Bivariant $K$- and Cyclic Theories

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Summary. Bivariant $K$-theories generalize $K$-theory and its dual, often called $K$-homology, at the same time. They are a powerful tool for the computation of $K$-theoretic invariants, for the formulation and proof of index theorems, for classification results and in many other instances. The bivariant $K$-theories are paralleled by different versions of cyclic theories which have similar formal properties. The two different kinds of theories are connected by characters that generalize the classical Chern character. We give a survey of such bivariant theories on different categories of algebras and sketch some of the applications.

1 Introduction

Topological $K$-theory was introduced in the sixties, [2]. On the category of compact topological spaces it gives a generalized cohomology theory. It was used in the solution of the vector field problem on spheres [1] and in the study of immersion and embedding problems. A major motivating area of applications was the study of Riemann-Roch type and index theorems [4].

Soon it became clear that $K$-theory could be generalized without extra cost and keeping all the properties, including Bott periodicity, from commutative algebras of continuous functions on locally compact spaces to arbitrary Banach algebras.

Kasparov himself was initially motivated by the study of the Novikov conjecture. He used his equivariant theory to prove powerful results establishing the conjecture in important cases, [44], [45]. Another application of the theory to geometry is the work of Rosenberg on obstructions to the existence of metrics with positive scalar curvature, [70].

In the classification of nuclear $C^*$-algebras, the theory has been used to obtain a classification by $K$-theory that went beyond all expectations, [48], [49], [61].

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There are many different descriptions of the elements of bivariant $KK$-theory and $E$-theory. They can be defined using Kasparov-modules, asymptotic morphisms or classifying maps for extensions or in still other ways. Each picture has its own virtues. The point of view that elements of $KK$ are described by $n$-step extensions of $A$ by $\mathcal{K} \otimes B$, with Kasparov product corresponding to the Yoneda concatenation product of extensions, was for the first time developed by Zekri in [81]. Here $\mathcal{K}$ denotes the algebra of compact operators on a Hilbert space and $\otimes$ denotes the $C^*$- tensor product.

Already from the start it was clear that one did not have to restrict to the case where the objects of the category are just $C^*$-algebras. In fact, already the first paper by Kasparov on the subject treated the case of $C^*$-algebras with several additional structures, namely the action of a fixed compact group, a $\mathbb{Z}/2$-grading and a complex conjugation. In the following years, versions of $KK$-theory were introduced for $C^*$-algebras with the action of a locally compact group, for $C^*$-algebras fibered over a locally compact space, for projective systems of $C^*$-algebras and for $C^*$-algebras with a specified fixed primitive ideal space. A general framework that covers all the latter three cases has been described in [9].

Cyclic homology and cohomology was developed as an algebraic pendant to algebraic and topological $K$-theory [14], [75]. It can be used to accommodate characteristic classes for certain elements of $K$-theory and $K$-homology [14], [39]. Motivated by the apparent parallelism to $K$-theory, bivariant cyclic theories were introduced in [37], [30], [65], [54]. For instance, the periodic bivariant cyclic theory can be viewed also as an additive (here even linear) category $HP$, whose objects are algebras (over a field of characteristic 0) and whose morphism sets are the $\mathbb{Z}/2$-graded vector spaces $HP(A, B)$. This category has formally exactly the same properties as $KK$ or $E$. One way of describing these properties is to say that $KK$, $E$ and $HP$ all form triangulated categories, [74]. The formalism of triangulated categories allows one to form easily quotient categories which are again triangulated and thus have the relevant properties, in order to enforce certain isomorphisms in the category. In [74], this technique is used to introduce and study bivariant theories for $C^*$-algebras whose restrictions to the category of locally compact spaces give connective $K$-theory and singular homology.

It certainly seems possible to construct bivariant versions of algebraic (Quillen) $K$-theory - a very promising attempt is in [59]. However, if one wants to have the important structural element of long exact sequences associated with an extension, one has to make the theory periodic by stabilizing with a Bott extension. Since this extension is not algebraic, it appears that one cannot avoid the assumption of some kind of topology on the class of algebras considered. This assumption can be weakened to a large extent.

Since the construction of $KK$-theory and of $E$-theory used techniques which are quite specific to $C^*$-algebras (in particular the existence of central approximate units) it seemed for many years that similar theories for other topological algebras, such as Banach algebras or Fréchet algebras, would be
impossible. However, in [28] a bivariant theory $kk$ with all the desired properties was constructed on the category of locally convex algebras whose topology is described by a family of submultiplicative seminorms (“m-algebras”). The definition of $kk_\ast(A, B)$ is based on “classifying maps” for $n$-step extensions of $A$ by $\mathcal{K} \otimes B$, where this time $\mathcal{K}$ is a Fréchet algebra version of the algebra of compact operators on a Hilbert space and $\otimes$ denotes the projective tensor product. The product of the bivariant theory $kk_\ast$ corresponds to the Yoneda product of such extensions. This theory allows to carry over the results and techniques from $C^\ast$-algebras to this much more general category. It allows the construction of a bivariant multiplicative character into cyclic homology, i.e. of a functor from the additive category $kk$ to the linear category $HP$ which is compatible with all the structure elements. Since ordinary cyclic theory gives only pathological results for $C^\ast$-algebras such a character say from $KK_\ast$ or from $E_\ast$ to $HP_\ast$ cannot make sense.

Another bivariant cyclic theory $HE^{loc}$, the local theory, which does give good results for $C^\ast$-algebras, was developed by Puschnigg [65]. The local theory is a far reaching refinement of Connes’ entire cyclic theory [15]. As a consequence, Puschnigg defines a bivariant character from $KK$ to $HE^{loc}$ which even is a rational isomorphism for a natural class of $C^\ast$-algebras.

The general picture that emerges shows that the fundamental structure in bivariant $K$-theory is the extension category consisting of equivalence classes of $n$-step extensions of the form

$$0 \to B \to E_1 \to \ldots \to E_n \to A \to 0$$

with the product given by Yoneda product. The crucial point is of course to be able to compare extensions of different length. As a general rule, $K$-theoretic invariants can be understood as obstructions to lifting problems in extensions.

A very good source for the first two chapters is [8]. The choice of topics and emphasis on certain results is based on the preferences and the own work of the author. I am indebted to R.Meyer for his contribution to the chapter on local cyclic theory.

### 2 Topological $K$-theory and $K$-homology

#### 2.1 Topological $K$-theory

Topological $K$-theory was introduced, following earlier work of Grothendieck and Bott, by Atiyah and Hirzebruch in connection with Riemann-Roch type theorems [2]. For a compact space $X$, the abelian group $K^0(X)$ can be defined as the enveloping group for the abelian semigroup defined by isomorphism classes of complex vector bundles over $X$ with direct sum inducing addition.

The reduced $K$-theory group $\tilde{K}^0(X)$ is defined as the $K^0$-group of $X$, divided by the subgroup generated by the image in $K^0$ of the trivial line bundle
on $X$. Groups $K^{-n}(X)$ can then be defined as the reduced $K^0$-group of the reduced $n$-fold suspension $S^nX$ of $X$. As it turns out, using clutching functions for the gluing of vector bundles, $K^{-1}(X) = \widetilde{K^0}(SX)$ can be identified with the group of homotopy classes of continuous maps from $X$ to the infinite unitary group $U_\infty$.

The Bott periodicity theorem then asserts that $K^{-n-2}(X) \cong K^{-n}(X)$. This periodicity shows that the family of groups $K^{-n}(X)$ consists only of two different groups, denoted by $K^0(X)$ and $K^1(X)$.

Given a compact subspace $Y$ of $X$ one denotes by $K^i(X,Y)$ the reduced $K^i$-theory groups of the quotient space $X/Y$.

**Theorem 2.1.** Let $X, Y$ be as above. There is a periodic cohomology exact sequence of the following form

\[
\begin{array}{c}
K^0(X, Y) \xrightarrow{i^*} K^0(X) \xrightarrow{j^*} K^0(Y) \\
\uparrow \quad \quad \quad \quad \quad \downarrow \\
K^1(Y) \xrightarrow{i^*} K^1(X) \xrightarrow{j^*} K^1(X, Y)
\end{array}
\]  

(1)

Now the Serre-Swan theorem shows that isomorphism classes of finite-dimensional complex vector bundles over $X$ correspond to isomorphism classes of finitely generated projective modules over the algebra $C(X)$ of continuous complex-valued functions on $X$. Therefore $K^0(X)$ can be equivalently defined as the enveloping group of the semigroup of isomorphism classes of such projective modules over the $C^*$-algebra $C(X)$, i.e. as the algebraic $K_0$-group $K_0(C(X))$ of the unital ring $C(X)$.

The reduced suspension of a topological space corresponds to the following operation on $C^*$-algebras or Banach algebras. Given such an algebra $A$, the suspension $SA$ is defined as the (non-unital!) algebra $C_0((0,1),A)$ of continuous $A$-valued functions on the unit interval, vanishing at 0 and 1. Using $n$-fold suspensions we can, as for spaces, define higher groups $K_n(A)$ as $\widetilde{K_0}(S^nA)$. Of course, $S^nA$ is a non-unital algebra. Just as for non-compact spaces, we have to define $K_0(I)$ for a non-unital algebra $I$ in an awkward way as $\widetilde{K_0}(\tilde{I})$, where $\tilde{I}$ denotes $I$ with unit adjoined.

Some of the standard proofs of Bott periodicity carry over immediately from locally compact spaces to Banach algebras. So does the proof of the $K$-theory exact sequence (2.1) associated to an extension of the form

\[
0 \rightarrow I \xrightarrow{i} A \xrightarrow{q} B \rightarrow 0
\]

of Banach algebras. It takes the following form

\[
\begin{array}{c}
K_0(I) \xrightarrow{K(i)} K_0(A) \xrightarrow{K(q)} KK_0(B) \\
\uparrow \quad \quad \quad \quad \quad \downarrow \\
K_1(B) \xrightarrow{K(q)} K_1(A) \xrightarrow{K(i)} K_1(I)
\end{array}
\]  

(2)
This generalizes the sequence (2.1) if we take $A = \mathcal{C}(X)$, $B = \mathcal{C}(Y)$ and for $I$ the ideal in $\mathcal{C}(X)$ consisting of functions vanishing on $Y$. Note that of course $I$ is not unital in general, so again we have to define $K_0(I)$ in an artificial way as $\tilde{K}_0(I)$.

### 2.2 The dual theory: $\text{Ext}$ and $K$-homology

One of the major motivations for the interest in topological $K$-theory was of course its use in the formulation and the proof of the celebrated Atiyah-Singer index theorem [4]. It is natural to try to interpret the index of an elliptic operator on a vector bundle as a pairing between a $K$-theory class and a class in a dual “$K$-homology” theory (for instance between the $K$-theory class given by the symbol and the $K$-homology class given by the extension of pseudodifferential operators, or as the pairing between a $K$-homology class defined by an untwisted elliptic operator and the $K$-theory class of a vector bundle by which it is twisted). Atiyah proposed abstract elliptic operators over a space $X$ as possible cycles for a dual theory $\text{Ell}(X)$. This proposal was taken up and developed by Kasparov in [40].

 Independently and earlier however such a theory was discovered by Brown-Douglas-Fillmore in connection with the investigation of essentially normal operators. Their theory $\text{Ext}(X)$ is based on extensions of the form

$$0 \to \mathcal{K} \to E \to \mathcal{C}(X) \to 0$$

where $\mathcal{K}$ is the standard algebra of compact operators on a separable infinite-dimensional Hilbert space $H$ and $E$ is a subalgebra of $\mathcal{L}(H)$ (necessarily a $C^*$-algebra). As equivalence relation for such extensions they used unitary equivalence. There is a natural direct sum operations on such extensions, based on the fact that the algebra $M_2(\mathcal{K})$ of $2 \times 2$ matrices over $\mathcal{K}$ is isomorphic to $\mathcal{K}$.

Every essentially normal operator $T \in \mathcal{L}(H)$ (essentially normal means that $T^*T - TT^*$ is compact) defines such an extension by choosing for $E$ the $C^*$-algebra generated by $T$ together with $\mathcal{K}$ and taking for $X$ the essential spectrum of $T$, i.e. the spectrum of the image of $T$ in $\mathcal{L}(H)/\mathcal{K}$.

The theory $\text{Ext}(A)$ was also developed to some extent for more general $C^*$-algebras $A$ in place of $\mathcal{C}(X)$. A very important result in that connection is Voiculescu’s theorem. It asserts that, for separable $A$, any two trivial (i.e. admitting a splitting by a homomorphism) extensions are equivalent and shows that their class gives a neutral element in $\text{Ext}(A)$. Some important questions like the homotopy invariance of $\text{Ext}$ remained open.

The pairing between an element $e$ in $\text{Ext}(X)$ and an element $\zeta$ of $K^1(X)$, represented by an invertible element $z$ in $M_n(\mathcal{C}(X))$ can be nicely described in terms of the Fredholm index. In fact, any preimage of $z$ in the extension defining $e$ is a Fredholm operator. The pairing $\langle \zeta, e \rangle$ is exactly given by its index.
3 $KK$-theory and $E$-theory

It was Kasparov who revolutionized the subject by his fundamental work in [41] (independently, at about the same time, Pimsner-Popa-Voiculescu had started to develop a bivariant $Ext$-theory, [62]). Formally Kasparov’s bivariant theory is based on a combination of the ingredients of $K$-theory and $K$-homology. The elements of his bivariant groups $KK(A,B)$, for $C^*$-algebras $A$ and $B$ are represented by a “virtual” finitely generated projective module over $B$ (given as the “index” of an abstract elliptic operator) on which $A$ acts by endomorphisms.

More specifically, Kasparov works with Hilbert $B$-modules. This is a straightforward generalization of an ordinary Hilbert space over $\mathbb{C}$ with $\mathbb{C}$-valued scalar product to a space (i.e. module) over $B$ with a $B$-valued inner product $(\cdot \mid \cdot)$. The axioms for this inner product are quite natural and we don’t want to reproduce them here. One uses the notation $\mathcal{L}(H)$ to denote the algebra of all operators on $H$ that admit an adjoint (such operators are automatically bounded and $B$-module maps). The closed subalgebra of $\mathcal{L}(H)$ generated by all rank 1 operators of the form $\theta_{x,y}: z \mapsto x (y \mid z)$ is denoted by $K(H)$ (the algebra of “compact” operators on $H$). It is a closed ideal in $\mathcal{L}(H)$.

Kasparov then considers triples of the following form.

**Definition 3.1.** (a) An odd $A$ - $B$ Kasparov module is a triple $(H, \varphi, F)$ consisting of a countably generated Hilbert $B$-module $H$, a $*$-homomorphism $\varphi : A \to \mathcal{L}(H)$ and a selfadjoint $F \in \mathcal{L}(H)$ such that, for each $x \in A$ the following expressions are in $K(H)$:

- $\varphi(x)(F - F^2)$ \hspace{1cm} ($KM1$)
- $F\varphi(x) - \varphi(x)F$ \hspace{1cm} ($KM2$)

(b) An even $A$ - $B$ Kasparov module is a triple $(H, \varphi, F)$ satisfying exactly the same conditions as under (a) where however in addition $H = H_+ \oplus H_-$ is $\mathbb{Z}/2$ graded, $\varphi$ is of degree 0 (i.e. $\varphi(x)$ respects the decomposition of $H$ for each $x \in A$) and $F$ is odd (i.e. maps $H_+$ to $H_-$ and vice versa).

One denotes by $\mathcal{E}_0(A,B)$ and $\mathcal{E}_1(A,B)$ the sets of isomorphism classes of even, respectively odd $A$ - $B$ Kasparov modules.

Kasparov defines two equivalence relations on these sets of modules:

- compact perturbation of $F$ together with stabilization by degenerate elements (i.e. for which the expressions in ($KM1$), ($KM2$) are exactly 0)
- homotopy

He shows the quite non-trivial result that both equivalence relations do in fact coincide. If we divide $\mathcal{E}_0(A,B)$ and $\mathcal{E}_1(A,B)$ with respect to these equivalence relations, we obtain abelian groups $KK_0(A,B)$ and $KK_1(A,B)$ (where the addition is induced by direct sum of Kasparov modules).
Specializing to the case where one of the variables of $KK$ is $\mathbb{C}$, we obtain $K$-theory and $K$-homology:

$$KK_\ast(C, A) = K_\ast(A) \quad KK_\ast(A, \mathbb{C}) = K^\ast(A)$$

To understand the connection with the usual definition of $K_0$, as sketched in chapter 1, assume that $A$ is unital. An element of $K_0(A)$ is then represented by a finitely generated projective module $M$ over $A$. Considering the Kasparov module $(M \oplus 0, \varphi, 0)$, where $\varphi$ is the natural action of $A$ on $M$, we obtain an element of $KK_0(A, B)$. The crucial point of Kasparov’s theory is the existence of an intersection product (which of course generalizes the pairing between $K$-theory and $K$-homology).

**Theorem 3.2.** There is an associative product

$$KK_i(A, B) \times KK_j(B, C) \rightarrow KK_{i+j}(A, C)$$

$(i, j \in \mathbb{Z}/2; A, B$ and $C$ C*-algebras), which is additive in both variables.

Further basic properties are described in the following theorem.

**Theorem 3.3.** The bivariant theory $KK_\ast$ has the following properties

(a) There is a bilinear, graded commutative, exterior product

$$KK_i(A_1, A_2) \times KK_j(B_1, B_2) \rightarrow KK_{i+j}(A_1 \otimes A_2, B_1 \otimes B_2)$$

(using the minimal or maximal tensor product of C*-algebras).

(b) Each homomorphism $\varphi : A \rightarrow B$ defines an element $KK(\varphi)$ in the group $KK_0(A, B)$. If $\psi : B \rightarrow C$ is another homomorphism, then

$$KK(\psi \circ \varphi) = KK(\varphi) \cdot KK(\psi)$$

$KK_\ast(A, B)$ is a contravariant functor in $A$ and a covariant functor in $B$. If $\alpha : A' \rightarrow A$ and $\beta : B \rightarrow B'$ are homomorphisms, then the induced maps, in the first and second variable of $KK_\ast$, are given by left multiplication by $KK(\alpha)$ and right multiplication by $KK(\beta)$.

(c) $KK_\ast(A, A)$ is, for each C*-algebra $A$, a $\mathbb{Z}/2$-graded ring with unit element $KK(id_A)$.

(d) The functor $KK_\ast$ is invariant under homotopies in both variables.

(e) The canonical inclusion $\iota : A \rightarrow K \otimes A$ defines an invertible element in $KK_0(A, K \otimes A)$. In particular, $KK_\ast(A, B) \cong KK_\ast(K \otimes A, B)$ and $KK_\ast(B, A) \cong KK_\ast(B, K \otimes A)$ for each C*-algebra $B$ (recall that $K$ denotes the standard algebra of compact operators on a separable infinite-dimensional Hilbert space).

(f) (Bott periodicity) There are canonical elements in $KK_1(A, SA)$ and in $KK_1(SA, A)$ which are inverse to each other (recall that the suspension $SA$ of $A$ is defined as the algebra $C_0((0,1), A)$ of continuous $A$-valued functions on $[0,1]$ vanishing in 0 and 1).
Every odd Kasparov $A$-$B$ module $(H, \varphi, F)$ gives rise to an extension

$$0 \to \mathcal{K}(H) \to E \to A' \to 0$$

by putting $P = 1/2(F + 1)$, $E = P\varphi(A)P$ and taking $A'$ to be the image of $E$ in $\mathcal{L}(H)/\mathcal{K}(H)$. Using stabilization, i.e. adding a degenerate Kasparov module to $(H, \varphi, F)$, we can always arrange that $\mathcal{K}(H) \cong \mathcal{K} \otimes B$, $\varphi$ is injective and $A' \cong A$. We thus get an extension

$$0 \to \mathcal{K} \otimes B \to E \to A \to 0$$

Conversely, it is easy to see using the Stinespring theorem that every extension admitting a completely positive splitting arises that way. In particular, every such extension $E : 0 \to I \to A \to B \to 0$ defines an element of $\text{KK}_1(B, I)$, which we denote by $\text{KK}(E)$.

For computations of $\text{KK}$ and other $K$-theoretic invariants, the following long exact sequences associated to an extension are an indispensable tool.

**Theorem 3.4.** Let $D$ be any separable $C^*$-algebra. Every extension of $C^*$-algebras admitting a completely positive linear splitting

$$E : 0 \to I \xrightarrow{i} A \xrightarrow{q} B \to 0$$

induces exact sequences in $\text{KK}_*(D, \cdot)$ and $\text{KK}_*(\cdot, D)$ of the following form:

$$
\begin{array}{ccc}
\text{KK}_0(D, I) & \xrightarrow{\text{KK}(i)} & \text{KK}_0(D, A) \xrightarrow{\text{KK}(q)} \text{KK}_0(D, B) \\
\uparrow & & \downarrow \\
\text{KK}_1(D, B) & \xleftarrow{\text{KK}(q)} & \text{KK}_1(D, A) \xleftarrow{\text{KK}(i)} \text{KK}_1(D, I)
\end{array}
$$

and

$$
\begin{array}{ccc}
\text{KK}_0(I, D) & \xleftarrow{\text{KK}(i)} & \text{KK}_0(A, D) \xleftarrow{\text{KK}(q)} \text{KK}_0(B, D) \\
\downarrow & & \uparrow \\
\text{KK}_1(B, D) & \xleftarrow{\text{KK}(q)} & \text{KK}_1(A, D) \xleftarrow{\text{KK}(i)} \text{KK}_1(I, D)
\end{array}
$$

(3)

The vertical arrows in (3) and (4) are (up to a sign) given by right and left multiplication, respectively, by the class $\text{KK}(E)$ described above.

A standard strategy to establish these long exact sequences, used for the first time in [24], goes as follows. Establish first, for any star homomorphism $\alpha : A \to B$ mapping cone exact sequences of the form

$$\text{KK}_0(D, C_\alpha) \to \text{KK}_0(D, A) \xrightarrow{\text{KK}(\alpha)} \text{KK}_0(D, B)$$

and

$$\text{KK}_0(C_\alpha, D) \leftarrow \text{KK}_0(A, D) \xleftarrow{\text{KK}(\alpha)} \text{KK}_0(B, D)$$

Theorem 3.4. Let $D$ be any separable $C^*$-algebra. Every extension of $C^*$-algebras admitting a completely positive linear splitting

$$E : 0 \to I \xrightarrow{i} A \xrightarrow{q} B \to 0$$

induces exact sequences in $\text{KK}_*(D, \cdot)$ and $\text{KK}_*(\cdot, D)$ of the following form:

$$
\begin{array}{ccc}
\text{KK}_0(D, I) & \xrightarrow{\text{KK}(i)} & \text{KK}_0(D, A) \xrightarrow{\text{KK}(q)} \text{KK}_0(D, B) \\
\uparrow & & \downarrow \\
\text{KK}_1(D, B) & \xleftarrow{\text{KK}(q)} & \text{KK}_1(D, A) \xleftarrow{\text{KK}(i)} \text{KK}_1(D, I)
\end{array}
$$

and

$$
\begin{array}{ccc}
\text{KK}_0(I, D) & \xleftarrow{\text{KK}(i)} & \text{KK}_0(A, D) \xleftarrow{\text{KK}(q)} \text{KK}_0(B, D) \\
\downarrow & & \uparrow \\
\text{KK}_1(B, D) & \xleftarrow{\text{KK}(q)} & \text{KK}_1(A, D) \xleftarrow{\text{KK}(i)} \text{KK}_1(I, D)
\end{array}
$$

The vertical arrows in (3) and (4) are (up to a sign) given by right and left multiplication, respectively, by the class $\text{KK}(E)$ described above.
where $C_\alpha$ denotes the mapping cone for $\alpha$. Using suspensions, these sequences can be extended to long exact sequences. To prove the exact sequences in 3.4 it then remains to show that the natural inclusion map from the ideal $I$ into the mapping cone $C_q$ for the quotient map $q$ in the given extension $E$, gives an isomorphism in $KK$.

Kasparov in fact treats his theory more generally in the setting of $C^*$-algebras with a $\mathbb{Z}/2$-grading. Using Clifford algebras he gets a very efficient formalism leading for instance to an elegant proof of Bott periodicity which also admits useful generalizations. His original proof of the existence and associativity in this setting however is a technical tour de force which is very difficult to follow.

A simple construction of the product was based in [26] on a rather different description of $KK$ which also revealed some of the abstract properties of the theory.

Given a $C^*$-algebra $A$, let $QA$ denote the free product $A * A$. It is defined by the universal property that there are two inclusion maps $\iota, \bar{\iota}: A \to A * A$, such that, given any two homomorphisms $\alpha, \bar{\alpha}: A \to B$, there is a unique homomorphism $\alpha * \bar{\alpha}: A * A \to B$ such that $\alpha = (\alpha * \bar{\alpha}) \circ \iota$ and $\bar{\alpha} = (\alpha * \bar{\alpha}) \circ \bar{\iota}$.

Thus in particular there is a natural homomorphism $\pi = id * id : QA \to A$. We denote by $qA$ the kernel of $\pi$. We obtain an extension

$$0 \to qA \to QA \to A \to 0$$

which is trivial and in fact has two different natural homomorphism splittings given by $\iota$ and $\bar{\iota}$. The following theorem holds

**Theorem 3.5.** The group $KK_0(A, B)$ can be described as $[qA, K \otimes B]$ (where $[X, Y]$ denotes the set of homotopy classes of homomorphisms from $X$ to $Y$).

$KK_1$ can be obtained from this by taking suspensions in one of the two variables. The proof of the theorem uses the following observation. Given an even $A$-$B$ Kasparov module $(H, \varphi, F)$, one can always arrange that $F^2 = 1$ and that $K(H) \cong K \otimes A$. Then, setting $\bar{\varphi} = AdF \circ \varphi$, we get a pair of homomorphisms $\varphi, \bar{\varphi}$ from $A$ to $L(H)$, therefore a unique homomorphism from the free product $QA$ to $L(H)$. Since by the condition on a Kasparov module, $\varphi(x) - \bar{\varphi}(x)$ is in $K(H)$ for each $x \in A$, this homomorphism has to map the ideal $qA$ to $K(H) \cong K \otimes A$.

The existence and associativity of the product follows from the following theorem which can be proved using standard $C^*$-algebra techniques.

**Theorem 3.6.** The natural map $\pi : q(qA) \to qA$ is a homotopy equivalence after stabilizing by $2 \times 2$-matrices, i.e. there exists a homomorphism $\eta : qA \to M_2(q(qA))$ such that $\pi \circ \eta$ and $\eta \circ \pi$ are both homotopic to the natural inclusions of $qA$, $q(qA)$ into $M_2(q(qA))$, $M_2(q(qA))$, respectively.

The product between $KK_0(A_1, A_2) = [qA_1, K \otimes A_2]$ and $KK_0(A_2, A_3) = [qA_2, K \otimes A_3]$ is then defined as follows:
Let $\varphi : qA_1 \to K \otimes A_2$ and $\psi : qA_2 \to K \otimes A_3$ represent elements of these two groups. The product is defined as the homotopy class of the following composition

$$q(qA_1) \xrightarrow{q(\varphi)} q(K \otimes A_2) \xrightarrow{id \otimes \psi} K \otimes K \otimes A_3$$

The arrow in the middle is the natural map. If we now identify $K \otimes K \otimes A_3$ with $K \otimes A_3$ and $q(qA_1)$ with $qA_1$ using Theorem 3.6 we obtain the desired element of $KK_0(A_1, A_3)$. The associativity of this product is more or less obvious.

Building on this construction, Zekri, [81], used an algebra $\varepsilon A$ (which in fact is isomorphic to the crossed product $qA \rtimes \mathbb{Z}/2$) to describe $KK_n(A, B)$ as $[\varepsilon^n A, K \otimes A]$ (where $\varepsilon^n A = \varepsilon(\varepsilon(\ldots \varepsilon A \ldots)$). The algebra $\varepsilon A$ is the universal ideal in an extension of $A$ admitting a completely positive splitting. Therefore every $n$-step extension

$$0 \to B \to E_1 \to \cdots \to E_n \to A \to 0$$

with completely positive splittings has a classifying map $\varepsilon^n A \to B$ and gives an element in $KK_n(A, B)$. Zekri showed that the Kasparov product of such elements corresponds to the Yoneda product of the original extensions.

It was noted in [25] that $KK$ is a functor which is universal with respect to three natural properties in the following way. Let $E$ be a functor from the category of separable $C^*$-algebras to the category of abelian groups satisfying:

- $E$ is homotopy invariant, i.e. two homotopic homomorphisms $A \to B$ induce the same map $E(A) \to E(B)$
- $E$ is stable, i.e. the natural inclusion $A \to K \otimes A$ induces an isomorphism $E(A) \to E(K \otimes A)$
- $E$ is split exact, i.e. every extension $0 \to I \to A \overset{q}{\to} B \to 0$ which splits in the sense that there is a homomorphism $A \to E$ which is a right inverse for $q$ induces a split exact sequence $0 \to E(I) \to E(A) \to E(B) \to 0$

Then $KK$ acts on $E$, i.e. every element of $KK(A, B)$ induces a natural map $E(A) \to E(B)$.

A more streamlined formulation of this result was given by Higson. He noted that $KK$ defines an additive category (i.e. a category where the Hom-sets are abelian groups and the product of morphisms is bilinear) by taking separable $C^*$-algebras as objects and $KK_0(A, B)$ as set of morphisms between the objects $A$ and $B$. Then $KK$ is the universal functor into an additive category which is homotopy invariant, stable and split exact in both variables.

More importantly, using abstract ideas from category theory, Higson constructed a new theory, later called $E$-theory. One shortcoming of $KK$ is the fact that only extensions with a completely positive linear splitting induce long exact sequences. In fact, an important counterexample has been constructed by G.Skandalis, [73], showing that there exist extensions that do not give rise to a long exact sequence in $KK$ (this example also limits the range of validity...
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for some other important properties of $KK$). Higson now takes the additive category $KK$ and forms a category of fractions $E(A,B)$ by inverting in $KK$ all morphisms induced by an inclusion $I \to A$ of a closed ideal $I$ into a $C^*$-algebra $A$, for which the quotient $A/I$ is contractible. The category $E$ is additive with a natural functor from the category of separable $C^*$-algebras into $E$ (which factors over $KK$). In $E$, every extension of $C^*$-algebras (not necessarily admitting a completely positive splitting) induces long exact sequences in $E(\cdot, D)$ and $E(D, \cdot)$ as in (3) and (4). Moreover, $E$ is the universal functor into an additive category which is homotopy invariant, stable and half-exact.

Later, a more concrete description of $E(A,B)$ was given by Connes and Higson in terms of what they call asymptotic morphisms from $A$ to $B$. An asymptotic morphism from $A$ to $B$ is a family of maps $(\phi_t, t \in \mathbb{R})$ from $A$ to $B$ such that the expressions $\phi_t(x)\phi_t(y) - \phi_t(xy)$, $\phi_t(x) + \lambda \phi_t(y) - \phi_t(x+\lambda y)$, $\phi_t(x)^* - \phi_t(x^*)$ all tend to 0 for $t \to \infty$ and $x,y \in A$, $\lambda \in \mathbb{C}$. Connes and Higson then define $E(A,B)$ as

$$E(A,B) = \left[ [A \otimes C_0(\mathbb{R}), K \otimes B \otimes C_0(\mathbb{R})] \right]$$

where $[[\cdot, \cdot]]$ denotes the set of homotopy classes of asymptotic morphisms. The category $E$ has the properties of $KK$ listed in 3.3, except that the exterior product in 3.3 (a) only exists with respect to the maximal tensor product of $C^*$-algebras.

To understand the connection between the bivariant theories $KK$ and $E$ on the one hand, and (monovariant) $K$-theory on the other hand, the so-called universal coefficient theorem (UCT) is quite useful. Let $\mathcal{N}$ denote the class of $C^*$-algebras which are isomorphic in $KK$ to an abelian $C^*$-algebra. This class is in fact quite large. Using some of the standard computations of $KK$-groups it can be shown that it is invariant under extensions, inductive limits, crossed products by $\mathbb{Z}$ or amenable groups etc. Therefore it contains many if not most of the algebras occurring in applications, since those are often constructed using operations under which $\mathcal{N}$ is stable. Moreover, the convolution $C^*$-algebra for every amenable groupoid is in $\mathcal{N}$, [76]. It is however an open problem, if every nuclear $C^*$-algebra is in $\mathcal{N}$.

The universal coefficient theorem (UCT) is the following formula

**Theorem 3.7.** ([71]) Let $A$ and $B$ be separable $C^*$-algebras with $A$ in $\mathcal{N}$. Then there is an exact sequence

$$0 \to \text{Ext}_1(K_*, A, K_*) \to KK_0(A, B) \to \text{Hom}_0(K_* A, K_* B) \to 0$$

where $\text{Ext}_1$ is odd, i.e. pairs $K_0$ with $K_1$ and $K_1$ with $K_0$ and $\text{Hom}_0$ is even.

Of course there is an analogous formula for $KK_1$. If $A = B$, then the image of the $\text{Ext}$-term on the left is a nilpotent ideal (with square 0) in the ring $KK_0(A, A)$. The product of elements in the $\text{Ext}$-term with elements in the $\text{Hom}$ term is obvious. Moreover, the extension splits (unnaturally) as an extension of rings.
If $A$ is in $\mathcal{N}$, then $KK(A, B) = E(A, B)$, therefore the UCT holds in exactly the same generality for $E$-theory. The counterexample of Skandalis, [73], shows that there are $C^*$-algebras $A$ for which the UCT for $KK(A, \cdot)$ fails and which therefore are not in $\mathcal{N}$. On the other hand, a separable $C^*$-algebra $A$ satisfies the UCT for $KK(A, B)$ with arbitrary $B$ if and only if it is in $\mathcal{N}$.

4 Other bivariant theories on categories related to $C^*$-algebras

4.1 Equivariant $KK$-and $E$-theory

A new element however was introduced by Kasparov in [44] where he introduced equivariant $KK$-theory with respect to the action of any locally compact group and applied this theory to prove important cases of the Novikov conjecture (for discrete subgroups of connected Lie groups). The equivariant theory for locally compact groups is technically much more delicate than for compact groups (where for instance the operator $F$ in a Kasparov module can be assumed to be invariant). Equivariant $E$-theory can be defined in a very natural way using equivariant asymptotic morphisms [33].

The equivariant theory plays an important role in the study (in fact already in the formulation) of the Baum-Connes and Novikov conjectures.

Equivariant $KK$-theory for the action of a Hopf $C^*$-algebra $H$ has been studied in [5] where also the following elegant duality result is proved for crossed products by the action of the two Hopf algebras $H = C^*_{\text{red}}G$ and $\hat{H} = C_0(G)$ associated with a locally compact group $G$.

**Theorem 4.1.** Let $H$ and $\hat{H}$ be the two Hopf $C^*$-algebras associated with a locally compact group $G$ and let $A$ and $B$ be $C^*$-algebras with an action of $H$. Then there is an isomorphism

$$KK_H(A, B) \cong KK_{\hat{H}}(A \rtimes_r H, B \rtimes_r H)$$

The same holds if we interchange $H$ and $\hat{H}$ (and the action of $\hat{H}$ is non-degenerate).

4.2 $KK$-theory for $C^*$-algebras over a topological space

In his work on the Novikov conjecture, Kasparov used, besides the equivariant $KK$-theory for the action of a locally compact group in addition a $KK$-theory on a category of $C^*$-algebras which are in a well defined technical sense bundles over a fixed locally compact space $X$. A generalization of this equivariant theory to $T_0$-spaces was used by Kirchberg in his work on the classification of non-simple nuclear purely infinite $C^*$-algebras (the $T_0$-space in question here being the ideal space of the given $C^*$-algebra.

The equivariant theories for the action of a group and the action of a space can be generalized simultaneously to a $KK$-theory which is equivariant for the action of a groupoid, [51].
4.3 $KK$-theory for projective systems of $C^*$-algebras

If $X$ is a noncompact locally compact space, then the algebra $\mathcal{C}(X)$ is not a $C^*$-algebra, but an inductive limit of $C^*$-algebras (for instance it can be viewed as the projective limit of the projective system of $C^*$-algebras $(\mathcal{C}(K))_K$, where $K$ ranges over all compact subsets of $X$). There are many other natural examples of projective limits of $C^*$-algebras. It is therefore natural to look for a definition of $KK$ or $E$-theory to such algebras. This has been done first by Weidner and independently partially by Phillips in [79], [60]. Recently A. Bonkat has developed in his thesis [9] $KK$-theories on various categories of projective systems of $C^*$-algebras. The objects of the categories he considers, are projective systems of $C^*$-algebras admitting a cofinal countable subsystem. For different choices of morphism sets he obtains as special cases the bivariant theories of Weidner and Phillips but also the theories of Kasparov and Kirchberg for $C^*$-algebras fibered over a $T_0$-space mentioned in section 4.2. Another interesting example where Bonkat’s approach applies is the category of 1-step extensions of $C^*$-algebras. Bonkat proves a UCT for $KK$ on this category which allows him to compute these groups quite explicitly in interesting cases.

4.4 Bivariant theories as triangulated categories

Methods from category theory were first used by N. Higson, when he constructed $E$-theory as a category of fractions from $KK$-theory. It has turned out later that similar constructions of quotients of bivariant theories can be used in different instances. Usually one forms such quotients in order to enforce certain properties on a bivariant theory. This means that one inverts certain maps which one wants to induce isomorphisms in the theory. More or less equivalently (using mapping cones) one divides by a “null”-subcategory. The framework best suited for that purpose seems to be the one of triangulated categories. A triangulated category is an additive category with a suspension operation on objects and abstract mapping cone sequences (called “triangles”) satisfying a rather long list of compatibility relations. For triangulated categories there is a very smooth way to form quotient categories which are again triangulated. Technically, this is described as follows.

**Definition 4.2.** Let $F : \mathcal{T}_1 \to \mathcal{T}_2$ be a functor between triangulated categories preserving the triangulated structure. One denotes by $\ker(F)$ the full triangulated subcategory of $\mathcal{T}_1$ whose objects map to objects isomorphic to 0 in $\mathcal{T}_2$.

**Theorem 4.3.** Let $\mathcal{T}$ be an essentially small triangulated category and $\mathcal{R}$ a triangulated subcategory. Then there exists a triangulated quotient category (Verdier quotient) $\mathcal{T}/\mathcal{R}$ and a functor $F : \mathcal{T} \to \mathcal{T}/\mathcal{R}$ preserving the triangulated structure, with the universal property that $\mathcal{R} \subset \ker(F)$.
$KK$-theory and $E$-theory as well as some other variants of bivariant theories can be viewed as a triangulated category. The technology of triangulated categories and their quotient categories has been applied in this connection first by Puschnigg, [65] to construct his “local” cyclic homology as a quotient of the bivariant entire theory. Triangulated categories have also been used by Valqui [77] as a framework for bivariant periodic cyclic theory.

The method to use triangulated categories and their quotient categories in order to enforce certain properties on a bivariant theory has been used systematically recently also by A.Thom in his thesis, [74]. The basic triangulated category in his approach is stable asymptotic homotopy as defined by Connes and Dadarlat [16], [32]. The set of morphisms between two $C^*$-algebras $A$ and $B$ is defined in this category as

$$\lim_{n \to \infty} [[S^n A, S^n B]]$$

where $S^n$ denotes $n$-fold suspension and $[\cdot ; \cdot]$ homotopy classes of asymptotic morphisms. In the stable asymptotic homotopy category the mapping cone $C_q$ of the quotient map $q$ in an extension $0 \to I \to E \xrightarrow{q} A \to 0$ is isomorphic to $I$. Thom constructs various bivariant theories as quotient categories of the fundamental stable homotopy category. In that way he obtains for instance bivariant connective $K$-theory and bivariant singular homology.

Using this approach one can also construct $E$-theory or similar theories as quotients of the stable asymptotic homotopy category. We mention also that the approach in [28] gives another method to construct bivariant theories for many categories of $C^*$-algebras or other algebras with specified properties.

5 Applications

Many computations of $K$-theoretic invariants for $C^*$-algebras can be greatly simplified and generalized using bivariant $K$-theory. This is true for many of the computations of the early days, e.g. [63], [64], [13], [21] which were based at the beginning on more concrete considerations involving idempotents or invertible elements in algebras. The general method to compute the $K$-theoretic invariants for a given algebra $A$ consists in constructing an isomorphism in $KK$ with an algebra $B$ for which this computation is simpler.

On the other hand $KK_*$ defines a novel recipient for many new invariants that could not be defined before, such as bivariant symbol classes, equivariant $K$-homology classes, classes classifying extensions or bivariant classes associated with bundles and many more.

5.1 Index theorems

Every elliptic pseudodifferential operator $T$ of order 0 from sections of a vector bundle $E_1$ over $X$ to sections of another bundle $E_2$ determines, by the very
Theorem 5.1. We have $\sigma(T) \cdot [\partial_X]$ determined by $[T]$ in $KK_0(C(X), \mathbb{C})$. In fact, taking $H = H_1 \oplus H_2$, where $H_i$ denotes the Hilbert space of $L^2$-sections in $E_i$, we may always assume that $T$ is normalized so that $1 - T^* T$ and $1 - TT^*$ are compact. We let then act $C(X)$ by multiplication on $H$ and put

$$F = \begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix}$$

Kasparov proved an especially elegant and illuminating form of the index theorem which determines this $K$-homology class. 

If $X$ is a (not necessarily closed) manifold, then the cotangent bundle $T^* X$ considered as a manifold carries an almost complex structure. Therefore there is the Dolbeault operator $D = \partial + \bar{\partial} : V \to V$, where $V$ denotes the space of smooth sections with compact support of the bundle of differential forms $\Lambda^0, \ast$ associated to the almost complex structure. $D$ extends to a selfadjoint operator and we can define the bounded operator

$$F = \frac{D}{\sqrt{1 + D^2}}$$

on the Hilbert space $H$ of $L^2$-sections of $A^0, \ast$. In fact, $H$ splits naturally as a direct sum $H = H_+ \oplus H_-$ of sections of even and odd forms. Moreover, $C_0(X)$ acts on $H$ by multiplication. We therefore get a natural Kasparov module $(H, \varphi, F)$ and thus an element of $KK(C_0(T^* X), \mathbb{C})$, denoted by $[\partial_X]$ (this element and variants of it play an important role in the work of Kasparov on the Novikov conjecture as the so called Dirac element).

It is important to note that, in the case of $X = \mathbb{R}^n$, the product with this element induces the Bott periodicity map $K_\ast(C_0(\mathbb{R}^{2n})) \to K_\ast(\mathbb{C})$. In this case there is a natural inverse $\eta$ in $KK(\mathbb{C}, C_0(T^* X))$ to $[\partial_X] \in KK(C_0(T^* X), \mathbb{C})$, called the dual Dirac element (the “Fourier transform” of $[\partial_X]$).

Assume now that $X$ is a closed manifold and $T$ is an elliptic pseudodifferential operator of order 0 from $L^2$-sections of a vector bundle $E_1 \to X$ to sections of $E_2 \to X$. Let $T^* X \to X$ be the projection map for the cotangent bundle. The (full) symbol $\sigma(T)$ of $T$ can be viewed as a morphism of vector bundles from $\pi^* E_1$ to $\pi^* E_2$. We obtain a Kasparov module $(L, \psi, \sigma)$, where $L = L_1 \oplus L_2$ denotes the space of $L^2$-sections of $\pi^* (E_1 \oplus E_2)$, $\psi$ denotes action by multiplication and $\sigma$ is the sum of $\sigma(T) : L_1 \to L_2$ and $\sigma(T)^* : L_2 \to L_1$. This gives an element denoted by $[\sigma(T)]$ in $KK_0(C_0(X), C_0(T^* X))$. The product with the class $[1]$ of the trivial line bundle in $KK(\mathbb{C}, C_0(X))$ gives the usual $K$-theory symbol class $[\Sigma(T)]$ used in the formulation of the Atiyah-Singer theorem.

Kasparov now shows that the $K$-homology class $[T]$ in $KK_0(C(X), \mathbb{C})$ determined by $T$ is given by the following formula.

**Theorem 5.1.** We have $[T] = [\sigma(T)] \cdot [\partial_X]$.

Note that the usual index of $T$ is simply obtained by pairing the $K$-homology class $[T]$ with the $K_0$-class given by $1$ (the trivial line bundle). Thus
the standard form of the Atiyah-Singer theorem saying that the analytic index $\text{ind}_a(T)$ equals the topological index $\text{ind}_t(T)$ follows from 5.1.

In fact, the analytic index is given by $\text{ind}_a(T) = 1 \cdot [T]$ and therefore by Kasparov’s theorem by

$$1 \cdot [\sigma(T)] \cdot [\bar{\partial} X] = [\Sigma(T)] \cdot [\bar{\partial} X]$$

Using an embedding of $X$ in $\mathbb{R}^n$ and identifying the cotangent bundle of a tubular neighbourhood $N$ of $X$ in this embedding with a bundle over $T^*X$, we get the following diagram whose commutativity is easy to check:

$$K^*(T^*X) \cong K^*(T^*N) \to K^*(\mathbb{R}^{2n})$$

$$\downarrow [\bar{\partial} X] \quad \quad \quad \downarrow [\bar{\partial} \mathbb{R}^n]$$

$$\mathbb{Z} \quad \quad \quad \quad \mathbb{Z}$$

($K^*$ here means $K$-theory with compact supports, e.g. $K^*(\mathbb{R}^n) = K_0(C_0(\mathbb{R}^n))$).

According to Kasparov’s theorem 5.1, the first vertical arrow applied to $[\Sigma(T)]$ gives the analytic index, while the composition of the first horizontal and the second vertical arrow gives the usual definition of the topological index.

To make the connection with the formulation of the index formula using de Rham cohomology and differential forms, Kasparov notes that the Chern character $\text{ch}([\bar{\partial} X])$ is Poincaré dual in de Rham cohomology for $T^*X$ to the Todd class of the complexified cotangent bundle of $X$ (viewed as a bundle over $T^*X$). Thus applying the Chern character one obtains the usual formula

$$\text{ind}_a(T) = \int_{T^*M} Td(T^*M \otimes \mathbb{C}) \wedge \text{ch}([\Sigma(T)])$$

From this theorem or at least from its proof, one obtains many other index theorems. Instances are the index theorem for families or the Miscenko-Fomenko index theorem for pseudodifferential operators with coefficients in a $C^*$-algebra.

Another important and typical index theorem using $C^*$-algebras is the longitudinal index theorem for foliated manifolds [18]. A foliated manifold is a smooth compact manifold $M$ together with an integrable subbundle $F$ of the tangent bundle for $M$. The algebra of longitudinal pseudodifferential operators on $(M, F)$ (differentiation only in direction of the leaves) can be completed to a $C^*$-algebra $\Psi_t$. The principal symbol of an element in $\Psi_t$ is a function on the dual bundle $F^*$. The kernel of the symbol map $\sigma$ is the foliation $C^*$-algebra $C^*(M, F)$ which is something like a crossed product of $C(M)$ by translation by $\mathbb{R}^k$ in the direction of $F$ with holonomy resolved. One obtains an exact sequence

$$0 \to C^*(M, F) \to \Psi_t \xrightarrow{\sigma} C_0(F^*) \to 0$$

(5)

A longitudinal pseudodifferential operator $T$ is called elliptic if its image $\sigma(T)$ in (matrices over) $C_0(F^*)$ (its principal symbol) is invertible. The analytic
index of $T$ is by definition the image under the boundary map in the $K$-theory long exact sequence, of the class in $K_1(F^*)$ defined by $\sigma(T)$, in $K_0(C^*(M, F))$. The index theorem computes this boundary map and thereby allows to obtain explicit formulas for $\text{ind}_a(T)$.

The computation is in terms of the topological index $\text{ind}_t$ which is a map from $K_1(F^*)$ to $K_0(C^*(M, F))$ defined using an embedding procedure analogous to the classical case above. It is constructed as follows. Embed $M$ into $\mathbb{R}^n$, define, $M'$ as $M \times \mathbb{R}^n$ and define a foliation $F'$ on $M'$ as the product of $F$ with the trivial foliation by points on $\mathbb{R}^n$. For this new foliated manifold we have that $C^*(M', F')$ is isomorphic to $C^*(M, F) \otimes C(\mathbb{R}^n)$ and therefore has the same $K$-theory (with a dimension shift depending on the parity of $n$). It has the advantage however that $M'$ admits a submanifold $N$ which is transverse to the foliation $F'$. In fact, one can embed the bundle $F$ in the trivial bundle $M \times \mathbb{R}^n$ and take for $N$ the orthogonal complement to $F$. Now, for any submanifold $N$ transverse to the foliation $F'$, the $C^*$-algebra $C^*(M', F')$ contains $\mathcal{K} \otimes C_0(N)$ (crossed product of functions on a tubular neighbourhood of $N$ by translation by $\mathbb{R}^k$ in leaf direction).

The topological index is defined as the composition of the following maps

$$K_*(F^*) \cong K_*(N) \to K_*(C^*(M', F')) \cong K_*(C^*(M, F))$$

The index theorem then states that the boundary map in the long exact $K$-theory sequences associated to the extension 5 is exactly this map $\text{ind}_t$.

**Theorem 5.2.** ([18]) For every longitudinally elliptic pseudodifferential operator on the foliated manifold $(M, F)$ one has $\text{ind}_a(T) = \text{ind}_t(T)$.

In order to prove the theorem, Connes and Skandalis compute a specific Kasparov product.

### 5.2 $K$-theory of group-algebras, Novikov conjecture, Baum-Connes conjecture

Let $M$ be a compact connected oriented smooth manifold. The signature $\sigma(M)$ of $M$ can be written as $\langle L(M), [M] \rangle$, where $L(M)$ is the Hirzebruch polynomial in the Pontryagin classes. The signature is homotopy invariant: if two manifolds $M$ and $M'$ are homotopy equivalent via an orientation preserving map, then $\sigma(M) = \sigma(M')$. For simply connected manifolds the signature is the only characteristic number of $M$ with this property.

For non-simply connected manifolds however the “higher signatures” are further candidates for homotopy invariants. If $\pi$ denotes the fundamental group and $f : M \to B\pi$ the classifying map then for any $x \in H^*(B\pi, \mathbb{Q})$ we can define the twisted signature $\sigma_x(M) = \langle x, f_*L(M) \rangle$, where $L(M)'$ is the Poincaré dual to $L(M)$. The Novikov conjecture asserts that the numbers $\sigma_x(M)$ are homotopy invariants for all $x$ or, equivalently, that $f_*L(M)'$ is an oriented homotopy invariant.
Let \([d+\delta]\) denote the \(K\)-homology class in \(K_0(M) = K^0(\mathcal{C}(M))\) defined by the signature operator \(d + \delta\). Using the Chern character isomorphism between \(H^*(B\pi, \mathbb{Q})\) and \(RK_0(B\pi) \otimes \mathbb{Q} := \lim_{\longrightarrow} K_0(X) \otimes \mathbb{Q}\), where the limit is taken over all compact subsets \(X\) of \(B\pi\) and the fact that, by the Atiyah-Singer theorem, the index of \(d + \delta\) is given by pairing with \(L(M)\) (i.e. the image under the Chern character of the \(K\)-homology class defined by \(d + \delta\) is \(L(M)'\)) the conjecture is equivalent to the fact that \(f_\ast([d+\delta]) \in RK_0(B\pi) \otimes \mathbb{Q}\) is a homotopy invariant.

The \(K\)-theoretic approach to a proof of the Novikov conjecture now considers the reduced group \(C^*\)-algebra \(C_{red}^\ast\pi\) (i.e. the closure of the algebra of operators generated by the elements of \(\pi\) in the left regular representation). There is a natural construction that associates to every element in \(RK_0(B\pi)\) a projection in a matrix algebra over \(C_{red}^\ast\pi\) and therefore an element of \(K_0(C_{red}^\ast\pi)\). This defines a map \(\beta : RK_0(B\pi) \to K_0(C_{red}^\ast\pi)\). A construction of Miscenko using algebraic surgery shows that \(\beta(f_\ast([d+\delta]))\) is always a homotopy invariant.

The Novikov conjecture for \(\pi\) therefore follows from the following “strong Novikov conjecture” (Rosenberg) for \(\pi\).

Conjecture (SNC): the map \(\beta : RK_0(B\pi) \to K_0(C_{red}^\ast\pi)\) is rationally injective.

This strong Novikov conjecture was proved by Miscenko in the case that \(B\pi\) is a closed manifold with non-positive sectional curvature and by Kasparov in the case that \(B\pi\) is a (not necessarily compact) complete Riemannian manifold with non-positive sectional curvature (in both cases \(\beta\) itself is already injective). This covers the case where \(\pi\) is a closed torsion-free discrete subgroup of a connected Lie group \(G\), since in this case one can take \(\pi \cap G/K\) for \(B\pi\). Using the special structure of discrete subgroups of Lie groups one can reduce to the torsion-free case.

Kasparov’s proof of SNC for groups as above uses the following two theorems which are of independent interest.

Assume that \(G\) is a separable locally compact group acting on the complete Riemannian manifold \(X\) by isometries. The Dolbeault operator used in 5.1 is \(G\)-invariant and defines an element \([\bar{\partial}_X]\) in \(KK_0^G(\mathcal{C}(T^*X), \mathbb{C})\).

**Theorem 5.3.** Let \(X\) be simply connected with non-positive sectional curvature. Then the element \([\bar{\partial}_X]\) is right invertible, i.e. there exists a right inverse \(\delta_X\) in \(KK_0^G(\mathcal{C}(T^*X))\) such that \([\bar{\partial}_X]\cdot \delta_X = 1\) in \(KK_0^G(\mathcal{C}(T^*X)), \mathbb{C}(T^*X))\).

In the presence of a Spin\(^c\)-structure on \(X\), \(\bar{\partial}_X\) can also be viewed as a Dirac operator. The element \(\delta_X\) is constructed using a “Fourier transform” of \(\bar{\partial}_X\) and is therefore usually called the dual Dirac element.

**Theorem 5.4.** Let \(G\) be connected, \(K\) a maximal compact subgroup and \(X = G/K\). Then there is a right inverse \(\delta_X\) to \([\bar{\partial}_X]\) as in 5.3. The element \(\gamma_G = \delta_X \cdot [\bar{\partial}_X]\) in \(KK^G(\mathcal{C}, \mathbb{C})\) is an idempotent and does not depend on the choice of \(K\) or \(\delta_X\).

For all \(C^*\)-algebras \(A\) and \(B\) with an action of \(G\) by automorphisms, the natural restriction map
\[ KK^G_*(A, B) \rightarrow KK^K_*(A; B) \]

is an isomorphism on \( \gamma G \) and has kernel \( (1 - \gamma G)KK^G_*(A, B) \) (note that \( KK^G(G, G) \) acts on \( KK^G_*(A, B) \) by tensoring).

It is clear from the considerations above that the \( K \)-theoretic proof of the Novikov conjecture for a given group \( \pi \) depends on a partial computation of the \( K \)-theory of the group \( C^* \)-algebra \( C^*_red\pi \).

The Baum-Connes conjecture proposes a general formula for \( K_*(C^*_red\pi) \) by refining the map \( \beta : RK_0(B\pi) \rightarrow K_0(C^*_red\pi) \). For the Baum-Connes conjecture one uses a map whose construction is similar to the one of \( \beta \), but one modifies the left-hand side \([7]\). There is a universal contractible space \( E_G \) on which \( G \) acts properly. An action of a discrete group \( G \) on a Hausdorff space \( X \) is called proper if any two points \( x, y \), in \( X \) have neighbourhoods \( U \) and \( V \) such that only finitely many translates of \( U \) by elements in \( G \) intersect \( V \) (in particular all stabilizer groups are finite). The left hand side of the Baum-Connes conjecture then is the equivariant \( K \)-homology \( KK_*(E_G, C) \) (again defined using an inductive limit over all \( G \)-compact subspaces \( X \) of \( E_G \)). The map analogous to \( \beta \) is called \( \mu \) and the conjecture predicts that

\[ KK^G_*(E_G, C) \xrightarrow{\mu} K_*(C^*_red G) \]

is always an isomorphism. Since the left hand side is an object involving only the equivariant theory of ordinary spaces it can be understood using methods from ("commutative") topology and there are means to compute it, \([6]\). The construction also works for groups which are not discrete.

The Baum-Connes conjecture contains the Novikov conjecture and the generalized Kadison conjecture and plays an important motivating role in current research on topological \( K \)-theory. While counterexamples to the more general conjecture "with coefficients" have recently been announced by various authors (Higson, Lafforgue-Skandalis, \( \mu \)), it is known to hold in many cases of interest (see e.g. \([43]\), \([38]\), \([44]\), \([36]\), \([50]\), \([56]\), \([12]\)).

A general strategy which is used in basically all proofs of the Baum-Connes conjecture for different classes of groups, has been distilled in \([76]\), \([33]\). It uses actions on so-called proper algebras and abstract versions of "Dirac-" and "dual Dirac-" elements.

**Definition 5.5.** Let \( \Gamma \) be a discrete group and let \( A \) be a \( C^* \)-algebra with an action of \( \Gamma \). We say that \( A \) is proper if there exists a locally compact proper \( \Gamma \)-space \( X \) and a \( \Gamma \)-equivariant homomorphism from \( C^*_0(X) \) into the center of the multiplier algebra of \( A \) such that \( C^*_0(X)A \) is dense in \( A \).

**Theorem 5.6.** Let \( \Gamma \) be a countable group and let \( A \) be a proper \( \Gamma \)-\( C^* \)-algebra. Suppose that there are elements \( \alpha \) in \( KK^\Gamma_*(A, C) \) and \( \beta \) in \( KK^\Gamma_*(C, A) \) such that \( \beta \cdot \alpha = 1 \). Then the Baum-Connes conjecture holds for \( \Gamma \).

Let \( X \) be a complete Riemannian manifold with non-positive sectional curvature on which \( \Gamma \) acts properly and isometrically (an important special
case being $\Gamma$ a discrete subgroup of a connected Lie group and $X = G/K$. The elements constructed by Kasparov in 5.3 define elements $\alpha = \overline{\partial_X}$ and $\beta = \delta_X$ for the proper algebra $\mathcal{C}_0(X)$ such that $\alpha \cdot \beta = 1$. If for these elements the element $\gamma = \beta \cdot \alpha$ is also equal to 1, then the Baum-Connes conjecture holds for $\Gamma$.

Lafforgue has shown that in some cases where $\gamma$ is different from 1, this element can still act as the identity on the corresponding $K$-groups by mapping $KK_*$ to a Banach algebra version of bivariant $K$-theory which allows more homotopies and Morita equivalences using analogues of Kasparov modules involving Banach modules. He deduced from this the validity of the Baum-Connes conjecture for a class of groups that contains certain property $T$ groups.

5.3 Existence of positive scalar curvature metrics

Let $M$ be a closed smooth spin manifold. J. Rosenberg [70] used the Mishchenko-Fomenko index theorem to prove necessary conditions on $M$ for the existence of a Riemannian metric with positive scalar curvature on $M$. In particular he showed that, if SNC holds for the fundamental group $\pi$ of $M$, then the higher $\hat{A}$-genera of the form

$$(\hat{A}(M) \cup f^*(x), [M])$$

vanish for all $x \in H^*(B\pi, \mathbb{Q})$ (here $f : M \to B\pi$ is the natural classifying map). This necessary condition can easily be used to show that many manifolds (with fundamental group for which SNC is known to hold) cannot admit a metric with positive scalar curvature. Much more can be said, see [71].

5.4 Applications in the classification of nuclear $C^*$-algebras

The interest in $K$-theoretic methods among operator algebraists was strongly motivated by the fact that $K$-theoretic invariants allowed to distinguish $C^*$-algebras which looked otherwise very similar. One of the first computations of that kind was the one by Pimsner-Popa and, independently, Paschke-Salinas of the Ext-groups for the algebras $\mathcal{O}_n$. The algebra $\mathcal{O}_n$ is defined as the $C^*$-algebra with generators $s_1, \ldots, s_n$ and relations $s_i^* s_i = 1, \sum_i s_i s_i^* = 1$. The algebra $\mathcal{O}_\infty$ has generators $s_1, s_2, \ldots$ and relations $s_i^* s_i = 1, s_i^* s_j = 0, j \neq i.$ [20]. The result of Pimsner-Popa and Paschke-Salinas is that $\text{Ext}(\mathcal{O}_n) = \mathbb{Z}/(n-1)$, so that in particular, they are not isomorphic for different $n$. Another striking application of $K$-theory were the influential results of Pimsner-Voiculescu on the $K$-theory of noncommutative tori [63] and - later - of the reduced group $C^*$-algebras of free groups [64]. Again these computations showed that noncommutative tori with different twist or reduced group $C^*$-algebras of free groups with different number of generators could not be isomorphic.

The effectiveness of $K$-theory in the classification of nuclear $C^*$-algebras has however proved to go beyond all expectations of these early days. In fact
it turned out that up to a notion of stable isomorphism, nuclear simple C*-algebras are in a sense completely classified by KK-theory.

A simple C*-algebra $A$ is called purely infinite if for all $x, y \in A$ with $x$ nonzero, there are $a, b \in A$ such that $y = axb$. The most standard examples of purely infinite algebras are the algebras $\mathcal{O}_n$, $n = 2, 3, \ldots, \infty$ mentioned above. They have $K_0(\mathcal{O}_n) = \mathbb{Z}/n$, $n = 2, 3, \ldots$, $K_0(\mathcal{O}_\infty) = \mathbb{Z}$ and $K_1(\mathcal{O}_n) = 0$, $n = 2, 3, \ldots, \infty$. If $A$ is any simple C*-algebra, then $K_*(A \otimes \mathcal{O}_n) = K_*(A)$. Moreover $A \otimes \mathcal{O}_\infty$ is automatically purely infinite and $A \otimes \mathcal{O}_\infty \otimes \mathcal{O}_\infty \cong A \otimes \mathcal{O}_\infty$. Thus $A \otimes \mathcal{O}_\infty$ may be viewed as a purely infinite stabilization of $A$. This shows the interest of the following astonishing and deep theorem obtained independently by Kirchberg and Phillips after groundbreaking work of Kirchberg.

Recall that a C*-algebra is called stable if $A \cong K \otimes A$.

**Theorem 5.7.** (cf. [69], 8.4.1) Let $A$ and $B$ be purely infinite simple nuclear algebras.

(a) Assume that $A$ and $B$ are stable. Then $A$ and $B$ are isomorphic if and only if they are isomorphic in KK. Moreover, for each invertible element $x$ in KK there exists an isomorphism $\varphi : A \to B$ with $KK(\varphi) = x$.

(b) Assume that $A$ and $B$ are stable and belong to the UCT class $N$. Then $A$ is isomorphic to $B$ if and only if $K_0(A) \cong K_0(B)$ and $K_1(A) \cong K_1(B)$. Moreover, for each pair of isomorphisms $\alpha_i : K_i(A) \to K_i(B)$, $i = 1, 2$, there is an isomorphism $\varphi : A \to B$ with $K_i(\varphi) = \alpha_i$, $i = 1, 2$.

(c) Assume that $A$ and $B$ are unital. Then $A$ and $B$ are isomorphic if and only if there exists an invertible element $x$ in $KK(A, B)$ such that $[1_A, x = [1_B]$ (where $\cdot$ denotes Kasparov product and $[1_A], [1_B]$ the elements of $K_0(M) = KK(\mathbb{C}, A)$, $K_0(B) = KK(\mathbb{C}, B)$ defined by the units of $A$ and $B$). For each such element $x$, there is an isomorphism $\varphi : A \to B$ with $KK(\varphi) = x$.

(d) Assume that $A$ and $B$ are unital and belong to the UCT class $N$. Then $A$ is isomorphic to $B$ if and only if there exist isomorphisms $\alpha_i : K_i(A) \to K_i(B)$, $i = 1, 2$ such that $\alpha_0([1_A]) = [1_B]$. Moreover, for each such pair of isomorphisms, there is an isomorphism $\varphi : A \to B$ with $K_i(\varphi) = \alpha_i$, $i = 1, 2$.

There are many examples of purely infinite nuclear algebras in the UCT class with the same $K$-groups but constructed in completely different ways. By the theorem these algebras have to be isomorphic, but in general it is impossible to find an explicit isomorphism.

### 5.5 Classification of topological dynamical systems

A topological dynamical system is an action of $\mathbb{Z}$ on a compact space $X$ by homeomorphisms. With such systems one can associate various noncommutative $C^*$-algebras, the most obvious one being the crossed product $\mathcal{C}(X) \rtimes \mathbb{Z}$. 
Interesting for applications are in particular systems where $X$ is a Cantor space. The $K$-theory of the crossed product for such a system has been analyzed and been used to obtain results on various notions of orbit equivalence for such systems in [34].

Besides the crossed product one can also associate other $C^*$-algebras constructed from groupoids associated with the system. Such a construction can be applied to subshifts of finite type. A subshift of finite type is defined by an $n \times n$-matrix $A = (a(ij))$ with entries $a(ij)$ in \{0,1\}. The shift space $X_A$ consists of all families $(c_k)_{k \in \mathbb{Z}}$ with $c_k \in \{1, \ldots, n\}$ and $a(c_k, c_{k+1}) = 1$ for all $k$. The subshift is given by the shift transformation $\sigma_A$ on $X_A$. A groupoid associated with $(X_A, \sigma_A)$ gives the $C^*$-algebra $O_A$ considered in [23]. It is a homeomorphism invariant for the suspension flow space associated to the transformation $(X_A, \sigma_A)$. The $K$-groups for $O_A$ recover invariants of flow equivalence discovered by Bowen and Franks, [10]:

\[ K_0(O_A) = \mathbb{Z}^n/(1 - A)\mathbb{Z}^n \]
\[ K_1(O_A) = \text{Ker} (1 - A) \]

If a topological Markov chain $(X_C, \sigma_C)$ is not minimal, then it can be decomposed into two components $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$. Correspondingly, the matrix $C$ can be written in the form

\[ C = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix} \]

The corresponding $C^*$-algebra $O_C$ contains $\mathcal{K} \otimes O_B$ as an ideal with $O_A$ as quotient. The corresponding extension

\[ 0 \to \mathcal{K} \otimes O_B \to O_C \to O_A \to 0 \]

defines an element of $KK_1(O_A, O_B)$ which describes how the suspension spaces for $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ are glued to give the one for $(X_C, \sigma_C)$.

In [22], it was proved that

\[ KK_1(O_A, O_B) \cong K^1(O_A) \otimes K_0(O_B) \oplus \text{Hom}(K_0(O_A), K_1(O_B)) \]

The two summands can be described as equivalence classes of $n \times m$-matrices. In fact they are the cokernels and kernels, for right multiplication by $B$, in the cokernel for left multiplication by $A$, on the space $M_{n,m}(\mathbb{Z})$ of $n \times m$-matrices.

It has been shown that, for the case of reducible topological Markov chains $(X_C, \sigma_C)$ with two components, this extension invariant together with the flow equivalence invariants for the components give complete invariants of flow equivalence for $(X_C, \sigma_C)$.

We mention that in many similar cases such as the $C^*$-algebras associated with a non-minimal foliation the extension invariant for the corresponding extension of the associated $C^*$-algebras can in principle be used to describe the way that the big system is composed from its components.
6 Bivariant $K$-theory for locally convex algebras

$KK$-theory and $E$-theory both use techniques which are quite specific to the category of $C^*$-algebras (in particular, central approximate identities play a crucial role). Therefore similar bivariant theories for other categories of algebras seemed for many years out of reach. Since ordinary and periodic cyclic theory give only pathological results for $C^*$-algebras, this made it in particular impossible to compare bivariant $K$-theory with cyclic theory via a Chern character.

A bivariant $K$-theory was developed finally in [28] for a large category of locally convex algebras ("$m$-algebras"). Since in this category one has less analytic tools at one’s disposal, the construction had to be based on a better understanding of the underlying algebraic structure in bivariant $K$-theory. The definition is formally similar to the $qA$-picture described briefly in chapter 1. The underlying idea is to represent elements of the bivariant theory by extensions of arbitrary length and the product by the Yoneda product of extensions.

A locally convex algebra $A$ is, in general, an algebra with a locally convex topology for which the multiplication $A \times A \to A$ is (jointly) continuous. In the present survey we restrict our attention however to locally convex algebras that can be represented as projective limits of Banach algebras.

A locally convex algebra $A$ that can be represented as a projective limit of Banach algebras can equivalently be defined as a complete locally convex algebra whose topology is determined by a family $\{p_\alpha\}$ of submultiplicative seminorms, [55]. Thus for each $\alpha$ we have $p_\alpha(xy) \leq p_\alpha(x)p_\alpha(y)$. The algebra $A$ is then automatically a topological algebra, i.e. multiplication is (jointly) continuous. We call such algebras $m$-algebras. The unitization $\tilde{A}$ of an $m$-algebra is again an $m$-algebra in a natural way. Also the completed projective tensor product $\hat{A} \hat{\otimes} \hat{B}$ of two $m$-algebras is again an $m$-algebra.

Since cyclic theory is homotopy invariant only for differentiable homotopies (called diffotopies below), we have to set up the theory in such a way that it uses only diffotopies in place of general homotopies.

Definition 6.1. Two continuous homomorphisms $\alpha, \beta : A \to B$ between two $m$-algebras are called differentiably homotopic or diffotopic, if there is a family $\varphi_t : A \to B, t \in [0, 1]$ of continuous homomorphisms, such that $\varphi_0 = \alpha, \varphi_1 = \beta$ and such that the map $t \mapsto \varphi_t(x)$ is infinitely differentiable for each $x \in A$.

It is not hard to see (though not completely obvious) that diffotopy is an equivalence relation.

The $m$-algebra $\mathcal{K}$ of "smooth compact operators" consists of all $\mathbb{N} \times \mathbb{N}$-matrices $(a_{ij})$ with rapidly decreasing matrix elements $a_{ij} \in \mathbb{C}, i, j = 0, 1, 2, \ldots$. The topology on $\mathcal{K}$ is given by the family of norms $p_n, n = 0, 1, 2, \ldots$, which are defined by
\[ p_n((a_{ij})) = \sum_{i,j} |1+i+j|^n |a_{ij}| \]

It is easily checked that the \( p_n \) are submultiplicative and that \( \mathcal{K} \) is complete. Thus \( \mathcal{K} \) is an \( m \)-algebra. As a locally convex vector space, \( \mathcal{K} \) is isomorphic to the sequence space \( s \) and therefore is nuclear in the sense of Grothendieck. The algebra \( \mathcal{K} \) of smooth compact operators is of course smaller than the \( C^* \)-algebra of compact operators on a separable Hilbert space which we used in the previous sections. It plays however exactly the same role in the theory. We hope that the use of the same symbol \( \mathcal{K} \) will not lead to confusion.

The map that sends \((a_{ij}) \otimes (b_{kl})\) to the \( N^2 \times N^2 \)-matrix \((a_{ij}b_{kl})_{(i,k)(j,l)} \in N^2 \times N^2\) obviously gives an isomorphism \( \Theta \) between \( \mathcal{K} \dot{\otimes} \mathcal{K} \) and \( \mathcal{K} \).

**Definition 6.2.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be \( m \)-algebras. For any homomorphism \( \varphi : \mathcal{A} \to \mathcal{B} \) of \( m \)-algebras, we denote by \( \langle \varphi \rangle \) the equivalence class of \( \varphi \) with respect to diffotopy and we set

\[ \langle \mathcal{A}, \mathcal{B} \rangle = \{ \langle \varphi \rangle | \varphi \text{ is a continuous homomorphism } \mathcal{A} \to \mathcal{B} \} \]

For two continuous homomorphisms \( \alpha, \beta : \mathcal{A} \to \mathcal{K} \dot{\otimes} \mathcal{B} \) we define the direct sum \( \alpha \oplus \beta \) as

\[ \alpha \oplus \beta = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} : \mathcal{A} \to M_2(\mathcal{K} \dot{\otimes} \mathcal{B}) \cong \mathcal{K} \dot{\otimes} \mathcal{B} \]

With the addition defined by \( \langle \alpha \rangle + \langle \beta \rangle = \langle \alpha \oplus \beta \rangle \) the set \( \langle \mathcal{A}, \mathcal{K} \dot{\otimes} \mathcal{B} \rangle \) of diffotopy classes of homomorphisms from \( \mathcal{A} \) to \( \mathcal{K} \dot{\otimes} \mathcal{B} \) is an abelian semigroup with 0-element \( \langle 0 \rangle \).

Let \( V \) be a complete locally convex space. We define the smooth tensor algebra \( T^sV \) as the completion of the algebraic tensor algebra

\[ TV = V \oplus V \otimes V \oplus V \otimes V \oplus \ldots \]

with respect to the family \{\( \tilde{p} \)\} of seminorms, which are given on this direct sum as

\[ \tilde{p} = p \oplus p \otimes p \oplus p \otimes p \oplus \ldots \]

where \( p \) runs through all continuous seminorms on \( V \). The seminorms \( \tilde{p} \) are submultiplicative for the multiplication on \( TV \). The completion \( T^sV \) therefore is an \( m \)-algebra.

We denote by \( \sigma : V \to T^sV \) the map, which maps \( V \) to the first summand in \( TV \). The map \( \sigma \) has the following universal property: Let \( s : V \to \mathcal{A} \) be any continuous linear map from \( V \) to an \( m \)-algebra \( \mathcal{A} \). Then there is a unique continuous homomorphism \( \tau_s : T^sV \to \mathcal{A} \) of \( m \)-algebras such that \( \tau_s \circ \sigma = s \).

It is given by

\[ \tau_s(x_1 \otimes x_2 \otimes \ldots \otimes x_n) = s(x_1)s(x_2)\ldots s(x_n) \]
The smooth tensor algebra is differentiably contractible, i.e., the identity map on $T^*V$ is diffotopic to $0$. A differentiable family $\varphi_t : T^*V \to T^*V$ of homomorphisms for which $\varphi_0 = 0, \varphi_1 = \text{id}$ is given by $\varphi_t = \tau_{t\sigma}, t \in [0, 1]$.

If $\mathcal{A}$ is an $m$-algebra, by abuse of notation, we write $T\mathcal{A}$ (rather than $T^*\mathcal{A}$) for the smooth tensor algebra over $\mathcal{A}$. Thus $T\mathcal{A}$ is again an $m$-algebra. For any $m$-algebra $\mathcal{A}$ there exists a natural extension

$$0 \to J\mathcal{A} \to T\mathcal{A} \xrightarrow{\pi} \mathcal{A} \to 0.$$  

(6)

Here $\pi$ maps a tensor $x_1 \otimes x_2 \otimes \ldots \otimes x_n$ to $x_1 x_2 \ldots x_n \in \mathcal{A}$ and $J\mathcal{A} = \ker \pi$. This extension is universal in the sense that, given any extension $0 \to I \to E \to \mathcal{A} \to 0$ of $\mathcal{A}$, admitting a continuous linear splitting, there is a morphism of extensions

$$0 \to J\mathcal{A} \to T\mathcal{A} \to \mathcal{A} \to 0 \quad \downarrow \gamma \quad \downarrow \tau \quad \downarrow \text{id} \quad 0 \to I \to E \to \mathcal{A} \to 0$$

The map $\tau : T\mathcal{A} \to E$ is obtained by choosing a continuous linear splitting $s : \mathcal{A} \to E$ in the given extension and mapping $x_1 \otimes x_2 \otimes \ldots \otimes x_n$ to $s(x_1)s(x_2)\ldots s(x_n) \in E$. Then $\gamma$ is the restriction of $\tau$.

**Definition 6.3.** The map $\gamma : J\mathcal{A} \to I$ in this commutative diagram is called the classifying map for the extension $0 \to I \to E \to \mathcal{A} \to 0$.

If $s$ and $s'$ are two different linear splittings, then for each $t \in [0, 1]$ the map $s_t = ts + (1-t)s'$ is again a splitting. The corresponding maps $\gamma_t$ associated with $s_t$ form a diffotopy between the classifying map constructed from $s$ and the one constructed from $s'$. The classifying map is therefore unique up to diffotopy.

For an extension admitting a homomorphism splitting the classifying map is diffotopic to $0$.

An extension of $\mathcal{A}$ of length $n$ is an exact complex of the form

$$0 \to I \to E_1 \to \ldots \to E_n \to \mathcal{A} \to 0$$

where the arrows or boundary maps (which we denote by $\varphi$) are continuous homomorphisms between $m$-algebras. We call such an extension linearly split, if there is a continuous linear map $s$ of degree $-1$ on this complex, such that $s\varphi + \varphi s = \text{id}$. This is the case if and only if the given extension is a Yoneda product (concatenation) of $n$ linearly split extensions of length $1$: $I \to E_1 \to \text{Im} \varphi_1, \ker \varphi_2 \to E_2 \to \text{Im} \varphi_2, \ldots$.

$J\mathcal{A}$ is, for each $m$-algebra $\mathcal{A}$, again an $m$-algebra. By iteration we can therefore form $J^2\mathcal{A} = J(J\mathcal{A}), \ldots, J^n\mathcal{A} = J(J^{n-1}\mathcal{A})$.

**Proposition 6.4.** For any linearly split extension
of $\mathcal{A}$ of length $n$ there is a classifying map $\gamma : J^n\mathcal{A} \to \mathcal{I}$ which is unique up to diffeotopy.

Proof. Compare with the free $n$-step extension

$$0 \to J^n\mathcal{A} \to T(J^{n-1}\mathcal{A}) \to T(J^{n-2}\mathcal{A}) \to \ldots \to T\mathcal{A} \to \mathcal{A} \to 0$$

Proposition 6.5. $J$ (and $J^n$) is a functor, i.e. each continuous homomorphism $\mathcal{A} \to \mathcal{B}$ between $m$-algebras induces a continuous homomorphism $J\mathcal{A} \to J\mathcal{B}$.

Consider now the set $H_k = \langle J^k \mathcal{A}, \mathcal{K} \hat{\otimes} \mathcal{B} \rangle$, where $H_0 = \langle \mathcal{A}, \mathcal{K} \hat{\otimes} \mathcal{B} \rangle$. Each $H_k$ is an abelian semigroup with 0-element for the $K$-theory addition defined in 6.2. Morally, the elements of $H_k$ are classifying maps for linearly split $k$-step extensions. In applications all elements arise that way.

To define a map $S : H_k \to H_{k+2}$, we use the classifying map $\varepsilon$ for the two-step extension which is obtained by composing the so called Toeplitz extension

$$0 \to \mathcal{K} \hat{\otimes} \mathcal{A} \to T_0 \hat{\otimes} \mathcal{A} \to \mathcal{A}[0,1) \to \mathcal{A} \to 0$$

with the cone or suspension extension

$$0 \to \mathcal{A}(0,1) \to \mathcal{A}[0,1) \to \mathcal{A} \to 0$$

Here, $\mathcal{A}(0,1)$ and $\mathcal{A}[0,1)$ denote the algebras of smooth $\mathcal{A}$-valued functions on the interval $[0,1]$, that vanish in 0 and 1, or only in 1, respectively, and whose derivatives all vanish in both endpoints. The smooth Toeplitz algebra $T_0$ is a standard extension of $\mathbb{C}(0,1)$ where a preimage of an element $e^{ih}$ in $\mathbb{C}(0,1)\setminus \{0\}$ has Fredholm index 1, for each monotone function $h$ in $\mathbb{C}(0,1]$ such that $h(0) = 0$ and $h(1) = 1$.

Definition 6.6. For each $m$-algebra $\mathcal{A}$, we define the periodicity map $\varepsilon_\mathcal{A} : J^2 \mathcal{A} \to \mathcal{K} \hat{\otimes} \mathcal{A}$ as the classifying map for the standard two-step Bott extension

$$0 \to \mathcal{K} \hat{\otimes} \mathcal{A} \to T_0 \hat{\otimes} \mathcal{A} \to \mathcal{A}(0,1) \to \mathcal{A} \to 0$$

We can now define the Bott map $S$. For $\langle \alpha \rangle \in H_k$, $\alpha : J^k \mathcal{A} \to \mathcal{K} \hat{\otimes} \mathcal{B}$ we set $S \langle \alpha \rangle = ((\text{id}_\mathcal{K} \hat{\otimes} \alpha) \circ \varepsilon)$. Here $\varepsilon : J^{k+2} \mathcal{A} \to \mathcal{K} \hat{\otimes} J^k \mathcal{A}$ is the $\varepsilon$-map for $J^k \mathcal{A}$.

If an element $\gamma$ of $H_k$ is given as a classifying map for an extension of length $k$, then $S\gamma$ is the classifying map for the Yoneda product of the given extension with the Bott extension.

Let $\varepsilon_- : J^{k+2} \mathcal{A} \to \mathcal{K} \hat{\otimes} J^k \mathcal{A}$ be the map which is obtained by replacing, in the definition of $\varepsilon$, the Toeplitz extension by the inverse Toeplitz extension. The sum $\varepsilon \oplus \varepsilon_-$ is then diffeotopic to 0. Therefore $S \langle \alpha \rangle + S_- \langle \alpha \rangle = 0$, putting $S_- \langle \alpha \rangle = ((\text{id}_\mathcal{K} \hat{\otimes} \alpha) \circ \varepsilon_-)$. 
Definition 6.7. Let \( \mathcal{A} \) and \( \mathcal{B} \) be \( m \)-algebras and \( * = 0 \) or \( 1 \). We define

\[
kk_*(\mathcal{A}, \mathcal{B}) = \lim_k H_{2k+*} = \lim_k \langle J^{2k+*} \mathcal{A}, \mathcal{K} \hat{\otimes} \mathcal{B} \rangle
\]

The preceding discussion shows that \( kk_*(\mathcal{A}, \mathcal{B}) \) is not only an abelian semigroup, but even an abelian group (every element admits an inverse). A typical element of \( kk_*(\mathcal{A}, \mathcal{B}) \) is given by a classifying map of a linearly split extension

\[
0 \to \mathcal{K} \otimes \mathcal{B} \to \mathcal{E}_1 \to \mathcal{E}_2 \ldots \to \mathcal{E}_n \to \mathcal{A} \to 0
\]

In the inductive limit, we identify such an extension with its composition with the Bott extension for \( \mathcal{A} \) on the right hand side.

As usual for bivariant theories, the decisive element, which is also the most difficult to establish, is the composition product.

Theorem 6.8. There is an associative product

\[
 kk_i(\mathcal{A}, \mathcal{B}) \times kk_j(\mathcal{B}, \mathcal{C}) \longrightarrow kk_{i+j}(\mathcal{A}, \mathcal{C})
\]

\((i, j \in \mathbb{Z}/2; \mathcal{A}, \mathcal{B} \text{ and } \mathcal{C} \text{ } m\text{-algebras}), \text{ which is additive in both variables.}\)

Neglecting the tensor product by \( \mathcal{K} \), the product of an element represented by \( \varphi \in \langle J^k \mathcal{A}, \mathcal{K} \hat{\otimes} \mathcal{B} \rangle \) and an element represented by \( \psi \in \langle J^l \mathcal{B}, \mathcal{K} \hat{\otimes} \mathcal{C} \rangle \) is defined as \( \langle \psi \circ J^l(\varphi) \rangle \). Thus for elements of \( kk \) which are given as classifying maps for higher length extensions, the product simply is the classifying map for the Yoneda product of the two extensions. The fact that this is well defined, i.e. compatible with the periodicity map \( S \), demands a new idea, namely the “basic lemma” from [28].

Lemma 6.9. Assume given a commutative diagram of the form

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & \mathcal{I} & \mathcal{A}_0 & \mathcal{A}_1 \\
\downarrow & \downarrow & \downarrow & \\
0 & \mathcal{A}_0 & \mathcal{A}_1 & \mathcal{A}_2 \\
\downarrow & \downarrow & \downarrow & \\
0 & \mathcal{A}_1 & \mathcal{A}_2 & \mathcal{B} \\
\downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & 0
\end{array}
\]

where all the rows and columns represent extensions of \( m \)-algebras with continuous linear splittings.

Let \( \gamma_+ \) and \( \gamma_- \) denote the classifying maps \( J^2 \mathcal{B} \to \mathcal{I} \) for the two extensions of length 2

\[
\begin{align*}
0 & \to \mathcal{I} \to \mathcal{A}_0 \to \mathcal{A}_2 \to \mathcal{B} \\
0 & \to \mathcal{I} \to \mathcal{A}_1 \to \mathcal{B} \\
0 & \to \mathcal{I} \to \mathcal{A}_2 \to \mathcal{B}
\end{align*}
\]

associated with the two edges of the diagram. Then \( \gamma_+ \oplus \gamma_- \) is difftopic to 0.
This lemma implies that, the classifying maps for the compositions of a linearly split extension

$$0 \to B \to E_1 \to E_2 \to \ldots \to E_n \to A \to 0$$

with the Bott extension for $B$ on the left or with the Bott extension for $A$ on the right hand side are difftopic.

The usual properties of a bivariant $K$-theory as listed in 3.3 and in particular the long exact sequences 3 and 4 in 3.4 for linearly split extensions of $m$-algebras can then be deduced for $kk$ in a rather standard fashion.

Moreover, the following important theorem holds.

**Theorem 6.10.** For every Banach algebra $A$, the groups $kk_*(\mathbb{C}, A)$ and $K_*A$ are naturally isomorphic.

In particular $kk_0(\mathbb{C}, \mathbb{C}) = \mathbb{Z}$ and $kk_1(\mathbb{C}, \mathbb{C}) = 0$. Phillips, [60], has developed a topological $K$-theory $K_*$ for Fréchet $m$-algebras that extends the theory from Banach algebras. Also for that theory one finds that $kk_*(\mathbb{C}, A) \cong K_*A$ for each Fréchet $m$-algebra $A$.

**Remark 6.11.** A version of bivariant $K$-theory for general locally convex algebras - not just $m$-algebras - has been worked out in [29]. In [29] a slightly different (but basically equivalent) approach is used to define the bivariant theory which can also be used to construct the bivariant theory for $m$-algebras described above. It is motivated by the thesis of A.Thom [74]. The theory is constructed as noncommutative stable homotopy, i.e. as an inductive limit over suspensions of both variables (using a noncommutative suspension for the first variable), rather than as an inductive limit over the inverse Bott maps $\varepsilon$ as above. This simplifies some of the arguments and also clarifies the fact that Bott periodicity becomes automatic once we stabilize by (smooth) compact operators in the second variable.

## 7 Bivariant cyclic theories

### 7.1 The algebra $\Omega A$ of abstract differential forms over $A$ and its operators

There are many different but essentially equivalent descriptions of the complexes used to define cyclic homology. The most standard choice is the cyclic bicomplex. For our purposes it is more convenient to use the $(B,b)$-bicomplex. $B$ and $b$ are operators on the algebra $\Omega A$ of abstract differential forms over $A$.

Given an algebra $A$, we denote by $\Omega A$ the universal algebra generated by all $x \in A$ with relations of $A$ and all symbols $dx$, $x \in A$, where $dx$ is linear in $x$ and satisfies $d(xy) = xdy + d(x)y$. We do not impose $d1 = 0$, i.e., if $A$ has a unit,
$d_1 \neq 0$. $\Omega A$ is a direct sum of subspaces $\Omega^n A$ generated by linear combinations of $x_0 dx_1 \ldots dx_n$, and $dx_1 \ldots dx_n$, $x_j \in A$. This decomposition makes $\Omega A$ into a graded algebra. As usual, we write $\deg(\omega) = n$ if $\omega \in \Omega^n A$.

As a vector space, for $n \geq 1$,

$$\Omega^n A \cong \widetilde{A} \otimes A^{\otimes n} \cong A^{\otimes (n+1)} \oplus A^{\otimes n}$$

(7)

(Where $\widetilde{A}$ is $A$ with a unit adjoined, and $1 \otimes x_1 \otimes \ldots \otimes x_n$ corresponds to $dx_1 \ldots dx_n$). The operator $d$ is defined on $\Omega A$ by

$$d(x_0 dx_1 \ldots dx_n) = dx_0 dx_1 \ldots dx_n$$
$$d(dx_1 \ldots dx_n) = 0$$

(8)

The operator $b$ is defined by

$$b(\omega dx) = (-1)^{\deg\omega} [\omega, x]$$
$$b(dx) = 0, \quad b(x) = 0, \quad x \in A, \quad \omega \in \Omega A$$

(9)

Then, by definition, $d^2 = 0$ and one easily computes that also $b^2 = 0$.

Under the isomorphism in equation (7) $d$ becomes

$$d(x_0 \otimes \ldots \otimes x_n) = 1 \otimes x_0 \otimes \ldots \otimes x_n \quad x_0 \in A$$
$$d(1 \otimes x_1 \otimes \ldots \otimes x_n) = 0$$

while $b$ corresponds to the usual Hochschild operator

$$b(\tilde{x}_0 \otimes x_1 \otimes \ldots \otimes x_n) =$$

$$\tilde{x}_0 x_1 \otimes \ldots \otimes x_n + \sum_{j=2}^n (-1)^{j-1} \tilde{x}_0 \otimes \ldots \otimes x_{j-1} x_j \otimes \ldots \otimes x_n$$

$$+ (-1)^n x_n \tilde{x}_0 \otimes x_1 \otimes \ldots \otimes x_{n-1}, \quad \tilde{x}_0 \in \widetilde{A}, \quad x_1, \ldots, x_n \in A$$

Another important natural operator is the degree (or number) operator:

$$N(\omega) = \deg(\omega) \omega$$

(10)

We also define the Karoubi operator $\kappa$ on $\Omega A$ by

$$\kappa = 1 - (db + bd)$$

(11)

Explicitly, $\kappa$ is given by

$$\kappa(\omega dx) = (-1)^{\deg\omega} dx \omega$$

The operator $\kappa$ satisfies $(\kappa^n - 1)(\kappa^{n+1} - 1) = 0$ on $\Omega^n$. Therefore, by linear algebra, there is a projection operator $P$ on $\Omega A$ corresponding to the generalized eigenspace for 1 of the operator $\kappa = 1 - (db + bd)$. 
Lemma 7.1. Let $L = (Nd)b + b(Nd)$. Then $\Omega A = \text{Ker} L \oplus \text{Im} L$ and $P$ is exactly the projection onto $\text{Ker} L$ in this splitting.

Proof. This follows from the identity

$$L = (\kappa - 1)^2(\kappa^{n-1} + 2\kappa^{n-2} + 3\kappa^{n-3} + \ldots + (n-1)\kappa + n)$$

$\square$

The operator $L$ thus behaves like a “selfadjoint” operator. It can be viewed as an abstract Laplace operator on the algebra of abstract differential forms $\Omega A$. The elements in the image of $P$ are then “abstract harmonic forms”.

By construction, $P$ commutes with $b, d, N$. Thus setting $B = NPd$ one finds $Bb + bB = PL = 0$ and $B^2 = 0$.

Explicitly, $B$ is given on $\omega \in \Omega$ by the formula

$$B(\omega) = \sum_{j=0}^{n} \kappa^j d\omega$$

Under the isomorphism in equation (7), this corresponds to:

$$B(x_0dx_1\ldots dx_n) = dx_0dx_1\ldots dx_n + (-1)^n dx_n dx_0 \ldots dx_{n-1}$$

$$+ \ldots + (-1)^n dx_1 \ldots dx_n dx_0$$

The preceding identities show that we obtain a bicomplex - the $(B,b)$-bicomplex - in the following way

$$\begin{array}{c}
\Omega^3 A & \overset{B}{\longrightarrow} & \Omega^2 A & \overset{B}{\longrightarrow} & \Omega^1 A & \overset{B}{\longrightarrow} & \Omega^0 A \\
\downarrow b & & \downarrow b & & \downarrow b & & \downarrow b \\
\Omega^2 A & \overset{\theta}{\longrightarrow} & \Omega^1 A & \overset{B}{\longrightarrow} & \Omega^0 A \\
\downarrow b & & \downarrow b & & \downarrow b \\
\Omega^1 A & \overset{B}{\longrightarrow} & \Omega^0 A \\
\downarrow b & & \\
\Omega^0 A & & 
\end{array}$$

(12)

One can rewrite the $(B,b)$-bicomplex (12) using the isomorphism $\Omega^n A \cong A^{\otimes (n+1)} \oplus A^{\otimes n}$ in equation (7). An easy computation shows that the operator $b : \Omega^n A \rightarrow \Omega^{n-1} A$ corresponds under this isomorphism to the operator $A^{\otimes (n+1)} \oplus A^{\otimes n} \rightarrow A^{\otimes n} \oplus A^{\otimes (n-1)}$ which is given by the matrix

$$\begin{pmatrix}
\lambda & 1 - \lambda \\
0 & -b'
\end{pmatrix}$$
where $b', b$ and $\lambda$ are the operators in the usual cyclic bicomplex.

Similarly, the operator $B : \Omega^n A \to \Omega^{n+1} A$ corresponds to the operator $A \otimes (n+1) \oplus A \otimes n \to A \otimes (n+2) \oplus A \otimes (n+1)$ given by the $2 \times 2$-matrix

$$
\begin{pmatrix}
0 & 0 \\
Q & 0
\end{pmatrix}
$$

where $Q = 1 + \lambda + \lambda^2 + \ldots \lambda^n$.

This shows that the $(B,b)$-bicomplex is just another way of writing the usual cyclic bicomplex which is based on the operators $b, b', \lambda$ and $Q$. The total complex $D^2$ for the $(B,b)$-bicomplex is exactly isomorphic to the total complex for the cyclic bicomplex. We define

**Definition 7.2.** The cyclic homology $HC_n A$ is the homology of the complex

$$
\cdots \to D^n_1 B' - b \to \cdots D^n_2 - b \to D^n_1 - b \to D^n_0 \to 0
$$

where

$$
D^n_0 = \Omega^0 A \oplus \Omega^2 A \oplus \ldots \oplus \Omega^{2n} A
$$

$$
D^n_{2n+1} = \Omega^1 A \oplus \Omega^3 A \oplus \ldots \oplus \Omega^{2n+1} A
$$

and $B'$ is the truncated $B$-operator, i.e., $B' = B$ on the components $\Omega^k A$ of $D^n_0$, except on the highest component $\Omega^n A$, where it is 0.

The Hochschild homology $HH_n (A)$ is the homology of the first column in 12, i.e. of the complex

$$
\cdots \to \Omega^n A \to \Omega^{n-1} A \to \cdots \Omega^1 A \to \Omega^0 A \to 0
$$

**Remark 7.3.** Assume that $A$ has a unit 1. We may introduce in $\Omega A$ the additional relation $d(1) = 0$, i.e., divide by the ideal $M$ generated by $d(1)$ (this is equivalent to introducing the relation $1 \cdot \omega = \omega$ for all $\omega$ in $\Omega A$). We denote the quotient by $\Omega A$. Now $M$ is a graded subspace, invariant under $b$ and $B$, and its homology with respect to $b$ is trivial. The preceding proposition is thus still valid if we use $\Omega A$ in place of $\Omega A$. The convention $d(1) = 0$ is often used (implicitly) in the literature. In some cases it simplifies computations considerably. There are however situations where one cannot reduce the computations to the unital situation (in particular this is true for the excision problem).

### 7.2 The periodic theory

The periodic theory is the one that has the really good properties like diffeotopy invariance, Morita invariance and excision. It generalizes the classical de Rham theory to the non-commutative setting.
Periodic cyclic homology

Let $A$ be an algebra. We denote by $\hat{\Omega} A$ the infinite product

$$\hat{\Omega} A = \prod_n \Omega^n A$$

and by $\hat{\Omega}^{ev} A$, $\hat{\Omega}^{odd} A$ its even and odd part, respectively. $\hat{\Omega} A$ may be viewed as the (periodic) total complex for the bicomplex

$$\begin{array}{c}
\downarrow & \downarrow & \downarrow & \downarrow \\
\langle \Omega^3 A \to \Omega^2 A \to \Omega^1 A \to \Omega^0 A \rangle & \langle \Omega^2 A \to \Omega^1 A \to \Omega^0 A \rangle & \langle \Omega^1 A \to \Omega^0 A \rangle & \langle \Omega^0 A \rangle
\end{array}$$

Similarly, the (continuous for the filtration topology) dual $(\hat{\Omega} A)'$ of $\hat{\Omega} A$ is

$$(\hat{\Omega} A)' = \bigoplus_n (\Omega^n A)'$$

**Definition 7.4.** The periodic cyclic homology $HP_* (A)$, $* = 0, 1$, is defined as the homology of the $\mathbb{Z}/2\mathbb{Z}$-graded complex

$$\begin{array}{c}
\hat{\Omega}^{ev} A & \xrightarrow{B} & \hat{\Omega}^{odd} A
\end{array}$$

and the periodic cyclic cohomology $HP^* (A)$, $* = 0, 1$, is defined as the homology of the $\mathbb{Z}/2\mathbb{Z}$-graded complex

$$\begin{array}{c}
(\hat{\Omega}^{ev} A)' & \xlongleftarrow{B} & (\hat{\Omega}^{odd} A)'
\end{array}$$

Now by definition, $S$ is the projection

$$D^\Omega_n = \Omega^n \oplus \Omega^{n-2} \oplus \ldots \quad \rightarrow \quad D^\Omega_{n-2} = \Omega^{n-2} \oplus \Omega^{n-4} \oplus \ldots$$

where $D^\Omega_n$ is as in 7.2. Therefore we get
\[ \hat{\Omega}A = \lim_{S} (D_{2n}^\Omega + D_{2n+1}^\Omega) \]

and

\[ (\hat{\Omega}A)' = \lim_{S'} (D_{2n}^\Omega + D_{2n+1}^\Omega)' \]

We deduce

**Proposition 7.5.** For any algebra \( A \) and \( * = 0, 1 \) one has

\[ HP^*(A) = \lim_{S} HC^{2n+1}A \]

and an exact sequence

\[ 0 \rightarrow \lim_{S}^1 HC_{2n++1}A \rightarrow HP_*A \rightarrow \lim_{S} HC_{2n++1}A \rightarrow 0 \]

(where as usual \( \lim_{S}^1 HC_{2n++1}A \) is defined as

\[ \left( \prod_n HC_{2n++1}A \right) \big/ (1 - s) \left( \prod_n HC_{2n++1}A \right) \]

\( s \) being the shift on the infinite product).

**The bivariant theory**

Now \( \hat{\Omega}A \) is in a natural way a complete metric space (with the metric induced by the filtration on \( \Omega A \)—the distance of families \( (x_n) \) and \( (y_n) \) in \( \prod \Omega^n A \) is \( \leq 2^{-k} \) if the \( k \) first \( x_i \) and \( y_i \) agree). We call this the filtration topology and denote by \( \text{Hom}(\hat{\Omega}A, \hat{\Omega}B) \) the set of continuous linear maps \( \hat{\Omega}A \rightarrow \hat{\Omega}B \). It can also be described as

\[ \text{Hom}(\hat{\Omega}A, \hat{\Omega}B) = \lim_{m} \lim_{n} \text{Hom} \left( \bigoplus_{i \leq n} \Omega^i A, \bigoplus_{j \leq m} \Omega^j B \right). \]

It is a \( \mathbb{Z}/2 \)-graded complex with boundary map

\[ \partial \varphi = \varphi \circ \partial - (-1)^{\deg(\varphi)} \partial \circ \varphi \]

where \( \partial = B - b \).

**Definition 7.6.** Let \( A \) and \( B \) be algebras. Then the bivariant periodic cyclic homology \( HP_*(A, B) \) is defined as the homology of the Hom-complex

\[ HP_*(A, B) = H_*(\text{Hom}(\hat{\Omega}A, \hat{\Omega}B)) \quad *= 0, 1 \]
It is not difficult to see that the \( \mathbb{Z}/2 \)-graded complex \( \hat{\Omega}C \) is (continuously with respect to the filtration topology) homotopy equivalent to the trivial complex
\[
\mathbb{C} \xrightarrow{\cong} 0
\]
Therefore
\[
HP_*(\mathbb{C}, B) = HP_*(B) \quad \text{and} \quad HP_*(A, \mathbb{C}) = HP^*(A)
\]
There is an obvious product
\[
HP_i(A_1, A_2) \times HP_j(A_2, A_3) \to HP_{i+j}(A_1, A_3)
\]
induced by the composition of elements in \( \text{Hom}(\hat{\Omega}A_1, \hat{\Omega}A_2) \) and \( \text{Hom}(\hat{\Omega}A_2, \hat{\Omega}A_3) \), which we denote by \((x, y) \mapsto x \cdot y\). In particular, \( HP_0(A, A) \) is a unital ring with unit \( 1_A \) given by the identity map on \( \hat{\Omega}A \).

An element \( \alpha \in HP_*(A, B) \) is called invertible if there exists \( \beta \in HP_*(B, A) \) such that \( \alpha \cdot \beta = 1_A \in HP_0(A, A) \) and \( \beta \cdot \alpha = 1_B \in HP_0(B, B) \). An invertible element of degree 0, i.e., in \( HP_0(A, B) \) will also be called an \( HP \)-equivalence. Such an \( HP \)-equivalence exists in \( HP_0(A, B) \) if and only if the supercomplexes \( \hat{\Omega}A \) and \( \hat{\Omega}B \) are continuously homotopy equivalent. Multiplication by an invertible element \( \alpha \) on the left or on the right induces natural isomorphisms
\[
HP_*(B, D) \cong HP_*(A, D) \quad \text{and} \quad HP_*(D, A) \cong HP_*(D, B)
\]
for any algebra \( D \).

Remark 7.7. One can also define a \( \mathbb{Z} \)-graded version of the bivariant cyclic theory, [37], as follows. Say that a linear map \( \alpha : \hat{\Omega}A \to \hat{\Omega}B \), continuous for the filtration topology, is of order \( \leq k \) if \( \alpha(F^n\hat{\Omega}A) \subset F^{n-k}\hat{\Omega}B \) for \( n \geq k \), where \( F^n\hat{\Omega}A \) is the infinite product of \( b(\hat{\Omega}^{n+1}), \hat{\Omega}^{n+1}, \hat{\Omega}^{n+2}, \ldots \) We denote by \( \text{Hom}^k(\hat{\Omega}A, \hat{\Omega}B) \) the set of all maps of order \( \leq k \). This is, for each \( k \), a subcomplex of the \( \mathbb{Z}/2 \)-graded complex \( \text{Hom}(\hat{\Omega}A, \hat{\Omega}B) \). We can define
\[
HC_n(A, B) = H_i(\text{Hom}^n(\hat{\Omega}A, \hat{\Omega}B)) \quad \text{where} \quad i \in \{0, 1\}, i \equiv n \mod 2
\]
The bivariant theory \( HC_n(A, B) \) has a product \( HC_n(A, B) \times HC_m(B, C) \to HC_{n+m}(A, C) \) and satisfies
\[
HC_n(C, B) = HC_n(B) \quad \text{and} \quad HC_n(A, C) = HC^n(A)
\]
In general, there exist elements in \( HP_*(A, B) \) which are not in the range of the natural map \( HC_{2n+m}(A, B) \to HP_*(A, B) \) for any \( n \), [30].

The bivariant periodic theory \( HP_* \) defines a linear category that has formally exactly the same properties (cf. 3.3) as the bivariant \( K \)-theories described above.

**Theorem 7.8.** \( HP_* \) has the following properties
(a) There is an associative product

$$HP_i(A, B) \times HP_j(B, C) \rightarrow HP_{i+j}(A, C)$$

with $$i, j \in \mathbb{Z}/2$$, which is additive in both variables.

(b) There is a bilinear, graded commutative, exterior product

$$HP_i(A_1, A_2) \times HP_j(B_1, B_2) \rightarrow HP_{i+j}(A_1 \hat{\otimes} A_2, B_1 \hat{\otimes} B_2)$$

(c) Each homomorphism $$\varphi : A \rightarrow B$$ defines an element $$HP(\varphi)$$ in $$HP_0(A, B)$$. If $$\psi : B \rightarrow C$$, is another homomorphism, then

$$HP(\psi \circ \varphi) = HP(\varphi) \cdot HP(\psi)$$

$$HP_*(A, B)$$ is a contravariant functor in $$A$$ and a covariant functor in $$B$$. If $$\alpha : A' \rightarrow A$$ and $$\beta : B \rightarrow B'$$ are homomorphisms, then the induced maps, in the first and second variable of $$HP_*$$, are given by left multiplication by $$HP(\alpha)$$ and right multiplication by $$HP(\beta)$$.

(d) $$HP_*(A, A)$$ is, for each algebra $$A$$, a $$\mathbb{Z}/2$$-graded ring with unit element $$HP(id_A)$$.

(e) The functor $$HP_*$$ is invariant under diffeotopies in both variables.

(f) The canonical inclusion $$i : A \rightarrow K \otimes A$$ defines an invertible element in $$HP_0(A, K \otimes A)$$, where $$K$$ denotes the algebra of smooth compact operators considered in the previous. In particular, $$HP_*(A, B) \cong HP_*(K \otimes A, B)$$ and $$HP_*(B, A) \cong HP_*(B, K \otimes A)$$ for each algebra $$B$$.

(g) Let $$0 \rightarrow S \rightarrow P \rightarrow Q \rightarrow 0$$ be an extension of algebras and $$A$$ an algebra. There are two natural six-term exact sequences

$$HP_0(A, S) \rightarrow HP_0(A, P) \rightarrow HP_0(A, Q)$$

$$\uparrow$$

$$HP_1(A, Q) \leftarrow HP_1(A, P) \leftarrow HP_1(A, S)$$

and

$$HP_0(S, A) \leftarrow HP_0(P, A) \leftarrow HP_0(Q, A)$$

$$\downarrow$$

$$HP_1(Q, A) \rightarrow HP_1(P, A) \rightarrow HP_1(S, A)$$

where the horizontal arrows are induced by the maps in the given extension and the vertical arrows are products by a canonical element in $$HP_1(Q, S)$$ associated with the extension.

The bivariant cyclic theory $$HP_*$$ makes sense without any basic modifications for arbitrary locally convex algebras. One only has to impose continuity on all maps and use completed projective tensor products instead of algebraic tensor products everywhere. The theorem above then remains valid (again replacing tensor products by completed projective tensor products). Only in the proof of the long exact sequences one has to be a little more careful than in the purely algebraic case, cf. [27], [77].

An equivariant version, for the action of a discrete group, of bivariant cyclic theory has been developed by C.Voigt, [78].
7.3 Local cyclic cohomology and bivariant local theory

Connes used inductive limit topologies with respect to bounded (if $A$ is e.g. a Banach algebra) or finite subsets of $A$ on the algebra $\Omega(A)$ to define his entire cyclic cohomology. This idea can be considerably extended. R. Meyer developed an elegant framework of bivariant entire theory for bornological algebras (i.e. algebras with a system of “bounded” sets). Even more significantly, Puschnigg took Connes’ idea as a basis to establish the bivariant “local” theory which furnishes the “correct” homological invariants for Banach algebras and in particular for $C^*$-algebras.

If $A$ is a Fréchet algebra, for each precompact subset $C$ of $A$, we take on $\Omega(A)$ the seminorm (Minkowski functional) defined by the absolutely convex hull of all sets of the form $CdC \ldots dC$ or $dCdC \ldots dC$ in $\Omega A$. We denote the Banach space obtained as the completion of $\Omega A$ with respect to this seminorm (after first dividing by its nullspace) as $((\Omega A))$, and thus obtain an inductive system $((\Omega A)_C)$ of Banach spaces. It is well known that such inductive systems form a category, the category of “Ind-spaces”, with morphism sets between two such systems $(K_i)_I$ and $(L_j)_J$ defined by

$$\text{Hom}((K_i), (L_j)) = \lim_{\leftarrow I} \lim_{\rightarrow J} \text{Hom}(K_i, L_j)$$

On $\Omega A$ we now have to consider the boundary operator $\frac{\partial}{\partial N}B - \frac{\partial}{\partial b}$, where $N$ denotes the degree operator on $\Omega A$ (see (10)), rather than just $B - b$. In the algebraic setting this makes no difference, since the complex $(\Omega A, B - b)$ is isomorphic to $(\Omega A, \frac{\partial}{\partial N}B - \frac{\partial}{\partial b})$. In the topological setting however the difference is crucial. The conceptual explanation for the choice of $\frac{\partial}{\partial N}B - \frac{\partial}{\partial b}$ is the description of the cyclic complex in terms of the “$X$-complex” for a quasifree resolution of $A$, [30].

The operator $\frac{\partial}{\partial N}B - \frac{\partial}{\partial b}$ defines maps of Ind-spaces

$$((\Omega^e A)_C) \Rightarrow ((\Omega^o A)_C)$$

and thus a $\mathbb{Z}/2$-graded Ind-complex.

To define $H^s_{loc}$ we observe the following. The category $\text{Ho}(\text{Ind})$ of inductive systems of $\mathbb{Z}/2$-graded complexes of normed spaces with homotopy classes of chain maps as morphisms is a triangulated category. The subclass $\mathcal{N}$ of inductive systems isomorphic to a system of contractible complexes is a “null system” in $\text{Ho}(\text{Ind})$. Thus the corresponding quotient category $\text{Ho}(\text{Ind})/\mathcal{N}$ is again a triangulated category. In this quotient category all elements of $\mathcal{N}$ become isomorphic to the zero object.

In the following definition, we view $\mathbb{C}$ as a constant inductive system of complexes with zero boundary as usual.

**Definition 7.9.** Let $K = (K_i)_{i \in I}$ and $L = (L_j)_{j \in J}$ be two inductive systems of complexes. Define $H_{0}^{loc}(K, L)$ to be the space of morphisms $K \to L$ in the category $\text{Ho}(\text{Ind})/\mathcal{N}$ and $H_{n}^{loc}(K, L)$ the space of morphisms $K[n] \to L$. Let

$$H_{s}^{loc}(K) := H_{s}^{loc}(K, \mathbb{C}), \quad H_{s}^{loc}(L) := H_{s}^{loc}(\mathbb{C}, L).$$
One can compute $H^\text{loc}_*(K, L)$ via an appropriate projective resolution of $K$ (see [65]). An analysis of this resolution yields that there is a spectral sequence whose $E^2$-term involves the homologies $H_*(\mathcal{L}(K_i, L_j))$ and the derived functors of the projective limit functor $\lim\leftarrow$, and which converges to $H^\text{loc}_*(K, L)$ under suitable assumptions. For countable inductive systems, the derived functors $R^p\lim\leftarrow$ with $p \geq 2$ vanish. Hence the spectral sequence degenerates to a Milnor $\lim\leftarrow$-exact sequence

$$0 \to \lim_{\leftarrow} \lim_{\leftarrow} H_{s-1}(\mathcal{L}(K_i, L_j)) \to H^\text{loc}_*(K, L)$$

$$\to \lim_{\leftarrow} \lim_{\leftarrow} H_*(\mathcal{L}(K_i, L_j)) \to 0$$

if $I$ is countable. In the local theory we have this exact sequence for arbitrary countable inductive systems $(K_i)$. In particular, for $K = \mathbb{C}$ we obtain

$$H^\text{loc}_*(L) = \lim_{\leftarrow} H_*(L_j).$$

Since the inductive limit functor is exact, $H^\text{loc}_*(L)$ is equal to the homology of the inductive limit of $L$.

The completion of an inductive system of normed spaces $(X_i)_{i \in I}$ is defined entry-wise: $(X^c_i) := (X^c_i)_{i \in I}$.

**Definition 7.10.** Let $A$ and $B$ be Fréchet algebras. We define local cyclic cohomology, local cyclic homology, and bivariant local cyclic homology by

$$HE^\text{loc}_*(A) := H^\text{loc}_*(\Omega A), \quad HE^\text{loc}_*(B) := H^\text{loc}_*(\Omega B), \quad HE^\text{loc}_*(A, B) := H^\text{loc}_*(\Omega A, \Omega B).$$

where $\Omega A$ denotes the $\text{Ind}$-space $(\Omega A)_C$.

As usual, we have $HE^\text{loc}_*(A, \mathbb{C}) \cong HE^\text{loc}_*(A)$ and $HE^\text{loc}_*(\mathbb{C}, A) \cong HE^\text{loc}_*(A)$. The composition of morphisms gives rise to a product on $HE^\text{loc}_*$.

The main advantage of local cyclic cohomology is that it behaves well when passing to “smooth” subalgebras.

**Definition 7.11.** Let $A$ be a Fréchet algebra, let $B$ be a Banach algebra with closed unit ball $U$ and let $j : A \to B$ be an injective continuous homomorphism with dense image. We call $A$ a smooth subalgebra of $B$ iff $S^\infty := \bigcup S^n$ is precompact whenever $S \subset A$ is precompact and $j(S) \subseteq rU$ for some $r < 1$.

Let $A \subseteq B$ be a smooth subalgebra. Then an element $a \in A$ that is invertible in $B$ is already invertible in $A$. Hence $A$ is closed under holomorphic functional calculus.

Whereas the inclusion of a smooth subalgebra $A \to B$ induces an isomorphism $K_*(A) \cong K_*(B)$ in $K$-theory, the periodic or entire cyclic theories of $A$ and $B$ may differ drastically. However, local theory behaves like $K$-theory in this situation:
Theorem 7.12. Let \( j: A \to B \) be the inclusion and let \( j_* \in HE^\text{loc}_0(A,B) \) be the corresponding element in the bivariant local cyclic homology.

If \( B \) has Grothendieck’s approximation property, then \( j_* \) is invertible.

Again \( HE^\text{loc} \) defines a linear category with objects Fréchet algebras and morphism sets \( HE^\text{loc}_*(A,B) \). It has the properties listed in 3.2, 3.3, 3.4 for \( C^* \)-algebras but in a significantly more flexible form. In particular, it applies also to arbitrary Fréchet algebras. Moreover, tensor products by \( C^* \)-algebras and \( C_0(0,1) \) can be replaced by smooth subalgebras of these tensor products, etc. In some important cases, such as the one of certain group \( C^* \)-algebras it can be computed directly [67]

8 Bivariant Chern characters

The construction of characteristic classes in cyclic cohomology or homology associated to \( K \)-theory or \( K \)-homology elements has been one of the major guidelines for the development of cyclic theory, [14],[39].

After the development of a well understood machinery for cyclic homology and also of a corresponding bivariant theory with properties similar to those of bivariant \( K \)-theory, [31], the principal obstruction to the definition of a bivariant Chern character from bivariant \( K \)-theory to bivariant cyclic homology consisted in the fact that both theories were defined on different categories of algebras.

Bivariant \( K \)-theories were defined for categories of \( C^* \)-algebras. For \( C^* \)-algebras however the standard cyclic theory gives only trivial and pathological results. (One basic reason for that is the fact that cyclic theory is invariant only under differentiable homotopies, not under continuous ones. Thus for instance the algebra \( C_0([0,1]) \) of continuous functions on the interval is not equivalent for cyclic theory to \( C \).) On the other hand, cyclic theory gives good results for many locally convex algebras where a bivariant \( K \)-theory was not available.

There are two ways out of this dilemma. The first one consists in defining bivariant \( K \)-theory with good properties for locally convex algebras, [28] and the second one in developing a cyclic theory which gives good results for \( C^* \)-algebras, [65]. In both cases, once the suitable theories are constructed, the existence of the bivariant character follows from the abstract properties of the theories.

8.1 The bivariant Chern-Connes character for locally convex algebras

In this section we describe the construction of a bivariant multiplicative transformation from the bivariant theory \( kk_* \) described in chapter 6 to the bivariant
theory $HP_*$ on the category of $m$-algebras. As a very special case it will furnish the correct frame for viewing the characters for idempotents, invertibles or Fredholm modules constructed by Connes, Karoubi and others.

Consider a covariant functor $E$ from the category of $m$-algebras to the category of abelian groups which satisfies the following conditions:

(E1) $E$ is diffeotopy invariant, i.e., the evaluation map $ev_t$ in any point $t \in [0, 1]$ induces an isomorphism $E(ev_t) : E(A[0, 1]) \to E(A)$.

(E2) $E$ is stable, i.e., the canonical inclusion $\iota : A \to \mathcal{K}\hat{\otimes}A$ induces an isomorphism $E(\iota) : E(A) \to E(\mathcal{K}\hat{\otimes}A)$.

(E3) $E$ is half-exact, i.e., each extension $0 \to I \to A \to B \to 0$ admitting a continuous linear splitting induces a short exact sequence $E(I) \to E(A) \to E(B)$.

(The same conditions can of course be formulated analogously for a contravariant functor $E$.) In (E1), $A[0, 1]$ denotes, as in section 6, the algebra of smooth $A$-valued functions on $[0,1]$ whose derivatives vanish in 0 and 1. Similarly, $A(0,1)$ consists of functions that, in addition, vanish at the endpoints. We note that a standard construction from algebraic topology, using property (E1) and mapping cones, permits to extend the short exact sequence in (E3) to an infinite long exact sequence of the form

$$
\cdots \to E(B(0,1)^2) \to E(I(0,1)) \to E(A(0,1)) \\
\to E(B(0,1)) \to E(I) \to E(A) \to E(B)
$$

see, e.g., [41] or [8].

**Theorem 8.1.** Let $E$ be a covariant functor with the properties (E1), (E2), (E3). Then we can associate in a unique way with each $h \in kk_0(A, B)$ a morphism of abelian groups $E(h) : E(A) \to E(B)$, such that $E(h_1 \cdot h_2) = E(h_2) \circ E(h_1)$ for the product $h_1 \cdot h_2$ of $h_1 \in kk_0(A, B)$ and $h_2 \in kk_0(B, C)$ and such that $E(kk(\alpha)) = E(\alpha)$ for each morphism $\alpha : A \to B$ of $m$-algebras (recall that $kk(\alpha)$ denotes the element of $kk_0(A, B)$ induced by $\alpha$).

An analogous statement holds for contravariant functors.

**Proof.** Let $h$ be represented by $\eta : J^{2n}A \to \mathcal{K}\hat{\otimes}B$. We set

$$
E(h) = E(\iota)^{-1}E(\eta)E(\varepsilon^n)^{-1}E(\iota)
$$

where $\varepsilon^n$ is the classifying map for the iterated Bott extension used in the definition of $kk_*$ and $\iota$ denotes the inclusion of an algebra into its tensor product by $\mathcal{K}$. It is clear that $E(h)$ is well-defined and that $E(kk(\alpha)) = E(\alpha)$. $\square$

The preceding result can be interpreted differently, see also [35], [8]. For this, consider again $kk_0$ as an additive category, whose objects are the $m$-algebras, and where the morphism set between $A$ and $B$ is given by $kk_0(A, B)$. 
This category is additive in the sense that the morphism set between two objects forms an abelian group and that the product of morphisms is bilinear. We denote the natural functor from the category of \(m\)-algebras to the category \(kk_0\), which is the identity on objects, by \(kk\).

**Corollary 8.2.** Let \(F\) be a functor from the category of \(m\)-algebras to an additive category \(C\), such that \(F(\beta \circ \alpha) = F(\alpha) \cdot F(\beta)\), for any two homomorphisms \(\alpha : A_1 \to A_2\) and \(\beta : A_2 \to A_3\) between \(m\)-algebras. We assume that for each \(B\), the contravariant functor \(C(F(\cdot), F(\cdot))\) and the covariant functor \(C(F(B), F(\cdot))\) on the category of \(m\)-algebras satisfy the properties (E1), (E2), (E3). Then there is a unique covariant functor \(F'\) from the category \(kk_0\) to \(C\), such that \(F = F' \circ kk\).

**Remark 8.3.** Property (E3) implies that any such functor \(F'\) is automatically additive:

\[
F'(h + g) = F'(h) + F'(g)
\]

As a consequence of the preceding corollary we get a bilinear multiplicative transformation from \(kk_0\) to \(HP_0\) — the bivariant Chern-Connes character.

**Corollary 8.4.** There is a unique (covariant) functor \(ch : kk_0 \to HP_0\), such that \(ch(kk(\alpha)) = HP_0(\alpha) \in HP_0(A, B)\) for every morphism \(\alpha : A \to B\) of \(m\)-algebras.

**Proof.** This follows from 8.2, since \(HP_0\) satisfies conditions (E1), (E2) and (E3) in both variables. \(\square\)

The Chern-Connes-character \(ch\) is by construction compatible with the composition product on \(kk_0\) and \(HP_0\). It also is compatible with the exterior product on \(kk_0\) (as in 3.3(a)), and the corresponding product on \(HP_0\), see [31], p.86.

It remains to extend \(ch\) to a multiplicative transformation from the \(\mathbb{Z}/2\)-graded theory \(kk_*\) to \(HP_*\) and to study the compatibility with the boundary maps in the long exact sequences associated to an extension for \(kk\) and \(HP\).

The natural route to the definition of \(ch\) in the odd case is the use of the identity \(kk_1(A, B) = kk_0(J(A, B))\).

Since \(HP\) satisfies excision in the first variable and since \(HP_0(T, A, B) = 0\) for all \(B\) (\(T, A\) is contractible), we find that

\[
HP_0(J, A, B) \cong HP_1(A, B)
\]

(13)

However, for the product \(kk_1 \times kk_1 \to kk_0\) we have to use the identification

\[
kk_0(J^2 A, B) \cong kk_0(A, B)
\]

which is induced by the \(\varepsilon\)-map \(J^2 A \to K \otimes A\). This identification is different from the identification \(HP_0(J^2 A, B) \cong HP_0(A, B)\) which we obtain by applying (13) twice. In fact, we have
**Proposition 8.5.** Under the natural identification

\[ HP_0(J^2 A, K \otimes A) \cong HP_0(A, A) \]

from (13), the element \( ch(\varepsilon) \) corresponds to \( (2\pi i)^{-1} \).

We are therefore lead to the following definition.

**Definition 8.6.** Let \( u \) be an element in \( kk(A, B) \) and let \( u_0 \) be the corresponding element in \( kk_0(JA, B) \). We set

\[ ch(u) = \sqrt{2\pi i} \ ch(u_0) \in HP_1(A, B) \cong HP_0(JA, B) \]

**Theorem 8.7.** The thus defined Chern-Connes character \( ch : kk_* \to HP_* \) is multiplicative, i.e., for \( u \in kk_i(A, B) \) and \( v \in kk_j(B, C) \) we have

\[ ch(u \cdot v) = ch(u) \cdot ch(v) \]

It follows that the character is also compatible with the boundary maps in the six-term exact sequences induced by an extension in both variables of \( kk_* \) and \( HP_* \).

For \( m \)-algebras, the bivariant character \( ch \) constructed here is a far reaching generalization of the Chern characters from \( K \)-Theory and \( K \)-homology considered by Connes, Karoubi and many others.

### 8.2 The Chern character for \( C^* \)-algebras

Bivariant local cyclic homology is exact for extensions with a bounded linear section, homotopy invariant for smooth homotopies and stable with respect to tensor products with the trace class operators \( \ell^1(H) \). Using Theorem 7.12, we can strengthen these properties considerably:

**Theorem 8.8.** Let \( A \) be a \( C^* \)-algebra. The functors \( B \mapsto HE_{loc}^*(A, B) \) and \( B \mapsto HE_{loc}^*(B, A) \) are split exact, stable homotopy functors on the category of \( C^* \)-algebras.

For separable \( C^* \)-algebras, there is a natural bivariant Chern character

\[ ch : KK_*(A, B) \to HE_{loc}^*(A, B). \]

The Chern character is multiplicative with respect to the Kasparov product on the left and the composition product on the right hand side.

If both \( A \) and \( B \) satisfy the universal coefficient theorem in Kasparov theory, then there is a natural isomorphism

\[ HE_{loc}^*(A, B) \cong \text{Hom}(K_*(A) \otimes_{\mathbb{Z}} \mathbb{C}, K_*(B) \otimes_{\mathbb{Z}} \mathbb{C}). \]
Proof. Since $C^\infty([0,1],A) \subseteq C([0,1],A)$ is a smooth subalgebra, Theorem 7.12 and smooth homotopy invariance imply continuous homotopy invariance. The projective tensor product of $A$ by the algebra $K$ of smooth compact operators is a smooth subalgebra of the $C^*$-algebraic stabilization of $A$. Hence Theorem 7.12 and stability with respect to $K$ imply $C^*$-algebraic stability (i.e. invariance under $C^*$-tensor product by the $C^*$-version of $K$).

The existence of the bivariant Chern character follows from these homological properties by the universal property of Kasparov's $KK$-theory as in the previous section.

The last assertion is trivial for $A = B = \mathbb{C}$. The class of $C^*$-algebras for which it holds is closed under $KK$-equivalence, inductive limits and extensions with completely positive section. Hence it contains all $C^*$-algebras satisfying the universal coefficient theorem (see [8]).   

\[ \square \]

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