Contemporary Mathematics
Volume 55, Part I, 1986

K_2 AND L-FUNCTIONS OF ELLIPTIC CURVES
COMPUTER CALCULATIONS

S. Bloch and D. Grayson

What follows are computer calculations done in the fall of 1981 to compare the value of the regulator on K_2 of an elliptic curve with the value of the L-function at $s = 2$. With one important modification, the results confirm everything that Beilinson and Bloch had conjectured. They also provide evidence for vast numbers of exotic relations between special values of "Eisenstein-Kronecker-Lerch" series and values of Hasse-Weil L-functions for non C-M curves.

A good reference for these "E-K-L" series is Weil's book Elliptic Functions According to Eisenstein and Kronecker. We try to use Weil's notation insofar as possible.

Let $W \subset \mathbb{C}$ be a lattice with fundamental parallelogram of area $= \pi A$. Suppose given $x_0 \in \mathbb{C}$. The function

$$\chi(w) = \exp[A^{-1}(\bar{x}_0 w - x_0 \bar{w})]$$

is an additive character of the lattice which depends only on the image of x_0 in $\mathbb{C}/W$. For $a \in \mathbb{Z}$ and $s \in \mathbb{C}$, the Eisenstein-Kronecker-Lerch series is

$$K_a(x, x_0, s) = \Sigma^* \chi(w)(\overline{x+w})^a |x+w|^{-2s}$$

(here Σ^* means omit terms with $x = -w$). We shall be interested in the series

$$K_1(0, x_0, 2) = \Sigma^* \frac{\chi(w)}{w^2 \bar{w}}$$

as these are linked to K_2 of elliptic curves. However it seems plausible that the series

$$K_1(0, x_0, r), \quad r = 2, 3, 4, \ldots$$

0271-4132/86 $1.00 + $.25 per page

will be linked in a similar way to values of Hasse-Weil L-functions at points $s = r$. Note that such relations are no surprise when the elliptic curve in question has complex multiplication, e.g., by an imaginary quadratic field to class number 1. In that case the L-function is a Hecke L-series whose expression at $s = 2$ is similar to that of $K_1(0,x_0,2)$ above except that the additive character of W is replaced by a multiplicative character. Relations between the two kinds of series are obtained by letting x_0 run through points of some fixed finite order divisible by the conductor of the Hecke character and then relating the additive and multiplicative characters via Fourier theory. The spirit of the calculations we want to explain is that relations persist even when the elliptic curve does not have CM.

To be more precise, let E be an elliptic curve over $\mathbb{Q}$. Let ω be a global one-form defined over $\mathbb{R}$ and normalized so the fundamental real period = 1. Let $W \subset \mathbb{C}$ be the lattice of periods of ω, $W = \mathbb{Z} + \mathbb{Z}\tau$, $\operatorname{Im} \tau > 0$. Define

$$M(E,x_0) = (\operatorname{Im} \tau)^2 K_1(0,x_0,2) = (\operatorname{Im} \tau)^2 \Sigma^* \frac{\chi(w)}{w^2\bar{w}} .$$

If $x_0 \mapsto \alpha \in E(\mathbb{C})$, we also write $M(\alpha) = M(E,x_0)$. Let $L(E,s)$ be the Hasse-Weil L-function of E. (In cases we work with E is known to satisfy Weil's conjecture. We take L to be the Mellin transform of the corresponding modular form.) We are interested in relations of the form

$$C \cdot L(E,2) + \sum_{\alpha \in E(\mathbb{Q})_{tors}} c_\alpha M(\alpha) = 0$$

with $C, c_\alpha \in \mathbb{Z}$. One checks easily that $M(-\alpha) = -M(\alpha)$, so we fix $S \subset E(\mathbb{Q})_{tors}$ such that

$$E(\mathbb{Q})_{tors} = S \coprod (-S) \coprod {}_2E(\mathbb{Q})$$

and look for relations (*) with $c_\alpha = 0$ if $\alpha \notin S$. Actually, it is convenient to modify the notation slightly. For convenience, we calculate only with curves of negative discriminant, so the real locus is always connected and points in $E(\mathbb{Q})_{tors}$ correspond to a cyclic subgroup of fractions of the real period. Thus if $\#E(\mathbb{Q})_{tors} = n$, we can take $S = \{1,\dots,[\frac{n-1}{2}]\}$ and look for relations (N = Weil conductor. Introducing N and n makes B and $\{b_j\}$ small).

$$N\cdot B\cdot L(E,2) + \sum_{j=1}^{[\frac{n-1}{2}]} n\cdot b_j M(E,\tfrac{j}{n}), B, b_j \in \mathbb{Z} .$$

Grayson made such calculations for all curves on the Swinnerton-Dyer table (Lecture Notes in Math. no. 476) with Weil conductor ≤ 180, negative discriminant, and a rational torsion point of order ≥ 5 (a total of 37 curves). Typically the calculations were carried out to 25 decimal places. Occasionally, they were carried out to 80 or more decimal places. In the tables, N = Weil conductor, name = letter under which the curve is listed in the S-D table, d = order of torsion subgroup, and $B, b_1, b_2, \ldots$ are as in (*). The other columns will be explained anon.

For example, the curve 26D has a point of order 7. The computer gives relations

$$26\cdot L(E,2) + 28M(E,\tfrac{2}{7}) + 28M(E,\tfrac{3}{7}) = 0$$

$$5M(E,\tfrac{1}{7}) + 10M(E,\tfrac{2}{7}) + 8M(E,\tfrac{3}{7}) = 0$$

K-theory

We sketch how these computations can be explained by a modified form of a conjecture which Bloch (in the context of elliptic curves) and Beilinson more generally have advanced. First of all, let E be an elliptic curve over $\mathbb{C}$ and let f, g be rational functions on E with divisors

$$(f) = \Sigma\, p_i(\alpha_i); \quad (g) = \Sigma\, q_j(\beta_j) .$$

Define

$$(f)*(g)^- = \sum_{i,j} p_i q_j (\alpha_i - \beta_j)$$

$$M(f,g) = \sum_{i,j} p_i q_j M(E, \alpha_i - \beta_j) .$$

One of the basic results in Bloch's Irvine notes is that $M(f,1-f) = 0$, so there is an induced map

$$M: K_2(\mathbb{C}(E)) \to \mathbb{C}$$

which we compose with the restriction map to get

$$M: K_2(E) \to \mathbb{C} .$$

If the zeros and poles of f and g are points of some finite order dividing d, then there exists an element $S_{f,g} \in K_2(E)$ well defined up to Ker M such that

$$M(S_{f,g}) = d \cdot M(f,g).$$

Suppose for example $\alpha \in E(\mathbb{Q})_{tors}$ generates a subgroup of order $d > 2$,

$$(g) = 2 \sum_{r=1}^{d-1} (r\alpha) - 2(d-1)(0).$$

For $1 \leq s \leq d-1$, define a rational function f_s such that

$$(f_s) = d(s\alpha) - d(0), \ \alpha \in E(\mathbb{Q})_{tors} .$$

Then $S_{\alpha,s} \overset{=}{\text{dfn.}} S_{f_s,g} \in K_2(E_{\mathbb{Q}})$ and

$$M(S_{\alpha,s}) = -2d^2 M(E, s\alpha).$$

We had conjectured that $K_2(E_{\mathbb{Q}})$ would have rank 1 and that

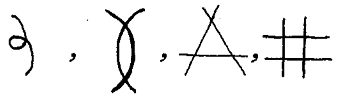
(**) $$\mathbb{Q}\cdot M(K_2(E_{\mathbb{Q}})) = \mathbb{Q}\cdot L(E,2) .$$

Assuming Weil's conjecture, it follows from Beilinson's work on modular curves that the right side in (**) is included in the left. On the other hand, the computer search for rational relations between the $M(E,\alpha)$, $\alpha \in E(\mathbb{Q})_{tors}$ strongly suggests that the rank of $K_2(E_{\mathbb{Q}})$ is in general > 1, e.g., the rank appears to be 2 for the curve 26D cited before.

This is as it should be! Let $E_{\mathbb{Z}}$ be the Neron model of E. What one wants to conjecture is that the rank of $K_2(E_{\mathbb{Z}}) = 1$ with generator related to the value of the L-function by the regulator. ($E_{\mathbb{Z}}$ is regular and proper over Spec $\mathbb{Z}$ just like spectra of rings of integers in the classical regulator formula.) Consider the localization sequence

$$K_2(E_{\mathbb{Z}}) \to K_2(E_{\mathbb{Q}}) \to \coprod_p K_1'(E_{\mathbb{F}_p}) .$$

For good reduction primes it is easy to show $K_1'(E_{\mathbb{F}_p})$ is torsion. However, if $E_{\mathbb{F}_p}$ is a bad fibre of Kodaira type I_ν,

$\nu \geq 1$, $E_{\mathbb{F}_p} =$ , ... then $K_1'(E_{\mathbb{F}_p}) \cong \mathbb{Z} \oplus$ torsion. (Other kinds of bad fibres have torsion K_1'. The point is that the dual graph of the I_ν fibres has $H_1 \cong \mathbb{Z}$.)

We calculated the boundary (tame symbol) for I_ν fibres

$$K_2(E) \xrightarrow{\partial_p} K_1'(E_{\mathbb{F}_p})/\text{tors} \cong \mathbb{Z} .$$

To explain the calculation, let $\alpha \in E(\mathbb{Q})_{\text{tors}}$ be a point of order $d > 2$. Assume the special fibre has m sides, numbered consecutively $0,1,\ldots,m-1$ with 0 corresponding to the component meeting the 0-section. Assume α meets the r-th component and write $t = r/m$. For $a \in \mathbb{Q}^+$ write $\langle a \rangle$ for the fractional part, so $0 \leq \langle a \rangle < 1$ and $a - \langle a \rangle \in \mathbb{Z}$. Let $S_{\alpha,s}$ be as on page 4. Then

$$\partial_p(S_{\alpha,s}) = (*) \sum_{\ell=1}^{d-1} (\langle ts\rangle - \langle t\ell\rangle)(-\langle ts\rangle\langle t\ell\rangle + \min(\langle t\ell\rangle, \langle ts\rangle))$$

where $(*)$ is a non-zero constant independent of s. Write $f_p(\alpha,s)$ for the right side (without the $(*)$). Then, up to torsion, an element $(a_s \in \mathbb{Z})$

$$\sum_{s=1}^{[\frac{d-1}{2}]} a_s S_{\alpha,s} \quad \text{(one can show } S_{\alpha,s} = -S_{\alpha,d-s})$$

comes from $K_2(E_{\mathbb{Z}})$ if and only if for all p with multiplicative reduction

$$(***) \qquad \sum_{s-1}^{[\frac{d-1}{2}]} a_s f_p(\alpha,s) = 0.$$

We compute tables of the $f_p(\alpha,s)$ and show that for all but two of the curves considered (the exceptions being 114B and 126C) the elements $\Sigma\, a_s S_{\alpha,s}$ with the a_s satisfying $(***)$ generate a rank one lattice in $K_2(E_{\mathbb{Z}})$, and the regulator map (denoted M above) applied to a generator of the lattice gives (up to 25 decimals) a simple non-zero fractional multiple of $L(E,2)$! In the two exceptional cases, there are no non-trivial solutions to $(***)$ so no linear combination of the $S_{\alpha,s}$ globalizes. One simply has to look elsewhere for an element in $K_2(E_{\mathbb{Z}})$! (*)

Relations with the dilogarithm and Milnor's conjecture

Let

$$D(z) = \log|z|\arg(1-z) - \mathrm{Im}(\int_0^z \log(1-t)\frac{dt}{t}) .$$

$D(z)$ is a single valued function of z (cf. the Irvine notes). Milnor considers $D(\zeta)$ for $\zeta^d = 1$. He conjectures (for geometric reasons) that the only relations

$$\sum_{r=1}^{d-1} a_r D(\zeta^r) = 0, \ a_r \in \mathbb{Z}$$

are those arising from the distribution relations

$$\frac{1}{s} D(x^s) = \sum_{\tau^s=1} D(\tau x)$$

together with

$$D(\overline{x}) = -D(x).$$

Now let E be an elliptic curve over $\mathbb{Q}$ corresponding to the lattice $1, \tau$ as above (so 1 = fundamental real period and $\mathrm{Im}\ \tau > 0$). Let $q = e^{2\pi i\tau}$ and define

$$D_q(z) = \sum_{n\in\mathbb{Z}} D(zq^n)$$

(the series converges, cf. the Irvine notes). A fourier series computation shows (notation as on p. 4)

$$\pi D_q(e^{2\pi i x_0}) = M(E,x_0) .$$

Assume $\Delta < 0$ so $E(\mathbb{Q})_{tors}$ is cyclic, and suppose $d = \#E(\mathbb{Q})_{tors} > 2$. Write Σ for the number of fibres of type I_ν with $\nu \geq 3$ in the Neron model, and suppose $[\frac{d-1}{2}] - \Sigma > 1$ (e.g. 11A, 14A, 20B, 26D, 42A, 48E, 54B, 66J, 90G, 150C on the enclosed list).

(*) Note the computer only verifies relations to 25 decimal places, it does not prove them. All relations in K_2 are mod Ker(M), which is conjecturally 0. Grayson has added to the table some curves with points of order 3 and 4. Results for these are consistent with everything said above.

Then there should be (and the computer verifies for the curves listed) at least $[\frac{d-1}{2}] - \Sigma - 1$ exotic relations $(a_r \in \mathbb{Z})$

$$\sum_{r=1}^{[\frac{d-1}{2}]} a_r D_q(\zeta^r) = 0 \quad \text{where } \zeta \text{ is a primitive d-th root of 1.}$$

Note $q \in \mathbb{R}$ so $D_q(\zeta^{-r}) = -D_q(\zeta^r)$. Also the function $f_p(\alpha, s)$ is identically zero when the reduction is of type I_1 or I_2, so these impose no conditions.

Tables by D. Grayson

Notation for tables

N = conductor

d = order of the rational point α

α = (real period)/n

$$B[N\cdot L(E,2)] + \sum_{j=1}^{[\frac{d-1}{2}]} b_j[d\ M(E,j\alpha)]$$

u = number of independent conditions imposed by the requirement that the tame symbol vanish on fibres with multiplicative reduction

p(r/m) means E has reduction of type I_m at p, $m > 1$, and α specializes to $\pm r \in \mathbb{Z}/m\mathbb{Z}$ = group of irreducible components of E(p). To find r, we use $p^r \| \wp'(\alpha)$.

N, name	d	B	b_1	b_2	b_3	b_4	b_5	u	reduction
11A	5	1	0	2				0	
		0	2	3					
11B	5	25	-8	-4				1	11(2/5)
14A	6	1	0	3				0	2(1/2)
		0	2	5					
14C	6	3	-1	-1				1	2(3/6)7(1/3)
15A	4	1	-4					0	

N, name	d	B	b_1	b_2	b_3	b_4	b_5	u	reduction
15F	4							1	2(1/2)5(2/8)
17C	4							1	17(1/4)
19A	3	1	-4					0	
19B	3							1	19(1/3)
20B	6	1	-36	90				0	5(1/2)
		0	5	-13					
26D	7	1	0	4	4			1	2(3/7)
		0	5	10	8				
30A	6	1	-2	-2				1	2(2/4)3(1/3)
36A	6	1	-4	0				0	
		0	0	1					
42A	8	1	-3	-1	1			1	2(3/8)3(1/2)
		0	1	5	5				
48E	8	1	-3	-1	1			1	3(3/8)
		0	1	-2	-3				
50A	5	1	-4	8				1	2(1/5)
50B	5	1	-4	-2				1	2(6/15)
54B	9	3	-8	-4	4	4		1	2(4/9)
		0	1	2	3	1			
		0	2	-4	-1	3			
57F	5	3	-8	-4				1	3(4/10)
58B	5	1	-8	-4				1	2(4/10)
66B	6	1	-8	10				1	3(1/6)11(1/2)
66J	10	1	-2	-2	-1	0		2	2(2/5)3(3/10)11(1/2)
		0	2	0	-3	-4			
84C	6	1	-4	-4				1	3(1/3)7(1/2)
90A	6	1	-8	10				1	2(1/6)

N, name	d	B	b_1	b_2	b_3	b_4	b_5	u	reduction
90G	12	3	-9	-8	-3	3	2		
		0	1	0	-1	1	2	2	2(5/12)5(1/3)
		0	0	1	2	3	2		
90M	6	1	-4	-4				1	2(1/2)5(1/3)
102B	6	1	-8	10				1	2(0/3) 3(2/12)17(1/2)
110C	5	1	-8	-4				1	2(2/5)5(2/5)
114B	6							2	2(1/3)3(1/6)19(1/2)
118B	5	1	-8	16				1	2(2/10)
123A	5	1	-4	8				1	3(1/5)
126C	6							2	2(1/6)7(1/3)
126E	6	1	-8	10				1	2(3/18)
130F	6	1	-8	10				1	2(0/2)5(1/6)13(1/2)
138G	6	1	-8	10				1	2(2/4)3(1/6)
150C	10	1	-8	-6	0	4		2	2(3/10)
		0	0	1	2	2			3(2/5)
155D	5	1	-8	-4				1	5(2/5)
170I	6	1	-8	10				1	2(0/4)5(2/4)17(1/6)
174G	7	1	-20	-20	-12			2	2(2/7)3(3/7)
175A	5	1	-4	8				1	7(1/5)
180D	6	1	-16	20				1	5(1/6)

Table for $\partial(S_{\alpha,s})$, where α is a point of order d, and meets the 1st component of a Neron fiber of type I_d. In fact, then $\partial(S_{\alpha,s}) = \frac{1}{6} s(d-s)((d-s)^2 - s^2)$. The general case $(d \neq m$ or $r \neq 1)$ can be reduced to this one.

d \ s	1	2	3	4	5	6
2	0					
3	1	-1				
4	4	0				
5	10	5				
6	20	16	0			
7	35	35	14			
8	56	64	40	0		
9	84	105	81	30		
10	120	160	140	80	0	
12	220	320	324	256	140	0

S. Bloch
Department of Mathematics
University of Chicago
Chicago, Illinois 60637

D. Grayson
Department of Mathematics
University of Illinois
Urbana, Illinois 61801