HIGHER ALGEBRAIC K-THEORY : II

[after Daniel Quillen]

by Daniel Grayson

The purpose of this paper is to prove three theorems announced in Higher Algebraic K-theory : I by Quillen.

The first theorem says that the two definitions of $K_i(R)$ offered by Quillen agree. The notion of monoidal category allows the construction of a fibration which shows that $K_0R \times BGl(R)^+$ is the loop space of $QP(R)$.

The localization theorem concerns the localization of a ring, $R \longrightarrow S^{-1}R$, and identifies the third term in the long exact sequence. It was first proved for K_0 and K_1 by Bass. Gersten gave a proof for the higher K_i and conjectured the non-affine version proved here.

The fundamental theorem describes extensions $R \longrightarrow R[t]$ and $R \longrightarrow R[t,t^{-1}]$, and was first proved by Bass. The proof here involves an application of the localization theorem to the projective line.

I thank Daniel Quillen for communicating to me his proofs of these theorems and for his helpful explanations of them.

Monoidal Categories and Localization

Suppose S is an abelian monoid which acts on a set X. S acts <u>invertibly</u> on X if each translation

$$\begin{aligned} X &\longrightarrow X \\ x &\longmapsto sx \end{aligned},$$

for $s \in S$, is a bijection. Define $S^{-1}X$ to be the quotient $S\times X/S$, where S acts on each factor of the product. Let S act on $S^{-1}X$ by $t\cdot(s,x) = (s,tx)$, and define $X \longrightarrow S^{-1}X$ to be $x \longmapsto (1,x)$. Then the translation $(s,x) \longmapsto (s,tx)$ has an inverse given by $(s,x) \longmapsto (ts,x)$, so S acts invertibly on $S^{-1}X$. The arrow $X \longrightarrow S^{-1}X$ respects the S-action, and is a universal arrow from X to a set upon which S acts invertibly.

If S is abelian, then the set $S^{-1}S$ is a group under the multiplication $(s,t)\cdot(u,v) = (su,tv)$. The map $S \longrightarrow S^{-1}S$ is a homomorphism, and is a universal arrow from S to a group.

This notion of localization is placed in the context of categories as follows.

<u>Def</u>: A <u>monoidal category</u> S is a category S with an operation $+ : S\times S \longrightarrow S$ and an object 0. There are natural isomorphisms $A+(B+C) \cong (A+B)+C$, $0+A \cong A$, $A+0 \cong A$. The following diagrams must commute :

$$\begin{array}{ccccc} A+(B+(C+D)) & \cong & (A+B)+(C+D) & \cong & ((A+B)+C)+D \\ \wr\| & & & & \wr\| \\ A+((B+C)+D) & & \cong & & (A+(B+C))+D \end{array},$$

$$\begin{array}{ccc} A+(0+C) & \cong & (A+0)+C \\ \wr\| & & \wr\| \\ A+C & = & A+C \end{array}. \quad \text{(see MacLane)}$$

<u>Def</u>: A <u>left action</u> of a monoidal category S on a category X is a functor $+ : S\times X \longrightarrow X$ with natural isomorphisms $A+(B+F) \cong (A+B)+F$ and $0+F \cong F$, where $A,B \in S$ and $F \in X$. Diagrams analogous to those above must commute.

<u>Def</u>: A <u>monoidal functor</u> is a functor $S \xrightarrow{f} T$ where S and T are monoidal categories, equipped with natural isomorphisms $f(A+B) \cong fA + fB$ and $f0 \cong 0$. The following diagrams must commute :

$$\begin{array}{ccccc} f((A+B)+C) & \cong & f(A+B)+fC & \cong & (fA+fB)+fC \\ \wr\| & & & & \wr\| \\ f(A+(B+C)) & \cong & fA+f(B+C) & \cong & fA+(fB+fC) \end{array}$$

$$\begin{array}{ccc} f(0+A) & \cong & f0 + fA \\ \wr\| & & \wr\| \\ fA & \cong & 0 + fA \end{array} \qquad \begin{array}{ccc} f(A+0) & \cong & fA + f0 \\ \wr\| & & \wr\| \\ fA & \cong & fA + 0 \end{array}.$$

<u>Def</u>: A functor $g : X \longrightarrow Y$ of categories with S-action, <u>preserves</u> the action if there is a natural isomorphism $A+gF \cong g(A+F)$, and

$$\begin{array}{ccccc} (A+B)+gF & \cong & g((A+B)+F) & \cong & g(A+(B+F)) \\ \wr\| & & & & \wr\| \\ A+(B+gF) & & \cong & & A+g(B+F) \end{array} \quad \text{and}$$

$$\begin{array}{ccc} 0 + gF & \cong & g(0 + F) \\ \wr\| & & \wr\| \\ gF & = & gF \end{array}$$ commute.

The commutativity of these diagrams yields the commutativity of every diagram which should commute. For details, see (MacLane). This commutativity assures us that the constructions to be made will satisfy the axioms for category or functor.

For details and notation about topological notions applied to categories, and for the theorem about constructing fibrations, the reader should refer to (Quillen).

<u>Def</u>: If S is a monoidal category which acts on a category X, then S acts <u>invertibly</u> on X if each translation

$$\begin{array}{ccc} X & \longrightarrow & X \\ F & \longmapsto & A+F \end{array}$$

is a homotopy equivalence.

<u>Def</u>: If S acts on X, the category $\langle S,X\rangle$ has the same objects as X. An arrow is represented by an isomorphism class of tuples $(F,G,A,A+F \longrightarrow G)$ with $A \in S$ and F,G in X. This arrow is an arrow from F to G. An isomorphism of tuples is an isomorphism $A \cong A'$ which makes

$$\begin{tikzcd} A+F \arrow[rr, phantom, "\cong"] \arrow[dr] & & A'+F \arrow[dl] \\ & G & \end{tikzcd}$$ commute.

<u>Def</u>: The category $S^{-1}X$ is $\langle S,S\times X\rangle$, where S acts on both factors of the product. The action of S on $S^{-1}X$ is given by $A + (B,F) = (B,A+F)$, if S is commutative up to natural isomorphism.

Notice that S acts invertibly on $S^{-1}X$. The translation $(B,F) \mapsto (B,A+F)$ has homotopy inverse $(B,F) \mapsto (A+B,F)$, in light of the natural transformation $(B,F) \longrightarrow (A+B,A+F)$.

<u>Note</u>: If every arrow is an isomorphism in S, then $\langle S,S\rangle$ has initial object 0 and is contractible. We now make the blanket assumption for the rest of the paper that this condition holds. In practice, S is usually the groupoid of isomorphisms in an exact category.

We consider now the projection on the first factor $S^{-1}X \to \langle S,S\rangle$. Call it ρ. The map ρ is given by

$$(B,F) \longmapsto (B) \qquad \text{on objects, and by}$$

$$\left(\begin{array}{ccc} A, & A+B, & A+F \\ & \downarrow & \downarrow \\ & B', & F' \end{array}\right) \longmapsto \left(\begin{array}{cc} A, & A+B \\ & \downarrow \\ & B' \end{array}\right) \qquad \text{on arrows.}$$

Suppose we are given an arrow $B \to B'$ in $\langle S,S\rangle$. It may be represented by some $(A, A+B \to B')$, and the arrow determines A up to isomorphism, but not up to unique isomorphism. An automorphism of the data giving the arrow is an automorphism $a: A \cong A$ such that

$$\begin{tikzcd} A+B \arrow[rr, "\overset{a+1}{\cong}"] \arrow[dr] & & A+B \arrow[dl] \\ & B' & \end{tikzcd} \qquad \text{commutes.}$$

We see that if $A+B \to B'$ is monic and $\mathrm{Hom}(A,A) \to \mathrm{Hom}(A+B,A+B)$ is injective, then the isomorphism a is necessarily the identity. So assume

1) every arrow of S is monic
2) translations $S \to S$ are faithful.

Under these conditions, every arrow in $\langle S,S\rangle$ determines its A up to unique isomorphism, and ρ is cofibred. The cobase-change map for an arrow $(A, A+B \to B')$ may be given by

$$\rho^{-1}B \longrightarrow \rho^{-1}B'$$

$$(B,F) \quad \mapsto \quad (B', A+F).$$

If we identify the fibers with X via the second projection, then the cobase-change map is just translation by A on X. If every translation on X is a homotopy equivalence, then all the cobase-change maps are, so the square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & S^{-1}X \arrow[d] \\ pt \arrow[r] & \langle S,S\rangle \end{tikzcd}$$

is homotopy cartesian. But $\langle S,S\rangle$ has initial object 0, so the map $X \to S^{-1}X$ given by $(F) \longmapsto (B,F)$ is a homotopy equivalence.

On the other hand, suppose $X \to S^{-1}X$ is a homotopy equivalence. This map is compatible with the action of S, and S acts invertibly on $S^{-1}X$. Therefore S acts invertibly on X.

We have shown :

<u>Th:</u> $X \to S^{-1}X$ is a homotopy equivalence if and only if S acts invertibly on X.

Homology Computation

Now $\pi_0 S$ acts on $H_p X$, and acts invertibly on $H_p S^{-1}X$, so $X \longrightarrow S^{-1}X$, the map given by $(F) \overset{p}{\longmapsto} (0,F)$, induces a map

$$(\pi_0 S)^{-1} H_p X \longrightarrow H_p(S^{-1}X)$$

<u>Th</u>: This map is an isomorphism.

<u>def</u>: If M is a $\pi_0 S$-module, define a functor $\overline{M} : \langle S,S\rangle \longrightarrow$ (ab gps) which sends each object (B) to the abelian group M, and sends an arrow $(A, A+B \xrightarrow{\sim} B')$ to multiplication by the class of A on M.

<u>Pf of Th</u>: If $\pi_0 S$ acts invertibly on M, then $\overline{M}$ is a morphism-inverting functor, and the homology group $H_p(\langle S,S\rangle, \overline{M})$ reduces to singular homology on the classifying space $B\langle S,S\rangle$ with coefficients in the local coefficient system determined by $\overline{M}$. Since $\langle S,S\rangle$ is contractible, we know that

$$H_p(\langle S,S\rangle, \overline{M}) = \begin{cases} M & \text{if } p = 0, \text{ and} \\ 0 & \text{if } p > 0. \end{cases}$$

Every fiber of the cofibred map $S\ X \xrightarrow{f} \langle S,S\rangle$ is identified with X, and the cobase-change maps are given by the action of S on X (see p.4). The spectral sequence for the map is thus :

$$E^2_{pq} = H_p(\langle S,S\rangle, \overline{H_q X}) \Longrightarrow H_{p+q}(S^{-1}X).$$

This spectral sequence is obtained from the bicomplex :

$$E^0_{pq} = \coprod_{\substack{B_0 \to \cdots \to B_p \\ \cap \\ N_p\langle S,S\rangle}} \ \coprod_{N_q(\rho\backslash B_0)} \mathbb{Z}$$

An action of S on this bicomplex is determined by the action of S on $S^{-1}X$ (via the X-component) and the action of S on $\langle S,S\rangle$ (the trivial action). Taking homology first in the q direction yields :

$$E^1_{pq} = \coprod_{B_0 \to \cdots \to B_p} H_q(\rho\backslash B_0) = \coprod H_q(\rho^{-1}B_0) = \coprod H_q X,$$

$$E^2_{pq} = H_p(\langle S,S\rangle, \overline{H_q X})$$

The action of S on the abutment and the abutment itself are computed using the degeneracy of the opposite spectral sequence, which begins :

$$E^1_{pq} = \begin{cases} \coprod_{N_q S^{-1}X} \mathbb{Z} & \text{if } p = 0, \text{ and} \\ 0 & \text{if } p > 0. \end{cases}$$

The action on the abutment $H_{p+q}S^{-1}X$ is the one induced by the action of S on $S^{-1}X$.

Localization with respect to a multiplicative subset of a ring is exact, so it preserves our spectral sequence. We localize with respect to $\pi_0 S$ inside its own integral group ring, and obtain :

$$E^1_{pq} = (\pi_0 S)^{-1} \coprod_{N_p\langle S,S\rangle} H_q X \Longrightarrow (\pi_0 S)^{-1} H_{p+q}(S^{-1}X).$$

Now S acts componentwise on E^1_{pq}, and acts invertibly on $H_{p+q}(S^{-1}X)$, so we get:

$$E^1_{pq} = \coprod (\pi_0 S)^{-1} H_q X, \text{ and}$$

$$E^2_{pq} = H_p(\langle S,S\rangle, \overline{(\pi_0 S)^{-1} H_q X}) \Longrightarrow H_{p+q}(S^{-1}X).$$

By the remark at the beginning of the proof, we know that this localized sequence degenerates from E^2 on, and the edge map is an isomorphism:

$$(\pi_0 S)^{-1} H_q X \xrightarrow{\sim} H_q(S^{-1}X).$$

That the edge map is the map induced by $X \longrightarrow S^{-1}X$ can be seen by comparing the degenerate spectral sequences which result from the following map of fibrations :

$$\begin{array}{ccccc} X & \longrightarrow & X & \longrightarrow & pt \\ \| & & \downarrow & & \downarrow 0 \\ X & \longrightarrow & S^{-1}X & \longrightarrow & \langle S,S\rangle \end{array}$$

QED

<u>Actions on fibers</u>

Suppose $f : X \longrightarrow Y$ is a map of categories on which S acts, f is compatible with the actions, and S acts trivially on Y. Then the action of S on X is said to be <u>fiberwise</u> with respect to f, and S does act on the fibers f G. If f is fibered and the base change maps respect the action on the fibers, then the action is said to be <u>cartesian</u>. In this case, $S^{-1}X$ is fibred over Y, its fibers are of the form $S^{-1}f^{-1}G$, and the base change maps are induced by those of f.

We consider now the projection on the second factor $S^{-1}X \longrightarrow \langle S,X\rangle$. Call it q. Assume

1) every arrow in X is monic, and
2) for each F in X the map $S \longrightarrow X$ given by $B \longmapsto B+F$ is a faithful functor.

Reasoning as before, we see that q is cofibred, each fiber may be identified with S, and the cobase-change maps are translations.

Let S act on $S^{-1}X$ via the first factor. This action is cartesian with respect to q, so localization yields a cofibred map $S^{-1}S^{-1}X \longrightarrow \langle S,X\rangle$, each fiber of which may be identified with $S^{-1}S$. Since S acts invertibly on $S^{-1}S$, the cobase-change maps are homotopy equivalences, so

$$\begin{tikzcd} S^{-1}S \arrow[r] \arrow[d] & S^{-1}S^{-1}X \arrow[d] \\ S^{-1}pt \arrow[r] & \langle S,X\rangle \end{tikzcd} \quad \text{is homotopy cartesian.}$$

The map $S^{-1}S \longrightarrow S^{-1}S^{-1}X$ is given by $(A,B) \longmapsto (A,(B,F))$ for some fixed F in X. Consider the following diagram :

$$\begin{tikzcd} & S^{-1}S \arrow[rr] \arrow[dd] & & S^{-1}S^{-1}X \arrow[dd] \\ S^{-1}S \arrow[ur, "\sim"] \arrow[ur, "sw"'] \arrow[rr] \arrow[dd] & & S^{-1}X \arrow[ur] \arrow[dd] & \\ & S^{-1}pt \arrow[rr] & & \langle S,X\rangle \\ \langle S,S\rangle \arrow[ur, equal] \arrow[rr] & & \langle S,X\rangle \arrow[ur, equal] & \end{tikzcd}$$

The back square is that which we just showed was homotopy cartesian. The map $S^{-1}S \xrightarrow{\sim} S^{-1}S$ is the switch isomorphism given by $(A,B) \longmapsto (B,A)$. The map $S^{-1}X \longrightarrow S^{-1}S^{-1}X$ is the usual map given by $(A,F) \longmapsto (0,(A,F))$. The map $pr_2 : S^{-1}S \longrightarrow \langle S,S\rangle$ is given by $(A,B) \longmapsto (B)$. Every square but the top is commutative, and the top square is homotopy commutative, as shown by the following natural transformations of functors $S^{-1}S \longrightarrow S^{-1}S^{-1}X$:

$$(0,(A,B+F)) \longrightarrow (B,(B+A,B+F)) \longleftarrow (B,(A,F)).$$

Notice that $S^{-1}X \longrightarrow S^{-1}S^{-1}X$ is a homotopy equivalence (see Th on p.4). Thus the front square is homotopy cartesian. Combine this with the fact that $\langle S,S\rangle$ is contractible, and we arrive at the following theorem :

<u>Th</u>: If $\langle S,X\rangle$ is contractible, then the map $S^{-1}S \longrightarrow S^{-1}X$ given by $(A,B) \longmapsto (A,B+F)$ for some fixed F in X is a homotopy equivalence.

The Plus Construction

The most important example of the previous constructions is the case where P is an exact category in which every exact sequence splits, and $S = \mathrm{Iso}(P)$ is the subcategory of P whose arrows are all isomorphisms of P. Direct sum is the operation which makes S into a monoidal category. $S^{-1}S$ becomes an H-space with multiplication $S^{-1}S \times S^{-1}S \longrightarrow S^{-1}S$ given by $((A,B),(C,D)) \longmapsto (A\oplus C, B\oplus D)$.

Suppose R is a ring, and let P be the category of finitely generated projective R-modules. We see easily that $\pi_0 S^{-1}S$ is K_0R. If A is a projective R-module, we can define a functor from $\mathrm{Aut}(A)$ to $S^{-1}S$ by sending $u : A \cong A$ to the arrow $(1_A, u) : (A,A) \cong (A,A)$. The natural transformation $(A,B) \longrightarrow (A\oplus R, B\oplus R)$ on $S^{-1}S$ shows that this diagram commutes up to homotopy :

$$\begin{tikzcd} \mathrm{Aut}(A) \arrow[rr] \arrow[dr] & & \mathrm{Aut}(A\oplus R) \arrow[dl] \\ & S^{-1}S. & \end{tikzcd}$$

Thus we can define a map from $\mathrm{BGl}(R) = \varinjlim \mathrm{Aut}(R^n)$ to $S^{-1}S$. In fact, this map lands in the connected component of the identity, $(S^{-1}S)_0$.

We can realize this map by using the telescope construction. If S_n is the component of S which contains R^n, then S_n is a groupoid equivalent to $\mathrm{Aut}(R^n) = \mathrm{Gl}_n(R)$. Define $S_n \longrightarrow S_{n+1}$ by $(B) \longmapsto (R \oplus B)$, and $S_n \longrightarrow S_{n+m}$ to be the composite of m of these functors. If N is the ordered set of positive integers, we have defined a functor from N to the category of categories, and can construct the corresponding cofibred category L over N. The objects of L are pairs (n,B) with B in S_n, and an arrow from (n,B) to $(n+m,C)$ is an isomorphism $R^m \oplus B \cong C$. L is homotopy equivalent to $\mathrm{BGl}(R)$.

Define $L \longrightarrow S^{-1}S$ by $(n,B) \longmapsto (R^n, B)$.

Let e in $\pi_0 S$ be the class of R. Since each projective module is a direct summand of a free one, the monoid generated by e is cofinal in $\pi_0 S$. Thus $H_p S^{-1}S \cong (\pi_0 S)^{-1} H_p S = H_p S[1/e]$. If $(R^n,B) \in (S^{-1}S)_0$, then for some m, $R^{n+m} \cong B \oplus R^m$. Thus any elememt of $H_p((S^{-1}S)_0)$ is of the form x/e^n for some n and some $x \in H_p S_n$. We see that $H_p((S^{-1}S)_0) \cong \varinjlim H_p S_n \cong H_p L$.

We can conclude that $L \longrightarrow (S^{-1}S)_0$ is an acyclic map. Since $(S^{-1}S)_0$ is an H-space as well, it must be $\mathrm{BGl}(R)^+$.

The multiplication on the H-space $S^{-1}S$ has a homotopy inverse given by $(A,B) \longmapsto (B,A)$ so the components must all be homotopy equivalent. We have proved :

<u>Th</u> : $S^{-1}S$ is homotopy equivalent to $K_0R \times \mathrm{BGl}(R)^+$.

Cofinality

Suppose $M \subseteq P$ are exact categories in which every exact sequence splits. Then M is cofinal in P if given $A \in P$ there exist $B \in P$ and $C \in M$ so $A \oplus B \cong C$, and if M is a full subcategory of P.

Th : If M is cofinal in P, then $QM \longrightarrow QP$ is a covering space, and $K_q M \longrightarrow K_q P$ is an isomorphism for $q > 0$ and is injective for $q = 0$.

For a proof of this theorem, see (Gersten).

Suppose $f : S \longrightarrow T$ is a monoidal functor. Then f is cofinal if given $A \in T$ there exist $B \in T$ and $C \in S$ so that $A + B \cong fC$. Suppose T acts on X, and S acts on X through f.

Th : If $f : S \longrightarrow T$ is cofinal, then $S^{-1}X = T^{-1}X$.

Pf: The point is that S acts invertibly on X if and only if T acts invertibly on X. Thus

$$S^{-1}X = T^{-1}(S^{-1}X) = S^{-1}(T^{-1}X) = T^{-1}X.$$

The Extension Construction

Let P be an exact category in which every exact sequence splits. Then S = Iso(P). Given C in P let E_C be the category whose objects are all exact sequences $(0 \to A \to B \to C \to 0)$ from P, and whose arrows are all isomorphisms which are the identity on C :

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\wr"] & B \arrow[r] \arrow[d, "\wr"] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C \arrow[r] & 0. \end{tikzcd}$$

We define a fibred category E over QP with fibres E_C. The base-change map $E_C \to E_{C'}$ for an arrow $C' \to C$ in QP can be described as follows :

a) for an injective arrow $C' \rightarrowtail C$, given $0 \to A \to B \to C \to 0$, construct the pullback $0 \to A \to B' \to C' \to 0$:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B' \arrow[r] \arrow[d, tail] & C' \arrow[r] \arrow[d, tail] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0. \end{tikzcd}$$

b) for a surjective arrow $C' \twoheadleftarrow C$, given $0 \to A \to B \to C \to 0$, compose the surjections to get a surjection $B \twoheadrightarrow C'$, and let A' be its kernel. We obtain $0 \to A' \to B \to C' \to 0$ in $E_{C'}$.

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] & B \arrow[r] & C' \arrow[r] & 0 \\ 0 \arrow[r] & A \arrow[r] \arrow[u, tail] & B \arrow[r] \arrow[u, equal] & C \arrow[r] \arrow[u, two heads] & 0. \end{tikzcd}$$

We see that E is the category whose objects are exact sequences $0 \to A \to B \to C \to 0$ from P, and whose arrows are represented by diagrams:

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \\ 0 \arrow[r] & A \arrow[r] \arrow[u, tail] \arrow[d, equal] & B' \arrow[r] \arrow[u, equal] \arrow[d, tail] & C_1 \arrow[r] \arrow[u, two heads] \arrow[d, tail] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0, \end{tikzcd}$$

but that isomorphisms of such diagrams involving C_1 give rise to the same arrow in E. The fibred map $E \to QP$ is the projection $(0 \to A \to B \to C \to 0) \mapsto (C)$, and every arrow of E is cartesian.

We let S act on E by setting $(A') + (0 \to A \to B \to C \to 0) = (0 \to A' \oplus A \to A' \oplus B \to C \to 0)$ and observe that $E \to QP$ is fibrewise and cartesian with respect to this action.

Notice that the map $S \to E_0$ given by $(A) \mapsto (0 \to A \to A \to 0 \to 0)$ is an equivalence of categories.

Th: For any C in P, $\langle S, E_C \rangle$ is contractible

Pf: Let M denote $\langle S, E_C \rangle$. We show

i) M is connected,
ii) M is an H-space,
iii) the multiplication on M has a homotopy inverse,
iv) the endomorphism $x \longmapsto x^2$ on M is homotopic to the identity, and
v) M is contractible.

We define the product on E_C using pullback in P : given $F_i = (0 \to A_i \to B_i \to C \to 0)$, set

$$F_1 * F_2 = (0 \to A_1 \oplus A_2 \to B_1 \times_C B_2 \to C \to 0).$$

Projection on one factor gives :

$$\begin{tikzcd}
(0 \arrow[r] & A_1 \oplus A_2 \arrow[r] \arrow[d, two heads] & B_1 \times_C B_2 \arrow[r] \arrow[d, two heads] & C \arrow[r] \arrow[d, equal] & 0) \\
(0 \arrow[r] & A_1 \arrow[r] & B_1 \arrow[r] & C \arrow[r] & 0).
\end{tikzcd} \tag{1}$$

We may choose a splitting for the surjections and obtain an isomorphism $A_2 + F_1 = F_1 * F_2$, and this determines an arrow $F_1 \to F_1 * F_2$ in $\langle S, E_C \rangle$. Similarly, we may construct an arrow $F_2 \to F_1 * F_2$, and we have connected F_1 and F_2 and proved i).

The constant functor to $(0 \to 0 \to C \to C \to 0)$ provides an identity for the operation just defined, so M is an H-space.

Any connected H-space has a homotopy inverse : consider

$$\begin{tikzcd}
M \arrow[r, "g"] \arrow[d, equal] & M \times M \arrow[r, "pr_2"] \arrow[d, "f"] & M \arrow[d, equal] \\
M \arrow[r, "g"] & M \times M \arrow[r, "pr_2"] & M,
\end{tikzcd}$$

where f is the map $(x,y) \longmapsto (xy,y)$, and g is $(x) \longmapsto (x,e)$, where e is the unit element. Since M is connected, the rows are fibrations, and the vertical maps on the fiber and on the base are homotopy equivalences, we know the total map f is a homotopy equivalence, with inverse h, say. One checks that the map $x \mapsto pr_1(h(e,x))$ is an inverse.

If $F_1 = F_2$ then in the diagram above (1), the diagonal map provides a canonical splitting of the surjections, and yields a natural arrow $F_1 \to F_1 * F_1$. This natural transformation gives the homotopy of iv).

Consider homotopy classes of maps $[M,M]$. By ii) this set is a monoid, by iii) this monoid is a group, by iv) the elements of this group satisfy the equation $x^2 = x$, and is therefore trivial. Thus M is contractible.

QED

Th: The square

$$\begin{tikzcd} S^{-1}S \arrow[r] \arrow[d] & S^{-1}E \arrow[d] \\ pt \arrow[r] & QP \end{tikzcd}$$

is homotopy cartesian.

Pf: We must show that the base change maps for the fibred map $S^{-1}E \longrightarrow QP$ are homotopy equivalences. It is enough to consider those associated to injective and surjective arrows of QP of the form $0 \rightarrowtail C$ and $0 \overset{j}{\twoheadleftarrow} C$. We will treat only the surjective case since the injective case is similar.

Identifying E_0 and S, the base change map $j^* : E_C \longrightarrow E_0$ is $(0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0) \mapsto (B)$. Consider $f : E_0 \longrightarrow E_C$ given by $(A) \longmapsto (0 \rightarrow A \rightarrow A \oplus C \rightarrow C \rightarrow 0)$. Since $\langle S, E_C \rangle$ is contractible, a previous theorem tells us that $S^{-1}f : S^{-1}E_0 \longrightarrow S^{-1}E_C$ is a homotopy equivalence. The composite $j^* \circ S^{-1}f : S^{-1}E_0 \longrightarrow S^{-1}E_0$ is given by $(A', A) \longmapsto (A', A \oplus C)$. This is a homotopy equivalence, as we have seen before, so j^* is a homotopy equivalence.

QED

Th: $S^{-1}E$ is contractible.

Pf: If X is a category, its <u>subdivision</u> $Sub(X)$ is the category whose objects are the arrows of X, and where an arrow from f to g is a pair of arrows from X, h and k, such that $kfh = g$. One sees that the codomain map $Sub(X) \longrightarrow X$ is a homotopy equivalence.

If X is the subcategory of QP of injective arrows, then E is equivalent to $Sub(X)$. X has initial object 0, so E is contractible. Then S acts invertibly on E, so we know that E and $S^{-1}E$ are homotopy equivalent. The Theorem is proved.

QED

Th: $\Omega QP \sim K_0R \times BGl(R)^+$.

This Theorem is a corollary of previous theorems. Here P is the category of finitely generated projective R-modules.

The Localization Theorem for projective modules

Suppose X is a quasi-compact scheme,

U is an affine open subscheme of X,

j is the inclusion $U \subseteq X$,

I is the sheaf of ideals defining the complement $X-U$ in X,

I is locally principal and generated by a non-zero-divisor,

H is the category of quasi-coherent sheaves on X which are zero on U and admit a resolution of length 1 by vector bundles on X.

Th : There is an exact sequence

$$\cdots \longrightarrow K_{q+1}U \longrightarrow K_qH \longrightarrow K_qX \longrightarrow K_qU$$

for $q \geqslant 0$.

Pf: Let P be the category of vector bundles on X,

V the category of vector bundles on U which extend to vector bundles on X, and

P_1 the category of quasi-coherent sheaves on X which have a resolution of length 1 by vector bundles.

U is affine, so every exact sequence in V splits. Let E be the extension construction over QV.

$\mathrm{Iso}(P)$ is cofinal in $\mathrm{Iso}(V)$, so we may use it instead of $\mathrm{Iso}(V)$; let $S = \mathrm{Iso}\ (P)$.

We will construct a diagram of categories with S-action :

$$\begin{tikzcd}
G \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, "g"] & F \arrow[r] \arrow[d, "p"] & E \arrow[d] \\
QH & QP \arrow[r] & QV
\end{tikzcd}$$

and show that f and h are homotopy equivalences. Localization will give :

$$\begin{tikzcd}
S^{-1}G \arrow[r, "\sim"] \arrow[d, "\wr"] & S^{-1}F \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S^{-1}E \arrow[d] & \sim \ \mathrm{pt} \\
QH & QP \arrow[r] & QV,
\end{tikzcd}$$

with the right-hand square homotopy cartesian. Combining this with the cofinality of V in all vector bundles on U gives the result. The map $K_qH \longrightarrow K_qX$ differs by a sign from the usual one.

In order to simplify notation, it is convenient to replace E by the equivalent category whose objects are surjections $(B \twoheadrightarrow C)$ with $B, C \in V$. That this category is equivalent is clear because a surjection determines its kernel up to unique isomorphism. An arrow in E is now represented by :

$$\begin{tikzcd} B' \arrow[r, two heads] \arrow[d, equal] & C' \\ B' \arrow[r, two heads] \arrow[d, tail] \arrow[dr, phantom, "\square"] & C_1 \arrow[u, two heads] \arrow[d, tail] \\ B \arrow[r, two heads] & C. \end{tikzcd}$$

F is defined as the pullback in $\begin{tikzcd} F \arrow[r] \arrow[d] & E \arrow[d] \\ QP \arrow[r] & QV. \end{tikzcd}$

Its objects are pairs $(B, Z \twoheadrightarrow j^*B)$ with $B \in P$, $Z \in V$. An arrow may be represented by

$$B' \quad \begin{tikzcd} B_1 \arrow[u, two heads] \arrow[d, tail] \\ B \end{tikzcd} \qquad \begin{tikzcd} Z' \arrow[r, two heads] \arrow[d, equal] & j^*B' \\ Z' \arrow[r, two heads] \arrow[d, tail] \arrow[dr, phantom, "\square"] & j^*B_1 \arrow[u, two heads] \arrow[d, tail] \\ Z \arrow[r, two heads] & j^*B. \end{tikzcd}$$

G is a sort of extension construction over $Q(H \times P)$. Its objects are surjections $(L \twoheadrightarrow M \oplus B)$ with $L, B \in P$ and $M \in H$. Its arrows are represented by diagrams :

$$\begin{tikzcd} L' \arrow[r, two heads] \arrow[d, equal] & M' \oplus B' \\ L' \arrow[r, two heads] \arrow[d, tail] \arrow[dr, phantom, "\square"] & M_1 \oplus B_1 \arrow[u, two heads] \arrow[d, tail] \\ L \arrow[r, two heads] & M \oplus B, \end{tikzcd}$$

and isomorphic diagrams give the same arrow. The vertical arrows on the right are each direct sums of arrows from H and P.

$G \rightarrowtail Q(H \times P)$ is defined by $(L \twoheadrightarrow M \oplus B) \mapsto (M, B)$. This map is fibred.

$g : G \rightarrowtail QP$ is defined by $(L \twoheadrightarrow M \oplus B) \mapsto (B)$, and
$h : G \rightarrowtail QH$ is defined by $(L \twoheadrightarrow M \oplus B) \mapsto (M)$. Both g and h are fibred.

$f : G \rightarrowtail F$ is defined by $(L \twoheadrightarrow M \oplus B) \mapsto (B, j^*L \twoheadrightarrow j^*B)$.

S acts on G via $(A) + (L \xrightarrow{\varphi} M \oplus B) = (A \oplus L \xrightarrow{(0\ \varphi)} M \oplus B)$. The action on F is similar, and is that induced by the action on E. S acts trivially on QH, QP, and QV.

Lemma 1 : $h : G \longrightarrow QH$ is a homotopy equivalence.

Pf: Let R be the category whose objects are surjections $(L \twoheadrightarrow M)$ with $L \in P$. M is a fixed object of H. The arrows of R are given by diagrams :

$$\begin{tikzcd} L' \arrow[r, two heads] \arrow[d, tail] & M \arrow[d, equal] \\ L \arrow[r, two heads] & M, \end{tikzcd}$$

where $L' \rightarrowtail L$ is an admissible monomorphism of P, i.e. its cokernel is a vector bundle.

There are natural transformations

$$(L \twoheadrightarrow M) \longrightarrow (L \oplus L' \twoheadrightarrow M) \longleftarrow (L' \twoheadrightarrow M),$$

so R is contractible.

Its subdivision, Sub(R), is equivalent to the fiber $h^{-1}(M)$, which is therefore contractible.

Since each fiber of h is contractible, h is a homotopy equivalence.

QED

Lemma 2 : If C is a vector bundle on X, then $C \subseteq j_*j^*C$, and and

$$j_*j^*C = \bigcup_n I^{-n}C.$$

Pf: The question is local on X, so we may assume X is affine and I is generated by the function w on X. Let R be the ring of X, so $X = \mathrm{Spec}\,(R)$, $U = \mathrm{Spec}(R[1/w])$. C is a projective R-module, and w is a non-zero-divisor in R, so w is a non-zero-divisor on C. Thus $C \subsetneq C_w = C \otimes_R R[1/w]$, and $C_w = \bigcup_n w^{-n}C$.

QED

Lemma 3 : $f : G \longrightarrow F$ is a homotopy equivalence.

Pf: Both g and p are fibred, so it is enough to show f is a homotopy equivalence on each fiber over QP. If $B \in QP$, consider the map $g^{-1}B \longrightarrow p^{-1}B$.

Let T be the category whose objects are surjections $(L \twoheadrightarrow B)$ with $L \in P$, and whose arrows are diagrams

$$\begin{tikzcd} L' \arrow[r, two heads] \arrow[d, tail] & B \arrow[d, equal] \\ L \arrow[r, two heads] & B, \end{tikzcd}$$

where $L' \rightarrowtail L$ is an admissible mono from P_1 whose cokernel is in H, i.e. any injection which is an isomorphism on U.

Then the functor $\mathrm{Sub}(T) \longrightarrow g^{-1}B$ given by

$$\left(\begin{tikzcd} L' \arrow[r, two heads] \arrow[d, tail] & B \arrow[d, equal] \\ L \arrow[r, two heads] & B \end{tikzcd}\right) \longmapsto (L \twoheadrightarrow (\mathrm{ckr}\ i) \oplus B)$$

is an equivalence of categories.

Let $W = j^*B$, so that $p^{-1}B = E_W$. We must show that $\mathrm{Sub}\ (T) \longrightarrow E_W$ is a homotopy equivalence. This map is

$$\left(\begin{tikzcd} L' \arrow[r, two heads] \arrow[d, tail] & B \arrow[d, equal] \\ L \arrow[r, dash] & B \end{tikzcd}\right) \longmapsto (j^*L \twoheadrightarrow j^*B = W).$$

It factors through the target map $\mathrm{Sub}(T) \longrightarrow T$, which is a homotopy equivalence, so it is enough to show that the map $T \xrightarrow{w} E_W$ given by $(L \twoheadrightarrow B) \mapsto (j^*L \twoheadrightarrow j^*B = W)$ is a homotopy equivalence. To do this we need only show that each fiber $w/(Z \twoheadrightarrow W)$ is contractible, where $(Z \twoheadrightarrow W)$ is an object of E_W.

An object of this fiber category is an object $(L \twoheadrightarrow B)$ of T with an isomorphism $w(L \twoheadrightarrow B) \cong (Z \twoheadrightarrow W)$ which is the identity on W, i.e.

$$\left(L \twoheadrightarrow B\ ,\ \begin{tikzcd} j^*L \arrow[r, two heads] \arrow[d, "\wr" , no head] & j^*B \arrow[d, equal] \\ Z \arrow[r, two heads] & W \end{tikzcd}\right).$$

Define an ordered set Lat to be the set of vector bundles L on X such that

1) $L \subseteq j_*Z$,
2) $j^*L = Z$,
3) the image of the map $(L \subset j_*Z \twoheadrightarrow j_*W = j_*j^*B)$ is B.

Elements of Lat will be called lattices.
The obvious map from Lat to the fiber $w/(Z \twoheadrightarrow W)$ is an equivalence of categories, so we need only to show that Lat is contractible ; we show it is actually filtering.

We have an exact sequence $0 \longrightarrow Y \longrightarrow Z \longrightarrow W \longrightarrow 0$ in V which splits. Now $Y = j^*C$ for some $C \in P$, so $Z = j^*(C \oplus B)$. Consider the lattice $C \oplus B \subseteq j_*Z$. If $(L \subseteq j_*Z)$ is another lattice, then condition 3) insures that $L \subseteq j_*j^*C \oplus B$. Now by lemma 2 $j_*j^*C = \bigcup_n I^{-n}C$, and L is finitely generated locally on X, which is quasi-compact, so for large n, $L \subseteq I^{-n}C \supseteq C$.

Thus Lat is filtering, and Lemma 3 is proved.

QED

The end of the proof of the theorem is now near. S acts trivially on QH, so by lemmas 1 and 3 it acts invertibly on G and F. Thus G and F are homotopy equivalent to $S^{-1}G$ and $S^{-1}F$, respectively, and h and f remain homotopy equivalences after localization. We know that $S^{-1}E \longrightarrow QV$ is a fibration, so $S^{-1}F \longrightarrow QP$ is, too, since it has the same fibers. Since the homotopy fibers of these two maps are the same, the square

$$\begin{array}{ccc} S^{-1}F & \longrightarrow & S^{-1}E \\ \downarrow & & \downarrow \\ QP & \longrightarrow & QV \end{array} \quad \text{is homotopy cartesian.}$$

As indicated earlier, we now know that

$$BQH \longrightarrow BQP \longrightarrow BQV$$

has the homotopy type of a fibration, and the cofinality of V in the category of all vector bundles on U gives the long exact sequence we want.

It only remains to compute the map $K_qH \longrightarrow K_qP$. To do this we show that the square

$$\begin{array}{ccc} G & \longrightarrow & QP \\ \downarrow & & \downarrow \\ QH & \longrightarrow & QP_1 \end{array} \quad \text{commutes up to sign.}$$

The two functors $G \rightrightarrows QP_1$ are given by $(L \twoheadrightarrow M\oplus B) \longmapsto (M)$ and by $(L \twoheadrightarrow M\oplus B) \longmapsto (B)$. The map $(L \twoheadrightarrow M\oplus B) \longmapsto (M\oplus B)$ is their sum, so we must show this map is homotopic to a constant map. The functor $(L \twoheadrightarrow M\oplus B) \longmapsto (L)$ maps all arrows of G to injective arrows of QP_1, so $0 \rightarrowtail L \twoheadrightarrow M\oplus B$ exhibits two natural transformations which give the desired homotopy.

The theorem is proved.

Suppose R is a ring,
$S \subseteq R$ is a multiplicative set of central non-zero-divisors,
H is the category of finitely generated R-modules M of projective dimension ≤ 1 such that $M_S = 0$.

Th : There is an exact sequence

$$\cdots \longrightarrow K_{q+1}R_S \longrightarrow K_qH \longrightarrow K_qR \longrightarrow K_qR_S$$

for $q \geq 0$.

The proof is formally the same as the proof of the previous theorem, except that Lemma 2 is replaced by:

Lemma 2' : If C is a projective R-module, then $C \subseteq C_S$, and

$$C_S = \bigcup_{s \in S} s^{-1}C.$$

The Suspension of a ring

This section is included so we can complete the proof of the fundamental theorem in the next section. We must ensure that a certain computation of Loday's involving products using the +-construction is compatible with our use of the Q-construction.

Suppose A is a ring. Then the cone of A, CA, is the ring of infinite matrices with entries in A which have only a finite number of non-zero entries in any given row or column. The matrices which have only a finite number of non-zero entries form a two-sided ideal $I \subsetneq CA$. The suspension of A, SA, is the quotient ring CA/I.

Let e be the element of CA whose only non-zero entry is a 1 in the corner :

$$\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots \end{pmatrix}$$

It is idempotent, so $CA = eCA \oplus (1-e)CA$ is a decomposition of CA into A-CA-bimodules. We use this to define w as the composite

$$Gl_n A \longrightarrow Aut((eCA)^n) \longrightarrow Aut(CA^n) = Gl_n CA.$$

It sends a matrix (a_{ij}) to the matrix (b_{ij}) where $b_{ij} = a_{ij}e$ if $i \neq j$, and $b_{ii} = a_{ii}e + (1-e)$.

Th: [Gersten-Wagoner] (i) $K_0CA \times BGl(CA)^+$ is contractible.

(ii) $K_0A\times BGl(A)^+ \xrightarrow{w} K_0CA\times BGl(CA)^+ \longrightarrow K_0SA\times BGl(SA)^+$

is a fibration.

Let P(R) denote the category of finitely generated projective right R-modules.

Def: $v : P(A) \longrightarrow P(CA)$ is the exact functor $B \mapsto B \otimes_A (eCA)$.

Th: $QP(A) \xrightarrow{v} QP(CA) \longrightarrow QP(SA)$ is a fibration which is a

delooping of the Gersten-Wagoner fibration.

Pf: We use the naturality of the extension construction to loop this sequence, yielding

$$S^{-1}S(A) \xrightarrow{v} S^{-1}S(CA) \longrightarrow S^{-1}S(SA).$$

Here $S^{-1}S(A)$ denotes $S^{-1}S$ where $S = Iso(P(A))$ (for any ring A).

The two functors $\quad \mathrm{Aut}(A^n) \longrightarrow S^{-1}S(CA)$

$$V : \quad a \longmapsto (1_{eCA^n}, a \otimes 1_{eCA^n})$$

$$W : \quad a \longmapsto (1_{CA^n}, a \otimes 1_{eCA^n} \oplus 1_{(1-e)CA^n})$$

are homotopic, so we identify the looped sequence with the Gersten-Wagoner fibration. The sequence in the statement of the theorem consists of connected H-spaces, so it, too, must be a fibration.

QED

The Fundamental Theorem

Suppose A is a (not necessarily commutative) ring.

Def: $NK_qA = \mathrm{ckr}\,(K_qA \longrightarrow K_qA[t])$

Def: Nil(A) is the exact category whose objects are pairs (P,f), where P is a finitely generated projective A-module, and f is a nilpotent endomorphism of P.

Def: $\mathrm{Nil}_q(A) = \ker\,(K_q\underline{\mathrm{Nil}}(A) \longrightarrow K_qA)$

Th: 1) $NK_qA \cong \mathrm{Nil}_{q-1}(A)$

2) $K_q(A[t,t^{-1}]) \cong K_qA \oplus K_{q-1}A \oplus NK_qA \oplus NK_qA$

Pf : Let X be the projective line over A. Then X has open subsets Spec(A[t]) and Spec($A[t^{-1}]$) which satisfy the conditions of the localization theorem for projective modules. We get :

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & K_qH & \longrightarrow & K_qX & \longrightarrow & K_qA[t^{-1}] & \longrightarrow & K_{q-1}H & \longrightarrow & \cdots \\ & & \| & & \downarrow & & \downarrow & & \| & & \\ \cdots & \longrightarrow & K_qH & \longrightarrow & K_qA[t] & \longrightarrow & K_qA[t,t^{-1}] & \longrightarrow & K_{q-1}H & \longrightarrow & \cdots \end{array} \qquad (*)$$

The naturality of the long exact sequence with respect to flat maps is clear from the proof of the localization theorem. The vertical equalities involving H arise from the fact that the category of abelian sheaves on X which vanish on Spec($A[t^{-1}]$) is equivalent to the category of abelian sheaves on Spec(A[t]) which vanish on Spec($A[t,t^{-1}]$).

If $(P,f) \in \underline{\mathrm{Nil}}(A)$, we have the characteristic sequence of f :

$$0 \longrightarrow P[t] \xrightarrow{t-f} P[t] \longrightarrow P_f \longrightarrow 0,$$

where P_f is the A[t]-module P with t acting as f. Since f is nilpotent, P_f is zero on Spec($A[t,t^{-1}]$), so determines an object of H.

If M is an A[t]-module of projective dimension ≤ 1 killed by some power t^n of t, then M is a projective A-module. For, let $0 \to P \to Q \to M \to 0$ be a projective resolution of M by A[t]-modules. Then

$$0 \longrightarrow M \xrightarrow{t^n} P/t^nP \longrightarrow P/t^nQ \longrightarrow 0$$

is exact, and P/t^nP is a projective A-module, P/t^nQ is an A-module of projective dimension 1. Then M is projective.

Thus
$$\begin{array}{ccc} \underline{\mathrm{Nil}}(A) & \longrightarrow & H \\ (P,f) & \longmapsto & P_f \end{array}$$

is an equivalence of categories.

The K-theory of the projective line was computed in (Quillen). We know

$$K_qX \cong K_qA \cdot 1 \oplus K_qA \cdot z,$$

where $1 = cl(O_X)$, $z = cl(O_X(-1))$ in K_0X. We alter this basis slightly :

$$K_qX = K_qA \cdot (1-z) \oplus K_qA \cdot 1.$$

Let $U = \mathrm{Spec}\,(A[t])$, and $V = \mathrm{Spec}\,(A[t^{-1}])$.

Now,

$$K_qA \xrightarrow{1-z} K_qX \longrightarrow K_qV$$ is zero, since $O_X|_V = O_X(-1)|_V$,

and

$$K_qA \xrightarrow{1} K_qX \longrightarrow K_qV$$ is the usual split injection induced by $A \longrightarrow A[t^{-1}]$. Thus the top row above splits into pieces :

$$0 \longrightarrow K_qA \longrightarrow K_qV \longrightarrow K_{q-1}\underline{\mathrm{Nil}}(A) \longrightarrow K_{q-1}A \longrightarrow 0. \qquad (\#)$$

If $(P,f) \in \underline{\mathrm{Nil}}(A)$, then the characteristic sequence extends to all of X as :

$$0 \longrightarrow P_X(-1) \longrightarrow P_X \longrightarrow P_f \longrightarrow 0.$$

Thus, the square

$$\begin{tikzcd}
K_{q-1}\underline{\mathrm{Nil}}(A) \arrow[r, "\sim"] \arrow[d, "pr_1"'] & K_{q-1}H \arrow[d] \\
K_{q-1}A \arrow[r, "1-z"] & K_{q-1}X
\end{tikzcd}$$ commutes, and

the last map of (#) is the usual projection. Splitting off the first and last terms of (#) gives $NK_qA \cong \mathrm{Nil}_{q-1}(A)$, proving 1).

From (*) we derive the Mayer-Vietoris sequence :

$$\cdots \longrightarrow K_qX \longrightarrow K_qA[t] \oplus K_qA[t^{-1}] \longrightarrow K_qA[t,t^{-1}] \longrightarrow K_{q-1}X \longrightarrow \cdots$$

and, as above, split it into shorter pieces :

$$0 \longrightarrow K_qA \longrightarrow K_qA[t] \oplus K_qA[t^{-1}] \longrightarrow K_qA[t,t^{-1}] \longrightarrow K_{q-1}A \longrightarrow 0.$$

According to (Loday, Coro 2.3.7), the map $K_qA[t,t^{-1}] \longrightarrow K_{q-1}A$ is split by the map induced by cup-product with t. All we need to do is verify that his definition of this map agrees with ours, so we must check that

$$\begin{tikzcd}
K_qA[t,t^{-1}] \arrow[r, "t \mapsto T"] \arrow[d] & K_qSA \arrow[dr, "\sim"] & \\
K_{q-1}H = K_{q-1}\underline{\mathrm{Nil}}(A) \arrow[rr, "pr_1"'] & & K_{q-1}A
\end{tikzcd}$$ commutes.

Loday uses the +-construction for his definition of the isomorphism $K_q SA = K_{q-1} A$, but we saw in the previous section that we may as well use the Q-construction. Let

$$\tau = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & 1 & \ddots \end{pmatrix} \quad \text{and } \sigma = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & \ddots \end{pmatrix} \text{ be elements of } CA.$$

They satisfy $\tau\sigma=1$, $\sigma\tau=1-e$. We have the commutative diagram :

$$\begin{tikzcd} A[t] \arrow[r] \arrow[d, "t \mapsto \tau"'] & A[t,t^{-1}] \arrow[d] \\ CA \arrow[r] & SA. \end{tikzcd}$$

We now refer to the proof of the localization theorem. If $P_1(R)$ denotes the exact category of finitely-generated R-modules of projective dimension $\leqslant 1$, we may conclude that

$$QH \longrightarrow QP_1(A[t]) \longrightarrow QP_1(A[t,t^{-1}])$$

is a fibration homotopy equivalent to the one produced by the theorem except for a change in sign of the left hand map. Notice that in the first part of the proof of the fundamental theorem we have implicitly used the maps with this more natural sign-sense.

Let H' be the exact category of right CA-modules B of projective dimension $\leqslant 1$ such that

$$B \otimes_{CA} SA = 0.$$

Since $eCA \in H'$, the map $v : QP(A) \longrightarrow QP(CA)$ yields a map $QP(A) \xrightarrow{v} QH'$.

Consider this diagram :

$$\begin{tikzcd} Q\underline{Nil}(A) \arrow[r, "\simeq", dash] \arrow[d] & QH \arrow[r] \arrow[d] & QP'(A[t]) \arrow[r] \arrow[d] & QP'(A[t,t^{-1}]) \arrow[d] \\ QP(A) \arrow[r, "v"] & QH' \arrow[r] & QP_1(CA) \arrow[r] & QP_1(SA). \end{tikzcd}$$

A functor $H \longrightarrow H'$ is defined by $M \longmapsto M \otimes_{A[t]} CA$, but we must check that this CA-module has the right projective dimension, and that this functor is exact. It is enough to see that the characteristic sequence of an element of $\underline{Nil}(A)$ remains exact under this tensor product. At issue is the injectivity of $\tau - f : P \otimes_A CA \longrightarrow P \otimes_A CA$, and this is true because the sum of an injective endomorphism and a nilpotent endomorphism which commute is injective.

We must also check that the left-hand square commutes. If (P,f) is in $\underline{Nil}(A)$ then there is a natural isomorphism

$$P_f \otimes_{A[t]} CA \cong P \otimes_A (eCA)$$

defined by the diagram

$$\begin{tikzcd} 0 \arrow[r] & P\otimes_A CA \arrow[r, "\tau"'] & P\otimes_A CA \arrow[r] & P\otimes_A (eCA) \arrow[r] & 0 \\ 0 \arrow[r] & P\otimes_A CA \arrow[r, "\tau-f"] \arrow[u, equal] & P\otimes_A CA \arrow[r] \arrow[u, "\wr"] & P_f\otimes_{A[t]} CA \arrow[r] \arrow[u, dotted] & 0 \end{tikzcd}$$

where $1 + f\sigma + f^2\sigma^2 + f^3\sigma^3 + \ldots$ is the vertical isomorphism. This isomorphism yields the commutativity of the square in question.

Finally we define the categories $P'(A[t])$ and $P'(A[t,t^{-1}])$. $P'(A[t])$ is the full exact subcategory of $P_1(A[t])$ consisting of modules M satisfying

$$\mathrm{Tor}_1^{A[t]}(M,CA) = 0.$$

We saw above that this category contains H. $P'(A[t,t^{-1}])$ is defined in a similar fashion relative to SA. It is clear then that these categories fit into the diagram as indicated, and the resolution theorem says that the top row still contains a fibration equivalent to the original one.

We conclude that we have a map of fibrations, so the naturality of the boundary map in the long exact sequence of homotopy groups yields the commutativity of

$$\begin{tikzcd} K_q A[t,t^{-1}] \arrow[r] \arrow[d] & K_{q-1}\underline{\mathrm{Nil}}(A) \arrow[d] \\ K_q SA \arrow[r, "\sim"] & K_{q-1}A \end{tikzcd} .$$

This concludes the proof of the fundamental theorem.

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