ideal generated by the Steinberg relations. This ring is a graded ring, and we let $K_n^M F$ denote its degree n part. It follows from what we have said that there is a natural map $K_n^M F \longrightarrow K_n F$ which is an isomorphism for $0 \leq n \leq 2$. We define the indecomposable part $K_n^{\text{ind}} F$ to be the cokernel of the map $K_n^M F \longrightarrow K_n F$. It is the quotient of K_nF by those elements which are products of elements of $K_1F = F^\times$, and is discussed in sections 17 and 18 below.

For finite fields, the K-groups are completely known.

THEOREM 1.2. **[35]**

$$\begin{aligned} K_0\mathbb{F}_q &= \mathbb{Z} \\ K_{2i+1}\mathbb{F}_q &= \mathbb{Z}/(q^{i+1} - 1) \\ K_{2i+2}\mathbb{F}_q &= 0 \end{aligned}$$

ome results are known for number fields F.

THEOREM 1.3. **[31**, Theorem 11.6 of Tate] $K_2\mathbb{Q} \simeq \mathbb{Z}/2 \oplus \bigoplus_{p \text{ prime}} (\mathbb{Z}/p)^\times$

THEOREM 1.4. **[20]** $K_3\mathbb{Q} = \mathbb{Z}/48$.

The following theorem of Borel is proved by a detailed study of harmonic forms on symmetric spaces associated to arithmetic groups, and depends on earlier work of Borel and Serre on compactifying these symmetric spaces.

THEOREM 1.5. **[6]** *Let F be a number field with r_1 real places and r_2 complex places. Then*

$$\operatorname{rank} K_nF = \begin{cases} 1 & \text{if } n = 0 \\ \infty & \text{if } n = 1 \\ 0 & \text{if } n = 2k \text{ and } k > 0 \\ r_1 + r_2 & \text{if } n = 4k+1 \text{ and } k > 0 \\ r_2 & \text{if } n = 4k+3 \text{ and } k \geq 0 \end{cases}$$

When phrased in terms of the ring $\mathcal{O}_F$ the result looks a bit better, and can be seen to be a generalization of Dirichlet's unit theorem, which states that $\operatorname{rank} \mathcal{O}_F^\times = r_1 + r_2 - 1$.

Soulé **[42]** proved surjectivity in certain cases for a Chern character map from K-theory to étale cohomology. For example, it follows that $K_{22}\mathbb{Q}$ contains an element of order 691 (the numerator of the 12th Bernoulli number).

EXAMPLE 1.6. Notice that $\operatorname{rank} K_2\mathbb{Q} = 0$ but $\operatorname{rank} K_5\mathbb{Q} = 1$. It follows that $\operatorname{rank} K_n^M\mathbb{Q} = 0$ for all $n > 2$, and thus the map $K_5^M\mathbb{Q} \to K_5\mathbb{Q}$ is not surjective. The possible injectivity of this map is dealt with by Corollary 16.2 below.

2. Hilbert's Theorem 90 for K_1.

THEOREM 2.1. *Let L/K be a cyclic field extension of degree n with Galois group G generated by σ. Let $N = 1 + \sigma + \cdots + \sigma^{n-1} \in K[G]$. The sequence*

$$L^\times \xrightarrow{1-\sigma} L^\times \xrightarrow{N} F^\times$$

is an exact sequence of abelian groups.

PROOF: We let $L_{\mathrm{tw}}[G]$ be the twisted group ring, where multiplication is defined so that $\sigma \cdot w = \sigma(w) \cdot \sigma$ for all $w \in L$. One of the fundamental results of Galois theory is that the evident map $L_{\mathrm{tw}}[G] \to \mathrm{Hom}_F(L, L)$ is an isomorphism of F-algebras. Now given $z \in L^\times$ with $Nz = 1$ we find

$$(z \cdot \sigma)^n = z \cdot \sigma(z) \cdots \sigma^{n-1}(z) \cdot \sigma^n = Nz = 1$$

and thus

$$(1 - z \cdot \sigma)(1 + z \cdot \sigma + \cdots + (z \cdot \sigma)^{n-1}) = 0$$

Letting $t = (1 + z \cdot \sigma + \cdots + (z \cdot \sigma)^{n-1})(w)$ for a suitable $w \in L$ we have $t \in L^\times$ and $(1 - z \cdot \sigma)(t) = 0$, which implies that $z = t/\sigma(t)$. ∎

The salient feature of this proof for us is the appearance of the element $y = z \cdot \sigma$ in some noncommutative F-algebra, satisfying $y^n = Nz$.

3. Cyclic Algebras.

Let L/K be a cyclic field extension of degree n with Galois group G generated by σ. Let $N = 1 + \sigma + \cdots + \sigma^{n-1} \in K[G]$.

Under the further assumption that n is prime to the characteristic of F and that F contains a primitive n^{th} root of unity, ζ, we will manage to extend the exact sequence of Theorem 2.1 further to the right.

Consider an element $b \in F^\times$. Our goal is to find the obstruction to b being the norm of some element of $L^\times$. To achieve this we construct a certain noncommutative algebra over F in which b acquires an n^{th} root .

We define the cyclic algebra $A_{L,\sigma}(b) = L[y]/(y^n - b)$ with the multiplication twisted so that $y \cdot w = \sigma(w) \cdot y$ for all $w \in L$. If we pick $a \in F$ so that $L = F(a^{1/n})$, then an equivalent presentation for this algebra is given by

$$A_\zeta(a, b) = F[x, y]/(x^n - a, y^n - b, yx - \zeta xy)$$

This algebra has dimension n^2 over F, with a basis given by $\{x^i y^j | 0 \le i, j < n\}$. Conjugation by x and y on these basis elements

$$x(x^i y^j)x^{-1} = \zeta^{-j} x^i y^j$$
$$y(x^i y^j)y^{-1} = \zeta^i x^i y^j$$

yields one-dimensional common eigenspaces, and allows one to see that the algebra is a central simple algebra over F.

We let $M_n F$ denote the $n \times n$ matrix algebra over F.

THEOREM 3.1. *$A_\zeta(a,b)$ is isomorphic to M_nF if and only if $b \in \operatorname{im}(L^\times \xrightarrow{N} F^\times)$.*

PROOF: Suppose $b = Nz$ for some $z \in L$. If we write $y_1 = z^{-1}y$, then we have $y_1^n = N(z^{-1})y^n = b^{-1}y^n = 1$. An alternate presentation of the algebra using x and y_1 shows that $A_\zeta(a,b) \simeq A_\zeta(a,1)$, and we have $A_\zeta(a,1) \simeq A_{L,\sigma}(1) \simeq L_{\mathrm{tw}}[G] \simeq \operatorname{Hom}_F(L,L) \simeq M_nF$.

Suppose, on the other hand, that $A_\zeta(a,b) \simeq M_nF$. Let $V = F^n$; it an M_nF-module, and thus an $A_\zeta(a,b)$-module. Since $L \subseteq A_\zeta(a,b)$, V is also an L-vector space (of dimension 1); choosing a basis for it amounts to identifying V with L, and gives us an F-algebra isomorphism $\phi : A_\zeta(a,b) \simeq \operatorname{Hom}_F(L,L)$. Let $f = \phi(y)$, then we deduce that $b = f^n(1) = f(f^{n-1}(1) \cdot 1) = \sigma(f^{n-1}(1))f(1) = \cdots = N(f(1))$. ∎

Phrasing the result in terms of the Brauer group gives us the following result.

COROLLARY 3.2. *The sequence*

$$0 \longrightarrow F^\times \longrightarrow L^\times \xrightarrow{1-\sigma} L^\times \xrightarrow{N} F^\times \xrightarrow{A_{L,\sigma}} \operatorname{Br} F \longrightarrow \operatorname{Br} L$$

is exact.

PROOF: One must check that the function $A_{L,\sigma}$ is a homomorphism of groups. We refer to [**10**, p.72] for the assertion that a central simple algebra over K split by the cyclic extension L is Brauer-equivalent to a cyclic algebra $A_{L,\sigma}(b)$ for some $b \in F$. ∎

Observe that the image of the map $A_{L,\sigma}$ is n-torsion.

We refer to [**1**] for a modern introduction to the theory of the Brauer group and Brauer-Severi varieties.

4. The norm residue map for K_2.

With $L = F(a^{1/n})$ as in the previous section, let us consider the case $b = 1-a$. Since $1 - a = N(1 - a^{1/n})$ we see that the class $[A_\zeta(a,b)] \in \operatorname{Br} F$ is trivial. Since the Steinberg relation $\{a, 1-a\}$ is the defining relation for K_2, we have a homomorphism

$$K_2F \xrightarrow{\alpha_F} \operatorname{Br} F$$

defined by $\alpha_F(\{a,b\}) = [A_\zeta(a,b)]$. This map is called the "norm residue map".

There is another way to define it (or rather a related map) which does not require the n^{th}roots of 1 to be in F, using Galois cohomology. Let μ_n denote the n^{th}roots of 1 in $\bar{F}$ (the algebraic closure of F). Then it is shown in [**52**] that the composite[1]

$$F^\times \otimes F^\times \to F^\times_{/n} \otimes F^\times_{/n} = H^1(F,\mu_n) \otimes H^1(F,\mu_n) \xrightarrow{\text{cup product}} H^2(F,\mu_n^{\otimes 2})$$

factors through K_2F. Moreover, if $\mu_n \subseteq F$ then there is a natural isomorphism $H^2(F,\mu_n^{\otimes 2}) \simeq {}_n\operatorname{Br} F \otimes \mu_n$. A choice of a primitive n^{th} root of 1 is all that is required to identify this definition with our previous one.

[1] We introduce the notation ${}_nA$ to denote the n-torsion in an abelian group A, and $A_{/n}$ to denote the quotient A/nA.

5. Hilbert's Theorem 90 for K_2 in a special case.

We follow the approach of Merkuriev and Suslin, as modified by Merkuriev in **[26]**. Our ultimate goal is Theorem 9.1 below.

We let $L = F(a^{1/n})$ be a cyclic extension of degree n, with $\mu_n \subseteq F$, as before. We also assume:

(1) The norm map $N : L^{\times} \xrightarrow{N} F^{\times}$ is surjective;
(2) All polynomials over F of degrees 2, 3, ..., and $n-1$ factor.

It follows from our assumptions that

$$\textit{the symbol map } L^{\times} \otimes F^{\times} \longrightarrow K_2L \textit{ is surjective} \tag{5.1}$$

To see this, we let $\alpha = a^{1/n}$. Property (2) tells us we only have to show that symbols of the form $\{r + s\alpha, t + u\alpha\}$ are in the image, but then there is a simple trick based on the Steinberg relation which allows us to write such symbols in terms of symbols of the form $\{r, t + u\alpha\}$.

THEOREM 5.2. *Under the hypotheses above, the sequence*

$$K_2L \xrightarrow{1-\sigma} K_2L \xrightarrow{N} K_2F$$

of abelian groups is exact.

Here N denotes the norm map of algebraic K-theory.

PROOF: We consider the following commutative diagram.

$$\begin{array}{ccccccc} L^{\times} \otimes F^{\times} & \xrightarrow{(1-\sigma)\otimes 1} & L^{\times} \otimes F^{\times} & \xrightarrow{N\otimes 1} & F^{\times} \otimes F^{\times} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ K_2L & \xrightarrow{1-\sigma} & K_2L & \xrightarrow{N} & K_2F & & \end{array}$$

The exactness of the top row follows from Hilbert's Theorem 90 for K_1 together and property (1), and we proved surjectivity of the middle vertical arrow above. Letting $C = \mathrm{ckr}(K_2L \xrightarrow{1-\sigma} K_2L)$, we see that it suffices to show that the symbol induces a map $K_2F \to C$, and by property (1) this is equivalent to showing that

$$\textit{for all } x \in L^{\times} \textit{ we have } \{x, 1 - Nx\} \in \mathrm{im}(1-\sigma). \tag{5.3}$$

This latter fact can be easily shown in the special case where $T^n - Nx$ splits in F as follows. Write $T^n - Nx = \prod_i (T - y_i)$. Now $Nx = y_i^n = Ny_i$ so by Hilbert's Theorem 90 we may write $x = y_i z_i^{1-\sigma}$ for some $z_i \in L$. Then $\{x, 1 - Nx\} = \{x, \prod_i (1 - y_i)\} = \prod_i \{x, 1 - y_i\} = \prod_i \{y_i z_i^{1-\sigma}, 1 - y_i\} = \prod \{z_i, 1 - y_i\}^{1-\sigma}$. In the general case, one considers the irreducible factors of $T^n - Nx$, the corresponding fields, and the various norm maps involved. ∎

6. The Severi-Brauer variety.

To proceed further with the proof of Merkuriev-Suslin, we introduce the Severi-Brauer variety. As usual, L is a cyclic extension of F, and we are wondering whether some element $b \in F$ is a norm from L. It is natural to consider the equation $Nz = b$ where z is unknown, and to write down its "generic" solution. We can write this equation as a system of n polynomial equations in n unknowns with coefficients in F by choosing a basis $L = Fe_1 \oplus \cdots \oplus Fe_n$ for L, introducing variables $t_1, \ldots, t_n$, and considering the n components of the equation $N(t_1 e_1 + \cdots + t_n e_n) = b$. The solution set of these equations is an algebraic variety $W = W(L/F, b)$, can be defined naturally without choosing a basis for L, and is compatible with base change.

A fundamental result of Galois theory says that there is an algebra isomorphism $L \otimes_F L \simeq \prod_G L$ which respects the G-action; it sends $x \otimes y$ to $(y, \sigma(x) \cdot y, \ldots, \sigma^{n-1}(x) \cdot y)$. Using it and the fact that the norm map $\prod_G L \xrightarrow{N} L$ sends $(u_1, \ldots, u_n)$ to the product $u_1 \cdot \ldots \cdot u_n$, we find that

$$\begin{aligned} W \otimes_F L &= W(L \otimes_F L/L, b) \\ &= W(\prod_G L/L, b) \\ &= \text{the zero set of } \; u_1 u_2 \cdots u_n = b \end{aligned}$$

If we define

$$\begin{aligned} x_1 &= u_1 \\ x_2 &= u_1 u_2 \\ &\vdots \\ x_{n-1} &= u_1 \cdots u_{n-1} \end{aligned}$$

then we see that the cyclic permutation $u_1 \mapsto u_2 \mapsto \cdots \mapsto u_n \mapsto u_1$ acts on the X variables by the following linear fractional transformations.

$$\begin{aligned} x_1 &\mapsto x_2/x_1 \\ x_2 &\mapsto x_3/x_1 \\ &\vdots \\ x_{n-2} &\mapsto x_{n-1}/x_1 \\ x_{n-1} &\mapsto b/x_1 \end{aligned}$$

Since the denominators are all the same, we can express this as an automorphism Φ of projective space $\mathbb{P}^{n-1}$, namely,

$$\Phi : (x_1 : \cdots : x_{n-1} : 1) \mapsto (x_2 : \cdots : x_{n-1} : b : x_1).$$

If we use Φ as descent data, we obtain a nonsingular variety $X_{L,\sigma} = X$ defined over F which is birationally equivalent to W, and such that $X \otimes_F L \simeq \mathbb{P}_L^{n-1}$. A variety such as X, which becomes isomorphic to projective space after some field extension, is called a "Severi-Brauer variety" — it is fortunate that we have come up with one because Quillen has explicitly computed its K-groups, as we will see later.

Consider the fraction field $F(X)$ of X. Since $X \otimes_F \bar{F} \simeq \mathbb{P}^{n-1}$ is an irreducible variety, we see that the ring $L(X) = L \otimes_F F(X)$ is a field, and is thus a cyclic field extension of $F(X)$ of degree n. Moreover, $b \in \mathrm{im}(L(X)^\times \xrightarrow{N} F(X)^\times)$, because the images of the variables $t_1, \ldots, t_n$ in $F(X)$ solve the equations defining W, which expressed the fact that b should be a norm for this extension. As a consequence, we see that $F(X)$ is a splitting field for the central simple algebra $A_{L,\sigma}(b)$, i.e.

$$[A_{L,\sigma}] \in \ker\left(\mathrm{Br}(F) \longrightarrow \mathrm{Br}(F(X))\right)$$

7. Useful explicit computations of K-groups.

We cite briefly some explicit computations of algebraic K-groups. These results are crucial for the sequel, and are sparkling gems in their own right.

THEOREM 7.1. (Quillen [36]) $K_*\mathbb{P}_F^{n-1} \simeq \underbrace{K_*F \oplus \cdots \oplus K_*F}_{n}$

THEOREM 7.2. (Quillen [36]) *Let X be a Severi-Brauer variety and let A be the corresponding central simple algebra. Then*

$$K_*(X) \simeq K_*(F) \oplus K_*(A) \oplus K_*(A \otimes_F A) \oplus \cdots \oplus K_*(A^{\otimes n-1})$$

THEOREM 7.3. (Swan [51]) *Let q be a quadratic form over F in n variables, let W be the zero set of q (it is a subvariety in $\mathbb{P}^n$ of dimension $n-1$), and let $C_0(q)$ be the even part of the Clifford algebra of q. Then*

$$K_*(W) \simeq \underbrace{K_*(F) \oplus \cdots \oplus K_*(F)}_{n-1} \oplus K_*(C_0(q))$$

8. Hilbert's Theorem 90 for K_2 in general.

We let L be a cyclic extension of F of degree n with a chosen generator σ for its Galois group, and assume that n is prime to the characteristic of F.

THEOREM 8.1. [23] *The sequence*

$$K_2L \xrightarrow{1-\sigma} K_2L \xrightarrow{N} K_2F$$

of abelian groups is exact.

This theorem was proved by Bass-Tate in the case where F is a purely transcendental extension of a local or global field.

PROOF: [26] One reduces readily to the case where n is a prime p, and $\mu_p \subseteq F$. For any field extension F' of F we let L' be the ring $L \otimes_F F'$. We let $V(F')$ denote the homology of the sequence

$$K_2L' \xrightarrow{1-\sigma} K_2L' \xrightarrow{N} K_2F'$$

The simple nature of L' in the case $F' = L$ shows that $V(L) = 0$, and then a standard argument involving transfer maps (go up and down) shows that $p \cdot V(F) = 0$. The transfer map argument also shows that when F' is a finite extension of F of degree prime to p, the map $V(F) \hookrightarrow V(F')$ is injective. The central point is then the following assertion: if X is a Severi-Brauer variety over F, then the map $V(F) \hookrightarrow V(F(X))$ is injective. The proof of this assertion involves Quillen's computation of K_*X (Theorem 7.2), the truth of Gersten's conjecture about the K-groups of local rings on a nonsingular algebraic variety **[36]**, and a result of Sherman which states that $H^0(\mathbb{P}^n_F, \mathcal{K}_2) = K_2F$ — the argument is too involved to present here, but forms the core of the proof, and justifies the introduction of the Severi-Brauer variety.

Now we take F' to be the very large field extension of F which is obtained from F by repeatedly

(a) adjoining all roots of polynomials of degree smaller than p, and
(b) taking the compositum of the function fields of all the Severi-Brauer varieties.

This field has the properties (1) and (2) stated above Theorem 5.2, and thus $V(F') = 0$. On the other hand, injectivity of the map $V(F) \hookrightarrow V(F')$ is assured by the results of the previous paragraph. We conclude that $V(F) = 0$, and thus the sequence of the theorem is exact. ∎

9. The main theorem of Merkuriev-Suslin.

THEOREM 9.1. **[23]** *Let F be a field whose characteristic is prime to n. Then the norm residue map gives an isomorphism*

$$\alpha_F : K_2F_{/n} \xrightarrow{\simeq} H^2(F, \mu_n^{\otimes 2})$$

PROOF: One reduces easily to the case where n is a prime ℓ, and $\mu_\ell \subseteq F$. Rephrasing now in terms of the Brauer group, we must show that the map

$$\alpha_F : K_2F_{/\ell} \xrightarrow{\simeq} {}_\ell \operatorname{Br} F$$

is an isomorphism.

We begin the proof of injectivity. Take an element x of the kernel and write it as a product of symbols.

$$x = \prod_{i=1}^{m} \{a_i, b_i\} \in \ker \alpha_F$$

We show by induction on m, the number of symbols, that $x = 0$.

In the case where $m = 1$ we have $0 = \alpha_F\{a, b\} = [L_\zeta(a, b)]$ and so there is an element $\beta \in F(a^{1/\ell})$ with $b = N\beta$. Letting $\alpha = a^{1/\ell}$ we have $\{a, b\} = \{a, N\beta\} = N\{a, \beta\} = N\{\alpha^\ell, \beta\} = (N\{\alpha, \beta\})^\ell \equiv 0 \bmod \ell$.

In the case where $m > 1$ we let $L = F(a_m^{1/\ell})$. The symbol $\{a_m, b_m\}$ goes to zero in $K_2L_{/\ell}$, so the inductive hypothesis applies to the image of x in $K_2L_{/\ell}$, showing that $x \in \ker(K_2F_{/\ell} \to K_2L_{/\ell})$. Referring to lemma 9.2 below, we see that we can express x as a single symbol $\{c, d\}$, reducing us to the case where $m = 1$.

LEMMA 9.2. *Suppose ℓ is a prime, F is a field of characteristic prime to ℓ, $\mu_\ell \subseteq F$, $a \in F$, $L = F(a^{1/\ell})$, and $x \in \ker\left(K_2F_{/\ell} \to K_2L_{/\ell}\right)$. Then there exists a $b \in F^\times$ such that $x = \{a, b\}$.*

PROOF: By an involved specialization argument, one reduces to the case where F is a field with a chain $F = F_N \supset \cdots \supset F_1$ of cyclic extensions of degree ℓ, and where F_1 is a purely transcendental extension of a local or global field F_0. Tate proved [52] that α_{F_0} is an isomorphism, and from Bloch's theorem in [3] it follows that α_{F_1} is an isomorphism. Now one passes up the tower of cyclic extensions, proving at each stage that α_{F_i} is injective; the argument for this makes heavy use of Hilbert's Theorem 90 for K_2, Theorem 5.2. Having deduced that α_F is injective, one may deduce the statement of this lemma by a diagram chase in the following diagram. The right hand column is exact by 3.2, and one proves the left hand column is exact, too.

$$\begin{array}{ccc}
K_2L_{/\ell} & \xrightarrow{\alpha_L} & {}_\ell\operatorname{Br} L \\
\uparrow & & \uparrow \\
K_2F_{/\ell} & \xrightarrow[\alpha_F]{\text{injective}} & {}_\ell\operatorname{Br} F \\
{\scriptstyle a\cup -}\uparrow & & \uparrow{\scriptstyle A_{L,a}} \\
F^\times_{/\ell} & \xrightarrow{=} & F^\times_{/\ell}
\end{array}$$

We continue the proof of Theorem 9.1, and take up the proof of surjectivity of

$$K_2F_{/\ell} \xrightarrow{\alpha_F} {}_\ell\operatorname{Br} F$$

So suppose we are given an element $u \in {}_\ell\operatorname{Br} F$. By the standard transfer argument used before, we may enlarge F sufficiently to ensure that $\operatorname{Gal}(\bar{F}/F)$ is a pro-ℓ-group. Now an induction argument based on the fact that ℓ-groups are nilpotent allows us to assume that we have some cyclic extension $L = F(a^{1/\ell})$ of F such that the image u_L of u in ${}_\ell\operatorname{Br} L$ lies in the image of α_L. Then, after a lot of diagram chasing which we omit here, one reduces to the case where $u_L = 0$, and conclude, as above, that for some $b \in F$ one has $u = \alpha(\{a, b\})$. ∎

As expected, this theorem says something concrete about the structure of central simple algebras. For example, we have the following interpretations of the surjectivity of the map in the theorem.

COROLLARY 9.3. [22] *Let F be any field of characteristic not 2, and let A be a central simple algebra over F such that there is an F-algebra isomorphism $F \simeq F^{\mathrm{op}}$. Then for some positive integer k the matrix ring $M_k(A)$ is isomorphic to a matrix algebra of a tensor product of quaternion algebras over F.*

COROLLARY 9.4. *Let n be a positive integer, prime to the characteristic of a field F containing the n^{th}roots of 1. If A is a central simple algebra over F such that $A^{\otimes n}$ is a matrix algebra over F, then for some positive integer k, the* *matrix ring $M_k(A)$ is isomorphic to a matrix algebra of a tensor product of cyclic F-algebras of degree n.*

Here we list some other applications of the theorem.

THEOREM 9.5. **[44]**, **[46]** *If X is a complete rational variety over F then*

$$H^0(X, \mathcal{K}_2) = K_2F.$$

THEOREM 9.6. [**7**] *Let X be a projective nonsingular variety over a finite field. Then the torsion subgroup of the Chow group $CH^2(X)$ is finite. Moreover, if ℓ is a prime not equal to the characteristic, then the ℓ-primary torsion of $CH^2(X)$ injects into the étale cohomology group $H^4(X, \mathbb{Z}_\ell(2))$.*

THEOREM 9.7. **[8]** *Let X be a complete nonsingular variety over a field F of characteristic zero. Assume also that X is a complete intersection in projective space, has a rational point, and has dimension larger than 2. Then the map $CH^2(X) \longrightarrow CH^2(X \otimes_F \bar{F})$ is injective.*

THEOREM 9.8. [Bloch, Gabber, Kato] *Let F be a field of characteristic $p > 0$, and let X be a nonsingular variety over F. Then there is an isomorphism of sheaves*

$$\mathcal{K}_2/p^n \xrightarrow{\simeq} \nu_n(2)$$

where $\nu_n(r) = W_n\Omega^r_{X,\log}$ is part of the logarithmic deRham-Witt complex.

The question concerning the situation when the field F does not contain the roots of 1 required for Theorem 9.1 has been answered for prime exponents by Merkuriev in the following theorem.

THEOREM 9.9. **[25]** *Let F be a field and let p be a prime number. Then ${}_p\mathrm{Br}\, F$ is generated by algebras of index p.*

The index of a central simple algebra is defined to be the square root of its dimension.

10. The Milnor K-groups of local and global fields.

We mention some results here for later refernce.

Let F be a nonarchimedean local field, and let μ_F denote its group of roots of unity. The Hilbert symbol gives a map $K_2F \to \mu_F$. Calvin Moore has shown in **[33]** that this map is split surjective, and its kernel, call it H, is a divisible group. Tate showed in **[53]** that if char $F \neq 0$ then H is torsion free, and in **[24]** Merkuriev removed the restriction on the characteristic. For the higher Milnor K-groups, Milnor showed in **[32]** that $K^M_n F$ is divisible ($n \geq 3$). Recently, Sivitskii **[41]** has generalized Merkuriev's result, and shown that $K^M_n F$ is torsion free for $n \geq 3$.

One sees easily that $K^M_n\mathbb{C}$ is a divisible group for $n \geq 1$ and is uniquely divisible for $n \geq 2$. From **[32]** we know that $K^M_n\mathbb{R} \simeq \mathbb{Z}/2 \oplus H$, where H is a uniquely divisible group. This latter result arises easily from the map $K^M_n\mathbb{R} \to \mathbb{Z}/2$ which sends $\{a_1, \dots, a_n\}$ to 0 if and only if one of the a_i is positive. The summand $\mathbb{Z}/2$ is generated by the nontrivial symbol $\{-1, \dots, -1\} \in K^M_n\mathbb{R}$.

We have the following theorem of Bass and Tate.

THEOREM 10.1. [2] *Let F be a global field with r_1 real places. Then $K_n^M F \simeq (\mathbb{Z}/2)^{r_1}$ for $n \geq 3$.*

11. The torsion in the Milnor K-groups.

Merkuriev has proved the following theorem.

THEOREM 11.1. [28] *Let F be a field, and let $\{-1\}$ denote the class of -1 in K_1F. If $K_n^M F$ is 2-divisible, then ${}_2K_n^M F = \{-1\} \cdot K_{n-1}^M F$ and for all $m > n$ we have ${}_2K_m^M F = 0$.*

Under a strong hypothesis on F and the assumption that Hilbert's Theorem 90 holds for K_n^M, Bogomolov [5] has proved the analogous assertion for p-torsion.

Now let F be any field of characteristic p. In [53] it was conjectured by Tate that $K_n^M F$ has no p-torsion — this result has been proved by Izhboldin in [14].

12. Milnor's K_3^M.

The following theorem has been proved independently by Rost and by Merkuriev-Suslin.

THEOREM 12.1. [37] [27] *Let F be a field, not of characteristic 2. Then there is an isomorphism*

$$K_3^M F_{/2} \xrightarrow{\simeq} H^3(F, \mu_2^{\otimes 3})$$

The proof of this theorem is analogous to the proof of the theorem of Merkuriev-Suslin (9.1). Recall that there the central issue was always the question of surjectivity of the norm map for a cyclic field extension (on K_1 groups of the fields, i.e. the units). Here, the central issue is the question of surjectivity of the norm map for quadratic extensions on K_2.

Suppose $F(\sqrt{a})$ is a quadratic extension of F, and suppose $\{b, c\} \in K_2F$. Then one shows that $\{b, c\}$ is a norm for this extension if and only if the equation

$$X^2 - aY^2 - bZ^2 + abU^2 - cV^2 = 0$$

has a nontrivial solution. The generic solution of this equation lies in the function field of the variety defined by this equation, which is a nonsingular quadric variety of dimension 3. Then Swan's computation of the K-groups of quadrics (Theorem 7.3) is used to achieve the result, in roughly the same way that Quillen's computation of the K-groups of Severi-Brauer varieties was used above.

13. K-groups of norm varieties.

We have now seen two examples where it has been useful to compute the K-groups explicitly for certain norm varieties, namely the Brauer-Severi varieties, and the quadric hypersurfaces. Marc Levine suggests that more computations of this type are needed to make further progress for higher K-groups. To be concrete, take a field F and let $X = S\ell_3(F)$ be the variety defined by the equation

$$\det\begin{pmatrix} x & y & z \\ r & s & t \\ u & v & w \end{pmatrix} = 1$$

Try to find a nonsingular model V for the field $F(X)$ (as a projective variety over F), for which it is possible to get an explicit computation of its K-groups. Having done that, do the same for all the twisted forms of V. This should be helpful in getting field elements to be norms of elements under degree 3 extensions, which would in turn help study the 3-part of various K-groups.

Another application of these ideas is Suslin's proof of Hilbert's Theorem 90 for $B(F)$, which is described in Theorem 17.8.

14. The obstruction to stability for the homology of $G\ell_n$.

THEOREM 14.1. **[45]** *Let F be an infinite field. Then we have isomorphisms $H_n(G\ell_n F, \mathbb{Z}) \simeq H_n(G\ell_{n+1} F, \mathbb{Z}) \simeq \cdots \simeq H_n(G\ell F, \mathbb{Z})$ as well as an exact sequence*

$$H_n(G\ell_{n-1} F, \mathbb{Z}) \longrightarrow H_n(G\ell_n F, \mathbb{Z}) \longrightarrow K_n^M F \longrightarrow 0$$

Suslin considers the homology product map

$$H_1(G\ell_1 F, \mathbb{Z}) \otimes \cdots \otimes H_1(G\ell_1 F, \mathbb{Z}) \to H_n(G\ell_n F, \mathbb{Z})$$

and shows that the composite map

$$F^\times \otimes \cdots \otimes F^\times \to \frac{H_n(G\ell_n F, \mathbb{Z})}{\text{image of } H_n(G\ell_{n-1} F, \mathbb{Z})}$$

factors through $K_n^M F$. He then shows the resulting map is an isomorphism.

15. The theory of Chern classes.

Let X be a variety and let $CH^i(X)$ denote the group of algebraic cycles of codimension i modulo rational equivalence. Grothendieck has constructed a useful theory of Chern classes, which appear as (nonadditive) functions

$$K_0(X) \xrightarrow{c_i} H^{2i}_{\text{ét}}(X, \mu_\ell^{\otimes i})$$

to étale cohomology (when $1/\ell$ exists in the base field F), or as functions

$$K_0(X) \xrightarrow{c_i} CH^i(X).$$

Soulé **[42]** has produced generalized Chern class maps for étale cohomology which appear as additive maps

$$K_n(X) \xrightarrow{c_i} H^{2i-n}_{\text{ét}}(X, \mu_\ell^{\otimes i})$$

for $n > 0$ (in the case where X is affine).

We introduce Bloch's formula $CH^i(X) \simeq H^i(X, \mathcal{K}_i X)$ **[36]**, which expresses the algebraic cycle groups in terms of sheaf cohomology for the Zariski topology of the sheafified K-groups. Gillet **[12**, Remark, p.234**]** and Shekhtman **[40]** have produced generalized Chern class maps which appear as additive maps

$$K_n(X) \xrightarrow{c_i} H^{i-n}(X, \mathcal{K}_i X)$$

for $n > 0$. In the case of interest to us, we may take $X = \operatorname{Spec} F$ to be the spectrum of a field, and then the relevant Zariski cohomology group vanishes unless $i = n$. These maps

$$K_p F \xrightarrow{c_n} K_p F$$

will be used in the next section.

16. The relation between K_n^M and K_n.

Recall that there is a natural map $\phi : K_n^M F \to K_n F$ which is defined using the product in K-theory (section 1). We have seen in Remark 1.6 that this map is far from surjective. Suslin **[43]** **[45]** has proved a good result concerning the extent to which injectivity may be true. Using Theorem 14.1 he constructs a map $\psi : K_n F \to K_n^M F$ which goes the other way, and is defined to be the following composite map.

$$K_nF = \pi_n BG\ell(F)^+ \xrightarrow{\text{Hurewicz}} H_n(G\ell F, \mathbb{Z})$$
$$\xleftarrow{\simeq} H_n(G\ell_n F, \mathbb{Z}) \to \frac{H_n(G\ell_n F, \mathbb{Z})}{\text{image of } H_n(G\ell_{n-1}F, \mathbb{Z})} \xrightarrow{\simeq} K_n^M F$$

He then proves the following theorem.

THEOREM 16.1. **[45]**

(1) $\phi \circ \psi = c_n$;
(2) $\psi \circ \phi =$ multiplication by $(-1)^{n-1}(n-1)!$;

COROLLARY 16.2. *The kernel of the map $\phi : K_n^M F \to K_n F$ is annihilated by the number $(n-1)!$.*

EXAMPLE 16.3. Weibel has pointed out that the kernel of ϕ is nontrivial for $n = 4$, as it kills the element

$$\{-1, -1, -1, -1\} = \{-1\}^4 \in K_4^M(\mathbb{Q})$$

of order two mentioned in section 10. To see this one observes that the map

$$\pi_n^s \to K_n(\mathbb{Q})$$

from stable homotopy is a ring homomorphism with the element $\{-1\} \in K_1(\mathbb{Q})$ in its image **[54]** and that $\pi_4^s = 0$ **[39]**.

17. Bloch's group and the indecomposable part of K_3F.

Let F be a field and recall that the Steinberg symbol gives a surjective map $F^\times \otimes F^\times \twoheadrightarrow K_2F$. Let us imagine for a moment that there is a space X such that

$$\pi_0 X = \mathbb{Z}$$
$$\pi_1 X = F^\times$$
$$\pi_2 X = F^\times \otimes F^\times$$

and imagine that there is a map $X \longrightarrow K(F)$ which does the expected thing on these lower homotopy groups. Then if we let Y be the homotopy fiber of the map, we deduce from the long exact sequence for homotopy groups that $\pi_0 Y = \pi_1 Y = 0$ and there is an exact sequence

$$K_3F \longrightarrow \pi_2 Y \longrightarrow F^\times \otimes F^\times \longrightarrow K_2F \longrightarrow 0$$

Since $\pi_2 Y = H_2(Y)$ is a homology group, one might be able to present it directly, without appealing to topology. Work of Bloch, Dupont-Sah and Suslin has accomplished something like this.

The definition of Bloch's group, as modified by Suslin, goes as follows. We let $F^\times \tilde{\otimes} F^\times = F^\times \otimes F^\times / \langle a \otimes b + b \otimes a \rangle$ be the symmetrized tensor product. We define

$$T(F) = \coprod_{F^\times - \{1\}} \mathbb{Z} \Big/ \Big\langle [x] - [y] + [\frac{y}{x}] - [\frac{1-x^{-1}}{1-y^{-1}}] + [\frac{1-x}{1-y}] \Big\rangle$$

It turns out that there is a map $T(F) \longrightarrow F^\times \tilde{\otimes} F^\times$ which sends a generator $[x]$ to $x \otimes (1-x)$. One defines Bloch's group $B(F)$ to be the kernel of this map. Notice that replacing $F^\times \tilde{\otimes} F^\times$ by the exterior power $\wedge^2 F^\times$ changes only the 2-torsion of the group.

An alternate definition for $T(F)$ is the following. One constructs the abelian group with generators 4-tuples (x_0, x_1, x_2, x_3) of distinct point in $\mathbb{P}^1_F$, and defining relations

$$(gx_0, gx_1, gx_2, gx_3) = (x_0, x_1, x_2, x_3) \qquad \text{for } g \in G\ell_2 F$$

$$\sum_{i=0}^{4} (-1)^i (x_0, \ldots, \hat{x}_i, \ldots, x_4) = 0 \qquad \text{for distinct } x_0, \ldots, x_4 \in \mathbb{P}^1_F$$

This is the group called $\mathcal{P}_F$ by Dupont-Sah [**9**]. The equivalence of the two definitions can be seen by considering the cross-ratio of the four points (x_0, x_1, x_2, x_3) as an element of $F^\times - \{1\}$, normalized so that the cross ratio of $(0, \infty, 1, x)$ is x (see [**9**, 4.9]).

THEOREM 17.1. (Bloch-Wigner, Dupont-Sah, [**9**, Theorem, Appendix A.]) *If F is an algebraically closed field of characteristic* 0, *then there is an exact sequence*

$$0 \to \mu_F^{\otimes 2} \to H_3(S\ell_2 F) \to B(F) \to 0$$

THEOREM 17.2. (Dupont-Sah, [**9**, Theorem 5.1]) *If F is algebraically closed, then $T(F)$ is a divisible group.*

Weibel announced in 1983 (but subsequently did not publish) a proof that if F is an algebraically closed field, then there is an exact sequence

$$0 \to K_3^M(F) \oplus \mu_F^{\otimes 2} \to K_3 F \to B(F) \to 0.$$

The proof was based on ideas of Bloch in [**4**].

THEOREM 17.3. (Sah, [**38**, Theorem 4.4]) *If F is an algebraically closed field of characteristic 0, then $T(F)$ is torsion free.*

It is easy to prove that $F^\times \tilde{\otimes} F^\times$ is uniquely divisible when F is algebraically closed, (as are $K_2 F$ and $\wedge^2 F^\times$), so it follows from 17.2 and 17.3 that $B(F)$ is uniquely divisible when F is algebraically closed of characteristic 0.

The following theorem fits in very nicely with Suslin's stability theorem 14.1.

THEOREM 17.4. (Sah, [**38**, Theorem 3.6]) *Let F be an infinite field with $F^\times = (F^\times)^6$, or let $F = \mathbb{R}$. Then the map $H_3 S\ell_2 F \to H_3 S\ell_3 F$ is injective.*

Appendices A and B of [**38**] contain the following illuminating remarks concerning this theorem. Firstly, if $K_2 F$ is 2-divisible (e.g. one may suppose that every element of F has square root in F) then the Hurewicz map provides an isomorphism

$$K_3 F \xrightarrow{\cong} H_3(S\ell(F)).$$

Moreover, the Suslin stability theorem 14.1 can be generalized slightly to provide an isomorphism

$$H_3(S\ell_3(F)) \xrightarrow{\cong} H_3(S\ell(F))$$

The most recent results concerning $B(F)$ are the following results of Suslin.

THEOREM 17.5. (Suslin, [**50**]) *Suppose F is an infinite field containing a primitive cube root of 1. Then $B(F)$ is a quotient of $K_3 F$.*

PROOF: A monomial matrix is one with exactly one nonzero entry in each row and column. We let $GM(F) \subseteq G\ell(F)$ be the group of monomial matrices, and we let $X = BGM(F)^+$ be the result of applying Quillen's plus-construction to its classifying space. By the Barrat-Priddy-Quillen theorem, one knows that $\pi_i X = \pi_i^s BF^\times$. One relates the cokernel of the map $\pi_3 X \to K_3 F$ to the cokernel of the map $H_3 GM_2(F) \to H_3 G\ell_2(F)$. The latter group is computed by considering the action of $G\ell_2 F$ on the simplicial complex whose k-simplices are the $k+1$-tuples of distinct points in $\mathbb{P}^1_F$, and turns out to be $B(F)$. ∎

THEOREM 17.6. (Suslin, [**29**]) *Let F be a field containing an algebraically closed field. Then there is an exact sequence*

$$0 \longrightarrow \operatorname{Tor}(\mu, \mu) \longrightarrow K_3^{\text{ind}} F \longrightarrow B(F) \longrightarrow 0$$

The following theorem removes the restriction on the characteristic from 17.3.

THEOREM 17.7. (Suslin [**50**]) *If F is algebraically closed, then $B(F)$ is a uniquely divisible group.*

Finally, we have the following theorem, whose proof involves global computations of norm varieties, which in this case turn out to be conics, i.e. Severi-Brauer varieties of dimension 1.

THEOREM 17.8. (Suslin, [**50**]) (Hilbert's Theorem 90 for $B(F)$, the case of a quadratic extension) *Let F be a field containing an algebraically closed field of characteristic not 2. Let L be a quadratic extension of F, and let σ be the generator of the Galois group. Then the sequence*

$$B(L) \xrightarrow{1-\sigma} B(L) \xrightarrow{N} B(F)$$

of abelian groups is exact. Here the map N is a certain norm map which can be defined in this case.

COROLLARY 17.9. *$B(F)$ is uniquely 2-divisible, and the map $K_3^M F \to K_3 F_{/2}$ is surjective.*

18. The indecomposable part of K_3.

Recently the structure of the group $K_3^{\text{ind}} F$ has been settled by Levine **[21]** and Merkuriev-Suslin **[30]** independently and simultaneously.

We describe first the results of Levine.

THEOREM 18.1. **[21]** *Let ℓ be a prime number, F be a field of characteristic not ℓ with $\mu_\ell \subseteq F$. Then ${}_\ell K_3^{\text{ind}} F$ is a cyclic group of order ℓ.*

THEOREM 18.2. **[21]** *Let ℓ be a prime number, F be a field of characteristic not ℓ. Then there is an isomorphism*

$$ {}_{\ell^\nu} K_3^{\text{ind}} F \xrightarrow{\simeq} H^0(F, \mu_{\ell^\nu}^{\otimes 2}) $$

THEOREM 18.3. **[21]** *Let ℓ be a prime number, F be a number field. Then there is an isomorphism*

$$ K_3^{\text{ind}} F \otimes \mathbb{Z}_\ell \xrightarrow[c_2]{\simeq} H^1(F, \mathbb{Z}_\ell(2)) $$

THEOREM 18.4. **[21]** *Let L be a Galois extension of F with Galois group G of order prime to the characteristic of F. Then K_3^{ind} satisfies Galois descent, i.e.*

$$ K_3^{\text{ind}} F = (K_3^{\text{ind}} L)^G $$

THEOREM 18.5. **[21]** *Let ℓ be a prime number, F be a field of characteristic not ℓ containing an algebraically closed field. Then Bloch's group $B(F)$ is ℓ-divisible.*

To prove these theorems, Levine begins with an idea of Bloch's for studying K_3. We let $Z = \{0, 1\} \subseteq \mathbb{P}^1$. Examining the relative K-theory sequence

$$ \begin{array}{ccccccc} K_3\mathbb{P}^1 & \longrightarrow & K_3 Z & \longrightarrow & K_2(\mathbb{P}^1, Z) & \longrightarrow & K_2\mathbb{P}^1 \\ \| & & \| & & & & \| \\ (K_3F)^2 & & (K_3F)^2 & & & & (K_2F)^2 \end{array} $$

(whose existence is virtually the definition of the relative K-group $K_2(\mathbb{P}^1, Z)$) and using the fact that the inclusions of the two points induce the same restriction map on K-groups convinces us that a copy of K_3F is a subgroup of $K_2(\mathbb{P}^1, Z)$. Let R be the semilocal ring of Z on $\mathbb{P}^1$, and let J be its Jacobson radical, rad(R). It turns out that the localization from $\mathbb{P}^1$ to R kills just the decomposable elements of that copy of K_3F, i.e. we have the following map of exact sequences.

$$ \begin{array}{ccccccc} 0 & \longrightarrow & K_3F & \longrightarrow & K_2(\mathbb{P}^1, Z) & \longrightarrow & K_2\mathbb{P}^1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & K_3^{\text{ind}}F & \longrightarrow & K_2(R, J) & \longrightarrow & K_2R \end{array} $$

Relative K_2-groups for ideals in the Jacobson radical were computed by Keune and Loday **[19]** in terms of certain generators and relations reminiscent of the Steinberg relations for K_2 of fields, and gives a handle on $K_2(R, J)$.

The next theorem is the main technical instrument used in the proof of the other theorems.

THEOREM 18.6. (Hilbert's Theorem 90 for relative K_2) **[21]** *Let ℓ be a prime number, F be a field of characteristic not ℓ with $\mu_\ell \subseteq F$. Let R be a semilocal F-algebra which is a principal ideal domain, and let $J = \mathrm{rad}(R)$. Given $a \in R^\times$ let $R' = R[a^{1/\ell}]$, let $J' = \mathrm{rad}(R')$, and let σ be a generator for the Galois group of R' over R. Then the sequence*

$$K_2(R', J') \xrightarrow{1-\sigma} K_2(R', J') \xrightarrow{N} K_2(R, J)$$

is exact.

The proof of this theorem follows the general outline of the proof of 5.2. The main difficulty in the proof is showing that $\{x, 1 - Nx\} \in \mathrm{im}(1 - \sigma)$ for all $x \in 1 + J'$, $x \neq 1$. Levine's solution is highly technical and involves Gersten's conjecture and the analysis of certain multiple blow-ups.

Theorem 18.6 together with diagram chasing analogous to that in the Merkuriev-Suslin proof yields the statement that

$${}_\ell K_2(R, J) = \{\{f, \zeta\} \mid f \in 1 + J \text{ and } \zeta^\ell = 1\}$$

from which Theorem 18.1 follows.

Theorem 18.2 follows from 18.1. Theorem 18.3 is deduced by considering the Chern class map of Soulé (Section 15)

$$c_2 : K_3^{\mathrm{ind}} F \otimes \mathbb{Z}_\ell \to H^1_{\text{ét}}(F, \mathbb{Z}_\ell(2)).$$

Soulé **[42]** has shown this map is surjective, and 18.1 and 18.2 together with the explicit computations 10.1 and 1.5 imply the map is an isomorphism.

The approach of Merkuriev and Suslin to K_3^{ind} in **[30]** is roughly the same as that of Levine described above. They isolate K_3^{ind} using the relative K_2, and state and prove the same Hilbert Theorem 90 for relative K_2. This then leads to the theorems in more or less the same way, but perhaps more efficiently. We list a few of the differences, as conveyed to me by Levine.

1. Both papers prove the relative analogue of the statement that when X is a Severi-Brauer variety over a field F, then the map

$$K_2(F) \to K_2(F(X))$$

is injective. Merkuriev and Suslin do this by reducing to a study of the Galois structure for cyclic extensions, whereas for Levine it somehow falls out at the end.

2. Merkuriev-Suslin use Karoubi-Villamayor K-groups to define norm maps, whereas Levine uses homotopy fibers, and must work harder.