1

Witt Groups

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Introduction

In his 1937 paper [86], Ernst Witt introduced a group structure – and even a ring structure – on the set of isometry classes of anisotropic quadratic forms, over an arbitrary field k. This object is now called the *Witt group* $\mathrm{W}(k)$ of k. Since then, Witt's construction has been generalized from fields to rings with involution, to schemes, and to various types of categories with duality. For the sake of efficacy, we review these constructions in a non-chronological order. Indeed, in Section 1.1, we start with the now "classical framework" in its most general form, namely over *exact categories with duality.* This folklore material is a basically straightforward generalization of Knebusch's scheme case [41], where the exact category was the one of vector bundles. Nevertheless, this level of generality is hard to find in the literature, like *e.g.* the "classical sublagrangian reduction" of Subsection 1.1.5. In Section 1.2, we specialize this classical material to the even more classical examples listed above : schemes, rings, fields. We include some motivations for the use of Witt groups.

This chapter focusses on the theory of Witt groups in parallel to Quillen's K-theory and is not intended as a survey on quadratic forms. In particular, the immense theory of quadratic forms over fields is only alluded to in Subsection 1.2.4; see preferably the historical surveys of Pfister [66] and Scharlau [72]. Similarly, we do not enter the arithmetic garden of quadratic forms : lattices, codes, sphere packings and so on. In fact, even Witt-group-like objects have proliferated to such an extent that everything could not be included here. However, in the intermediate Section 1.3, we provide a very short guide to various sources for the connections between Witt groups and other theories.

The second part of this chapter, starting in Section 1.4, is dedicated to the Witt groups of *triangulated categories with duality*, and to the recent developments of this theory. In Section 1.5, we survey the applications of triangular Witt groups to the above described classical framework.

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1.1 Usual Witt Groups: General Theory

1.1.1 Duality and Symmetric Spaces

Definition 1.1.1. A *category with duality* is a triple $(\mathcal{C}, *, \varpi)$ made of a category $\mathcal{C}$ and an involutive endo-functor $* : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{C}$ with given isomorphism

$$\varpi : \mathrm{Id}_{\mathcal{C}} \xrightarrow{\simeq} * \circ *$$

and subject to the condition below. Write as usual $M^* := *(M)$ for *the dual* of an object $M \in \mathcal{C}$ and similarly for morphisms. Then $M \mapsto M^*$ is a functor and $\varpi_M : M \xrightarrow{\sim} (M^*)^*$ is a natural isomorphism such that:

$$(\varpi_M)^* \circ \varpi_{M^*} = \mathrm{id}_{M^*} \qquad \text{for any object } M \in \mathcal{C}.$$

Definition 1.1.2. A *symmetric space* in $(\mathcal{C},*,\varpi)$ – or simply in $\mathcal{C}$ – consists of a pair (P,φ) where P is an object of $\mathcal{C}$ and where $\varphi : P \xrightarrow{\sim} P^*$ is a symmetric isomorphism, called *the symmetric form* of the space (P,φ). The symmetry of φ reads $\varphi^* \circ \varpi_P = \varphi$, *i.e.* $\varphi^* = \varphi$ when P is identified with P^{**} via ϖ_P :

$$\begin{tikzcd} P \arrow[r, "\varphi"] \arrow[d, "\varpi_P"', "\cong"] & P^* \arrow[d, equal] \\ P^{**} \arrow[r, "\varphi^*"'] & P^* \end{tikzcd}$$

Note that the notion of "symmetry" depends on the chosen identification ϖ of objects of $\mathcal{C}$ with their double dual. This allows us to treat *skew-symmetric* forms as symmetric forms in $(\mathcal{C},*,-\varpi)$. Nevertheless, when clear from the context, we drop ϖ from the notations and identify P^{**} with P.

Remark 1.1.3. We shall focus here on "non-degenerate" or "unimodular" forms, that is, we almost always assume that φ is an isomorphism. In good cases, one can consider the non-unimodular forms as being unimodular in a different category (of morphisms). See Bayer-Fluckiger – Fainsilber [16].

Definition 1.1.4. Two symmetric spaces (P,φ) and (Q,ψ) are called *isometric* if there exists an *isometry* $h : (P,\varphi) \xrightarrow{\sim} (Q,\psi)$, that is an isomorphism $h : P \xrightarrow{\sim} Q$ in the category $\mathcal{C}$ respecting the symmetric forms, *i.e.* $h^*\psi\, h = \varphi$.

Definition 1.1.5. A *morphism of categories with duality*

$$(\mathcal{C},*^{\mathcal{C}},\varpi^{\mathcal{C}}) \longrightarrow (\mathcal{D},*^{\mathcal{D}},\varpi^{\mathcal{D}})$$

consists of a pair (F,η) where $F : \mathcal{C} \to \mathcal{D}$ is a functor and $\eta : F\circ *^{\mathcal{C}} \xrightarrow{\simeq} *^{\mathcal{D}}\circ F$ is an isomorphism respecting ϖ, *i.e.* for any object M of $\mathcal{C}$, the following diagram commutes :

$$\begin{tikzcd} F(M) \arrow[r, "F(\varpi^{\mathcal{C}}_M)"] \arrow[d, "\varpi^{\mathcal{D}}_{F(M)}"'] & F(M^{**}) \arrow[d, "\eta_{M^*}"] \\ F(M)^{**} \arrow[r, "(\eta_M)^*"'] & F(M^*)^* \end{tikzcd}$$

where $(-)^*$ is $(-)^{*^{\mathcal{C}}}$ or $(-)^{*^{\mathcal{D}}}$ depending on the context, in the obvious way.

Definition 1.1.6. An *additive category with duality* is a category with duality $(\mathcal{A},*,\varpi)$ where $\mathcal{A}$ is additive and where $*$ is an additive functor, *i.e.* $(A\oplus B)^* = A^* \oplus B^*$ via the natural morphism.

Remark 1.1.7. The identification ϖ necessarily respects the additivity, *i.e.* $\varpi_{A\oplus B} = \varpi_A \oplus \varpi_B$. This is a general fact for natural transformations between additive functors. Similarly, we need not consider ϖ in the following :

Definition 1.1.8. A *morphism of additive categories with duality* is simply a morphism of categories with duality (F,η) in the sense of Def. 1.1.5, such that the functor F is additive.

Example 1.1.9. In an additive category with duality $(\mathcal{A}, *, \varpi)$, one can produce symmetric spaces (P, φ) as follows. Take any object $M \in \mathcal{A}$. Put $P := M \oplus M^*$ and

$$\varphi := \left(\begin{smallmatrix} 0 & \mathrm{id}_{M^*} \\ \varpi_M & 0 \end{smallmatrix}\right) : \quad \underbrace{M \oplus M^*}_{=P} \quad \xrightarrow{\simeq} \quad \underbrace{M^* \oplus M^{**}}_{=P^*}.$$

Note that the symmetry of φ uses the assumption $(\varpi_M)^* = (\varpi_{M^*})^{-1}$. This space (P, φ) is called the *hyperbolic space* (over M) and is denoted by $\mathrm{H}(M)$.

Definition 1.1.10. Let $(\mathcal{A}, *, \varpi)$ be an additive category with duality. Let (P, φ) and (Q, ψ) be symmetric spaces. We define *the orthogonal sum* of these spaces as being the symmetric space $(P, \varphi) \perp (Q, \psi) := \Big(P \oplus Q\,,\, \left(\begin{smallmatrix} \varphi & 0 \\ 0 & \psi \end{smallmatrix}\right)\Big)$.

Definition 1.1.11. Let $(F, \eta) : (\mathcal{C}, *^{\mathcal{C}}, \varpi^{\mathcal{C}}) \longrightarrow (\mathcal{D}, *^{\mathcal{D}}, \varpi^{\mathcal{D}})$ be a morphism of categories with duality and let (P, φ) be a symmetric space in $\mathcal{C}$. Then

$$F\big(P, \varphi\big) := \big(F(P)\,,\, \eta_P \circ F(\varphi)\big)$$

is a symmetric space in $\mathcal{D}$, called *the image by* F of the space (P, φ).

It is clear that two isometric symmetric spaces in $\mathcal{C}$ have isometric images by F. If we assume moreover that F is a morphism of *additive* categories with duality, it is also clear that the image of the orthogonal sum is isometric to the orthogonal sum of the images; similarly, the image of the hyperbolic space $\mathrm{H}(M)$ over any $M \in \mathcal{C}$ is then isometric to $\mathrm{H}\big(F(M)\big)$.

1.1.2 Exact Categories with Duality

Remark 1.1.12. The reader is referred to the original Quillen [68] or to the minimal Keller [40, App. A] for the definition of an *exact category.* The basic example of such a category is the one of vector bundles over a scheme. We denote by $\rightarrowtail$ and by $\twoheadrightarrow$ the *admissible* monomorphisms and epimorphisms, respectively. Note that being exact (unlike additive) is *not* an intrinsic property. By a *split exact category* we mean an additive category where the admissible exact sequences are exactly the split ones. The basic example of the latter is the split exact category of finitely generated projective modules over a ring.

Definition 1.1.13. An *exact category with duality* is an additive category with duality $(\mathcal{E}, *, \varpi)$ in the sense of Def. 1.1.6, where the category $\mathcal{E}$ is exact and such that the functor $*$ is exact. So, $\mathcal{E}$ is an exact category, $M \mapsto M^*$ is a contravariant endofunctor on $\mathcal{E}$, $\varpi_M : M \xrightarrow{\sim} M^{**}$ is a natural isomorphism such that $(\varpi_M)^* = (\varpi_{M^*})^{-1}$ and for any admissible exact sequence $A \overset{\alpha}{\rightarrowtail} B \overset{\pi}{\twoheadrightarrow} C$, the following (necessarily exact) sequence is admissible:

$$C^* \overset{\pi^*}{\rightarrowtail} B^* \overset{\alpha^*}{\twoheadrightarrow} A^*.$$

Example 1.1.14. The key example of an exact category with duality is the one of vector bundles over a scheme with the usual duality, see Subsection 1.2.1 below. Note also that any additive category with duality can be viewed as a (split) exact category with duality.

Definition 1.1.15. A *morphism of exact categories with duality* (F, η) is a morphism of categories with duality (Def. 1.1.5) such that F is exact, *i.e.* F sends admissible short exact sequences to admissible short exact sequences. Such a functor F is necessarily additive.

1.1.3 Lagrangians and Metabolic Spaces

Definition 1.1.16. Let $(\mathcal{E}, *, \varpi)$ be an exact category with duality (see Def. 1.1.13). Let (P, ϖ) be a symmetric space in $\mathcal{E}$. Let $\alpha : L \rightarrowtail P$ be an admissible monomorphism. *The orthogonal* in (P, φ) of the pair (L, α) is as usual

$$(L,\alpha)^{\perp} \;:=\; \ker(\alpha^*\varphi\colon P \to L^*)\,.$$

Explicitly, consider an admissible exact sequence $L \overset{\alpha}{\rightarrowtail} P \overset{\pi}{\twoheadrightarrow} M$ and dualize it to get the second line below:

$$\begin{tikzcd}
 & P \arrow[d, "\varphi"', "\simeq"] \arrow[dr, "\alpha^*\varphi"] & \\
M^* \arrow[ur, dashed] \arrow[r, tail, "\pi^*"'] & P^* \arrow[r, two heads, "\alpha^*"'] & L^*
\end{tikzcd}$$

This describes $(L,\alpha)^{\perp} := \ker(\alpha^*\varphi)$ as the pair $\big(M^*\ ,\ \varphi^{-1}\pi^* : M^* \rightarrowtail P\big)$. We shall write $L^{\perp}$ instead of M^* when the monomorphism α is understood.

Definition 1.1.17. An (admissible) *sublagrangian* of a symmetric space (P, φ) is an admissible monomorphism $\alpha\colon L \rightarrowtail P$ such that the following conditions are satisfied:

(a) the form φ vanishes on L, that is $\quad \alpha^*\varphi\,\alpha = 0 \quad : \quad L \to L^*$,
(b) the induced monomorphism $\beta : L \to L^{\perp}$ is admissible; in the above notations β is the unique morphism such that $(\varphi^{-1}\pi^*) \circ \beta = \alpha$.

Remark 1.1.18. For condition (b), consider the diagram coming from above:

$$\begin{tikzcd}
L \arrow[r, tail, "\alpha"] \arrow[d, "\beta"'] & P \arrow[r, two heads, "\pi"] \arrow[d, "\simeq"', "\varphi"] & (L^{\perp})^* \arrow[d, "\beta^*"] \\
L^{\perp} \arrow[r, tail, "\pi^*"'] & P^* \arrow[r, two heads, "\alpha^*"'] & L^*.
\end{tikzcd} \tag{1.1}$$

Since $\alpha^* \circ (\varphi\,\alpha) = 0$, there exists a unique $\beta : L \to L^{\perp}$ as claimed. Observe that β^* makes the right square commutative by symmetry (we drop the ϖ's).

This β is automatically a monomorphism since α is, and β^* is automatically an epimorphism. Condition (b) only requires them to be admissible. However, in many cases, it is in fact automatic, namely when the exact category $\mathcal{E}$ can be embedded into some abelian category $\iota : \mathcal{E} \hookrightarrow \mathcal{A}$ in such a way that a morphism q in $\mathcal{E}$ is an admissible epimorphism in $\mathcal{E}$ if and only if $\iota(q)$ is an epimorphism in $\mathcal{A}$. This can always be achieved if $\mathcal{E}$ is *semi-saturated*, *i.e.* if any split epimorphism is admissible, in particular if $\mathcal{E}$ is idempotent complete (see [79, App. A]). So, in real life, condition (b) is often dropped.

Definition 1.1.19. An (admissible) *lagrangian* of a symmetric space (P, φ) is an admissible sublagrangian (L, α) such that $L = L^\perp$, *i.e.* a sublagrangian as in Def. 1.1.17 such that the morphism $\beta : L \rightarrowtail L^\perp$ is an isomorphism.

Note that (L, α) is a lagrangian of the space (P, φ) if and only if the following is an admissible exact sequence – compare diagram (1.1) :

$$L \overset{\alpha}{\rightarrowtail} P \overset{\alpha^*\varphi}{\twoheadrightarrow} L^* . \tag{1.2}$$

Definition 1.1.20. A symmetric space (P, φ) is called *metabolic* if it possesses an admissible lagrangian, *i.e.* if there exists an exact sequence as above.

Example 1.1.21. Assume that the exact sequence (1.2) is *split* exact. Such a metabolic symmetric space is usually called *split metabolic*. A symmetric space is split metabolic if and only if it is isometric to a space of the form

$$\big(L \oplus L^* \, , \begin{pmatrix} 0 & 1 \\ \varpi_L & \xi \end{pmatrix} \big)$$

for some object L and some symmetric morphism $\xi = \xi^*$. In particular, any hyperbolic space $\mathrm{H}(L)$ is split metabolic with $\xi = 0$. If we assume further that 2 is invertible in $\mathcal{E}$ (see 1.4.10), then any split metabolic space is isometric to a hyperbolic space $\mathrm{H}(L)$ via the automorphism h of $L \oplus L^*$ defined by :

$$\underbrace{\begin{pmatrix} 1 & 0 \\ -\frac{1}{2}\xi & 1 \end{pmatrix}}_{h^*} \cdot \begin{pmatrix} 0 & 1 \\ \varpi_L & \xi \end{pmatrix} \cdot \underbrace{\begin{pmatrix} 1 & -\frac{1}{2}\xi \\ 0 & 1 \end{pmatrix}}_{h} = \begin{pmatrix} 0 & 1 \\ \varpi_L & 0 \end{pmatrix} .$$

Note that a symmetric space is split metabolic if and only if it is metabolic for the split exact structure of the additive category $\mathcal{E}$. See Ex. 1.2.5 below for an exact sequence like (1.2) which does not split (when the category $\mathcal{E}$ is not split) and Ex. 1.2.4 for a split-metabolic space which is not hyperbolic (when 2 is not invertible).

Example 1.1.22. Let $(\mathcal{A}, *, \varpi)$ be an additive category with duality and let (P, φ) be a symmetric space. Then the sequence

$$P \overset{\alpha := \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)}{\rightarrowtail} P \oplus P \overset{\left(\begin{smallmatrix} -\varphi & \varphi \end{smallmatrix}\right)}{\twoheadrightarrow} P^*$$

is split exact and the second morphism is equal to $\alpha^* \circ \left(\begin{smallmatrix}\varphi & 0\\ 0 & -\varphi\end{smallmatrix}\right)$. This proves that the symmetric space $(P,\varphi)\perp(P,-\varphi)$ is split metabolic in $(\mathcal{A},*,\varpi)$ and hence in any exact category.

Remark 1.1.23. It is easy to prove that the only symmetric space structure on the zero object is metabolic, that any symmetric space isometric to a metabolic one is also metabolic, that the orthogonal sum of metabolic spaces is again metabolic and that the image (see Def. 1.1.11) of a metabolic space by a morphism of exact categories with duality is again metabolic. For the latter, the image of a lagrangian is a lagrangian of the image.

1.1.4 The Witt Group of an Exact Category with Duality

We only consider additive categories which are *essentially small*, *i.e.* whose class of isomorphism classes of objects is a set.

Definition 1.1.24. Let $(\mathcal{A},*,\varpi)$ be an additive category with duality (1.1.6). Denote by $\mathrm{MW}(\mathcal{A},*,\varpi)$ the set of *isometry classes of symmetric spaces* in $\mathcal{A}$.

The orthogonal sum gives a structure of abelian monoid on $\mathrm{MW}(\mathcal{A},*,\varpi)$.

Definition 1.1.25. Let $(\mathcal{E},*,\varpi)$ be an *exact* category with duality (1.1.13). Let $\mathrm{NW}(\mathcal{E},*,\varpi)$ be the subset of $\mathrm{MW}(\mathcal{E},*,\varpi)$ of the classes of metabolic spaces. This defines a submonoid of $\mathrm{MW}(\mathcal{E},*,\varpi)$ by Rem. 1.1.23.

Remark 1.1.26. Let $(M,+)$ be an abelian monoid (*i.e.* "a group without inverses") and let N be a submonoid of M (*i.e.* $0\in N$ and $N+N\subset N$). Consider the equivalence relation: for $m_1,m_2\in M$, define $m_1\sim m_2$ if there exists $n_1,n_2\in N$ such that $m_1+n_1=m_2+n_2$. Then the set of equivalence classes $M/\!\sim$ inherits a structure of abelian monoid via $[m]+[m']:=[m+m']$. It is denoted by M/N. Assume that for any element $m\in M$ there is an element $m'\in M$ such that $m+m'\in N$, then M/N is an abelian group with $-[m]=[m']$. It is then canonically isomorphic to the quotient of the Grothendieck group of M by the subgroup generated by N.

Definition 1.1.27 (Knebusch). Let $(\mathcal{E},*,\varpi)$ be an exact category with duality. The *Witt group* of $\mathcal{E}$ is the quotient of symmetric spaces modulo metabolic spaces, *i.e.*

$$\mathrm{W}(\mathcal{E},*,\varpi) := \frac{\mathrm{MW}(\mathcal{E},*,\varpi)}{\mathrm{NW}(\mathcal{E},*,\varpi)}\,.$$

This is an abelian group. We denote by $[P,\varphi]$ the class of a symmetric space (P,φ) in $\mathrm{W}(\mathcal{E})$, sometimes called *the Witt class of the symmetric space* (P,φ). We have $-[P,\varphi]=[P,-\varphi]$ by Ex. 1.1.22 and the above Remark.

Definition 1.1.28. Two symmetric spaces (P,φ) and (Q,ψ) which define the same Witt class, $[P,\varphi]=[Q,\psi]$, are called *Witt equivalent*. This amounts to the existence of metabolic spaces (N_1,θ_1) and (N_2,θ_2) and of an isometry $(P,\varphi)\perp(N_1,\theta_1)\simeq(Q,\psi)\perp(N_2,\theta_2)$.

Remark 1.1.29. A Witt class $[P,\varphi]=0$ is trivial in $\mathrm{W}(\mathcal{E})$ if and only if there exists a split metabolic space (N,θ) with $(P,\varphi)\perp(N,\theta)$ metabolic, or equivalently, if and only if there exists a metabolic space (N,θ) with $(P,\varphi)\perp(N,\theta)$ split metabolic. This follows easily from the definition, by stabilizing with suitable symmetric spaces inspired by Ex. 1.1.22. However, we will see in Ex. 1.2.6 below that a symmetric space (P,φ) with $[P,\varphi]=0$ needs not be metabolic itself, even when $\mathcal{E}$ is a split exact category.

Remark 1.1.30. It is easy to check that $\mathrm{W}(-)$ is a covariant functor from exact categories with duality to abelian groups, via the construction of Def. 1.1.11.

1.1.5 The Sublagrangian Reduction

We now explain why the Witt-equivalence relation (Def. 1.1.28) is of interest for symmetric spaces. Two spaces are Witt equivalent in particular if we can obtain one of them by "chopping off" from the other one some subspace on which the symmetric form is trivial, *i.e.* by chopping off a sublagrangian.

Let $(\mathcal{E},*,\varpi)$ be an exact category with duality. Let (P,φ) be a symmetric space in $\mathcal{E}$ and let (L,α) be an admissible sublagrangian (Def. 1.1.17) of the space (P,φ). Recall from (1.1) that we have a commutative diagram

$$\begin{tikzcd}
 & & Q^* \arrow[d, tail, "\mu^*"] \\
L \arrow[r, tail, "\alpha"] \arrow[d, tail, "\beta"'] & P \arrow[r, two heads, "\pi"] \arrow[d, "\varphi", "\simeq"'] & (L^{\perp})^* \arrow[d, "\beta^*"] \\
L^{\perp} \arrow[r, tail, "\pi^*"'] \arrow[d, "\mu"'] & P^* \arrow[r, two heads, "\alpha^*"'] & L^* \\
Q := L^{\perp}/L & &
\end{tikzcd} \tag{1.3}$$

where we also introduce the cokernel Q in $\mathcal{E}$ of the admissible monomorphism β, displayed in the first column. The third column is the dual of the first.

Now consider the morphism $s := \pi\,\varphi^{-1}\pi^* : L^{\perp} \longrightarrow (L^{\perp})^*$. This is nothing but the form φ "restricted" to the orthogonal $L^{\perp}$ via the monomorphism $\varphi^{-1}\pi^* \colon L^{\perp} \rightarrowtail P$ from Def. 1.1.16. Observe that the morphism s is symmetric: $s^*=s$, that $s\,\beta=0$ and that $\beta^*\,s=0$. From this, we deduce easily (in two steps) the existence of a *unique* morphism

$$\psi : Q \longrightarrow Q^* \qquad \text{such that} \qquad s=\mu^*\,\psi\mu. \tag{1.4}$$

One checks that ψ^* also satisfies equation (1.4). Therefore ψ is *symmetric*: $\psi=\psi^*$. Below, we shall get for free that ψ is an isomorphism, and hence defines a form on $Q=L^{\perp}/L$, but note that we could deduce it immediately from the Snake Lemma in some "ambient abelian category".

Lemma 1.1.31. *The following left hand square is a push-out:*

$$\begin{tikzcd} L \arrow[r, tail, "\alpha"] \arrow[d, tail, "\beta"'] & P \arrow[r, two heads, "\pi"] \arrow[d, tail, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] & (L^\perp)^* \arrow[d, equal] \\ L^\perp \arrow[r, tail, "\left(\begin{smallmatrix}\varphi^{-1}\pi^*\\ \mu\end{smallmatrix}\right)"] & P \oplus Q \arrow[r, two heads, "\left(\begin{smallmatrix}\pi & -\mu^*\psi\end{smallmatrix}\right)"] & (L^\perp)^* \end{tikzcd} \tag{1.5}$$

and the diagram commutes and has admissible exact lines.

Proof. One checks directly that the left-hand square satisfies the universal property of the push-out: use that if two test-morphisms $x : P \to Z$ and $y : L^\perp \to Z$ are such that $x\alpha = y\beta$ then the auxiliary morphism $w := y - x\,\varphi^{-1}\pi^* : L^\perp \to Z$ factors uniquely as $w = \bar{w}\mu$ because of $w\,\beta = 0$ and hence $(x\ \ \bar{w}) : P \oplus Q \to Z$ is the wanted morphism. It follows from the axioms of an exact category that the morphism $\gamma := \left(\begin{smallmatrix}\varphi^{-1}\pi^*\\ \mu\end{smallmatrix}\right) : L^\perp \rightarrowtail P \oplus Q$ is an admissible monomorphism. It is a general fact that the two monomorphisms α and γ must then have the same cokernel, and it is easy to prove (using that μ is an epimorphism) that $\mathrm{Coker}(\gamma)$ is as in the second line of (1.5). $\square$

Comparing that second line of (1.5) to its own dual and using symmetry of φ and ψ, we get the following commutative diagram with exact lines:

$$\begin{tikzcd} L^\perp \arrow[r, tail, "\left(\begin{smallmatrix}\varphi^{-1}\pi^*\\ \mu\end{smallmatrix}\right)"] \arrow[d, equal] & P \oplus Q \arrow[r, two heads, "\left(\begin{smallmatrix}\pi & -\mu^*\psi\end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix}\varphi & 0\\ 0 & -\psi\end{smallmatrix}\right)"] & (L^\perp)^* \arrow[d, equal] \\ L^\perp \arrow[r, tail, "\left(\begin{smallmatrix}\pi^*\\ -\psi\mu\end{smallmatrix}\right)"] & P^* \oplus Q^* \arrow[r, two heads, "\left(\begin{smallmatrix}\pi\,\varphi^{-1} & \mu^*\end{smallmatrix}\right)"] & (L^\perp)^*. \end{tikzcd}$$

This proves two things. First $\left(\begin{smallmatrix}\varphi & 0\\ 0 & -\psi\end{smallmatrix}\right)$ is an isomorphism and hence ψ is an isomorphism, *i.e.* (Q, ψ) is a symmetric space, as announced. Secondly, our monomorphism $\gamma : L^\perp \rightarrowtail P \oplus Q$ is a lagrangian of the space $(P, \varphi) \perp (Q, -\psi)$. This means that the space $(P, \varphi) \perp (Q, -\psi)$ is metabolic, *i.e.* $[P, \varphi] = [Q, \psi]$ in the Witt group. So we have proven the following folklore result:

Theorem 1.1.32. *Let $(\mathcal{E}, *, \varpi)$ be an exact category with duality. Let (P, φ) be a symmetric space in $\mathcal{E}$ and let (L, α) be an admissible sublagrangian of the space (P, φ). Consider the orthogonal $L^\perp$ and the quotient $L^\perp/L$. Then there is a unique form ψ on $L^\perp/L$ which is induced by the restriction of φ to $L^\perp$. Moreover, the symmetric space $(L^\perp/L\,,\ \psi)$ is Witt equivalent to (P, φ).* $\square$

Remark 1.1.33. A sort of converse holds: any two Witt equivalent symmetric spaces can be obtained from a common symmetric space by the above sublagrangian reduction, with respect to two different sublagrangians. This is obvious since a metabolic space with lagrangian L reduces to zero: $L^\perp/L = 0$.

Remark 1.1.34. Observe that $L^\perp/L$ is a *subquotient* of P and hence should be thought of as "smaller" than P. If $L^\perp/L$ still possesses an admissible sublagrangian, we can chop it out again. And so on. If the category $\mathcal{E}$ is reasonable, this process ends with a space possessing no admissible sublagrangian – this could be called *(admissibly) anisotropic*. Even then, such an admissibly anisotropic symmetric space needs not be unique up to isometry in the Witt class of the symmetric space (P, φ) that we start with. See more in 1.2.22.

1.2 Usual Witt Groups: Examples and Motivations

Still in a very anti-chronological order, we specialize the categorical definitions of the previous section to more classical examples.

1.2.1 Schemes

The origin is Knebusch [41]. The affine case is older: see the elegant Milnor-Husemoller [50]. A modern reference is Knus [42, Chap. VIII].

Let X be a scheme and let VB_X be the category of locally free coherent $\mathcal{O}_X$-modules (*i.e.* vector bundles). Let $\mathcal{L}$ be a line bundle over X. One defines a duality $* : \mathrm{VB}_X \to \mathrm{VB}_X$ by $E^* := \mathrm{Hom}_{\mathcal{O}_X}(E, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{L}$, which is the *usual duality twisted by the line bundle* $\mathcal{L}$. One defines the natural identification $\varpi : E \overset{\sim}{\to} E^{**}$ in the usual way. For $\mathcal{L} = \mathcal{O}_X$, this E^* is of course the usual dual and ϖ is locally given by mapping an element e to the evaluation at e. The triple $(\mathrm{VB}_X, *, \varpi)$ is an exact category with duality in the sense of Def. 1.1.13. We can thus apply Def. 1.1.27 to get Knebusch's original one [41]:

Definition 1.2.1. With the above notations, the *usual Witt group of a scheme* X with values in the line bundle $\mathcal{L}$ is the Witt group (Def. 1.1.27):

$$\mathrm{W}(X, \mathcal{L}) := \mathrm{W}(\mathrm{VB}_X, *, \varpi).$$

The special case $\mathcal{L} = \mathcal{O}_X$ is the usual usual Witt group $\mathrm{W}(X)$.

Remark 1.2.2. Let R be a commutative ring. We define $\mathrm{W}(R)$ as $\mathrm{W}(\mathrm{Spec}(R))$; this common convention of dropping the "$\mathrm{Spec}(-)$" applies everywhere below. The category VB_R is simply the category of finitely generated projective R-modules, which is a split exact category. So, here, metabolic spaces are the split-metabolic ones. If $\frac{1}{2} \in R$, these are simply the hyperbolic spaces, yielding the maybe better known definition of the Witt group of a commutative ring.

Remark 1.2.3. When $\mathcal{L} = \mathcal{O}_X$, then the group $\mathrm{W}(X)$ is indeed a ring, with product induced by the tensor product: $(E, \varphi) \cdot (F, \psi) = (E \otimes_{\mathcal{O}_X} F\, ,\, \varphi \otimes_{\mathcal{O}_X} \psi)$.

We now produce examples proving "strictness" of the trivial implications: hyperbolic $\Longrightarrow$ split metabolic $\Longrightarrow$ metabolic $\Longrightarrow$ trivial in the Witt group.

Example 1.2.4. Over the ring $R = \mathbb{Z}$, the symmetric space $(R^2, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right))$ is split metabolic but not hyperbolic (the hyperbolic space $\mathrm{H}(R) = (R^2, \psi)$ has the property that $\psi(v, v) \in 2R$ for any $v \in R^2$ but the above form represents 1).

Example 1.2.5. An example of a metabolic space which is not split-metabolic cannot exist in the affine case. Choose an exact sequence $\mathcal{O}_X \rightarrowtail P \twoheadrightarrow \mathcal{O}_X$, say on an elliptic curve X, with P indecomposable. Then $\wedge^2 P$ is trivial and hence P has a structure of *skew-symmetric* space. It is metabolic with the (left) $\mathcal{O}_X$ as lagrangian but cannot be split metabolic since P itself is indecomposable as module. An example of a *symmetric* such space can be found in Knus-Ojanguren [44, last remark]. They produce a metabolic symmetric space, which is not split metabolic, as can be seen on its Clifford algebra.

Example 1.2.6 *(Ojanguren).* Let $A := \mathbb{R}[X, Y, Z]/X^2 + Y^2 + Z^2 - 1$ and let P be the indecomposable projective A-module of rank 2 corresponding to the tangent space of the sphere. The rank 4 projective module $E := \mathrm{End}_A(P)$ is equipped with the symmetric bilinear form $\varphi : E \xrightarrow{\sim} E^*$, where $\varphi(f)(g) = \frac{1}{2}(q(f+g) - q(f) - q(g))$, for any $f, g \in E$, is the form associated to the quadratic form $q(f) := \det(f)$. Then $[E, \varphi] = 0$ in $\mathrm{W}(A)$ but the symmetric space (E, φ) is not metabolic.

If Q is the field of fractions of A, it is easy to write $\varphi \otimes_A Q$ and to check it is hyperbolic. Hence the class $[E, \varphi]$ belongs to the kernel of the homomorphism $\mathrm{W}(A) \to \mathrm{W}(Q)$, which is known to be injective (A is regular and $\dim(A) \le 3$, see *e.g.* Thm. 1.5.19 below). Hence $[E, \varphi] = 0$ and (E, φ) is stably metabolic. To see that this symmetric space is not metabolic, assume the contrary. Here, a metabolic space is hyperbolic (affine case and $\frac{1}{2} \in A$). So, we would have $(E, \varphi) \simeq \mathrm{H}(M)$ for some projective module M of rank 2. Using the hyperbolic form on $\mathrm{H}(M)$ and the presence of $\mathrm{id}_P \in E$ with $q(\mathrm{id}) = \det(\mathrm{id}) = 1$, one can find an element $f \in E$ such that $q(f) = -1$, that is an endomorphism $f : P \to P$ with determinant -1. Such an endomorphism cannot exist since it would yield a fibrewise decomposition of P into two eigenspaces, and hence would guarantee the unlikely triviality of the tangent space of the sphere.

(By the way, one can show that $\mathrm{W}(A) \cong \mathbb{Z} \oplus \mathbb{Z}/2$, see [42, § VIII.6.2].)

Remark 1.2.7. The simplest schemes are the points $X = \mathrm{Spec}(k)$ for k a field, or k a local ring, containing $\frac{1}{2}$. In these cases, the Witt group allows a complete classification of quadratic forms – see Subsection 1.2.4. With this in mind, several people got interested in the map $\mathrm{W}(A) \to \mathrm{W}(Q)$ for A a domain with field of fractions Q. This is commented upon in Subsection 1.5.2.

Example 1.2.8. For elementary examples of Witt groups of affine schemes (*i.e.* commutative rings) like for $\mathrm{W}(\mathbb{Z}) = \mathrm{W}(\mathbb{R}) = \mathbb{Z}$, or, q being a power of a prime, for $\mathrm{W}(\mathbb{F}_q) = \mathbb{Z}/2$ when q is even, $\mathrm{W}(\mathbb{F}_q) = \mathbb{Z}/2[\varepsilon]/(\varepsilon^2)$ or $\mathbb{Z}/4$ when $q \equiv 1$ or 3 mod 4 respectively, or for $\mathrm{W}(\mathbb{Q}) = \mathrm{W}(\mathbb{Z}) \oplus \bigoplus_{p \in \mathbb{P}} \mathrm{W}(\mathbb{F}_p)$, or for Witt groups of other fields, or of Dedekind domains, and so on, the reader is referred to the already mentioned [50] or to Scharlau [71].

Example 1.2.9. As a special case of Karoubi's Thm. 1.2.19, we see that for any commutative ring R containing $\frac{1}{2}$, for instance $R = k$ a field of odd characteristic, the Witt group of the affine space over R is canonically isomorphic to the one of R:

$$\mathrm{W}(\mathbb{A}^n_R) = \mathrm{W}(R).$$

(See also Thm. 1.5.10 below.) The case of the projective space over a field is a celebrated result of Arason [1] (compare Walter's Thm. 1.5.28 below):

Theorem 1.2.10 (Arason). *Let k be a field of characteristic not 2 and let $n \geq 1$. Then $\mathrm{W}(\mathbb{P}^n_k) = \mathrm{W}(k)$.*

This has been extended to Brauer-Severi varieties:

Theorem 1.2.11 (Pumplün). *Let k be a field of characteristic not 2. Let A be a central simple k-algebra and X the associated Brauer-Severi variety.*

(i) *The natural morphism $\mathrm{W}(k) \to \mathrm{W}(X)$ is surjective.*
(ii) *When A is of odd index, $\mathrm{W}(k) \to \mathrm{W}(X)$ is injective.*

See [67], where further references and partial results for twisted dualities are to be found. Injectivity fails for algebras with even index, in general.

Example 1.2.12. Here is an example of the possible use of Witt groups in algebraic geometry. The problem of Lüroth, to decide whether a unirational variety is rational, is known to have a positive answer for curves over arbitrary fields (Lüroth), for complex surfaces (Castelnuovo), and to fail in general. Simple counter-examples, established by means of Witt groups, are given in Ojanguren [57], where an overview can be found. See also [20, Appendix].

Here is another connection between Witt groups and algebraic geometry.

Theorem 1.2.13 (Parimala). *Let R be a regular finitely generated $\mathbb{R}$- or $\mathbb{C}$-algebra of Krull dimension 3. Then the Witt group $\mathrm{W}(R)$ is finitely generated if and only if the Chow group $\mathrm{CH}^2(R)/2$ is finite.*

See [64, Thm. 3.1] where examples are given; compare also Totaro's Thm. 1.5.23 below. This important paper of Parimala significantly contributed to the study of connections between Witt groups and étale cohomology. Abundant work resulted from this, among which the reader might want to consider Colliot-Thélène –Parimala [21], which relates to the subject of real connected components discussed below in Subsection 1.2.2. In this direction, see also Scheiderer [73].

We end this Subsection by a short guide to the literature for a selection of results in Krull dimension 1 and 2. The reader will find additional information in Knus [42, § VIII.2]. For Witt groups of fields (dim = 0), see Subsection 1.2.4.

In dimension 1, we have:

Dedekind domains: If D is a Dedekind domain with field of fractions Q, there is an exact sequence: $0 \to \mathrm{W}(D) \longrightarrow \mathrm{W}(Q) \xrightarrow{\partial} \oplus_p \mathrm{W}(D/p)$ where the sum runs over the non-zero primes p of D and where ∂ is the classical *second residue homomorphism*, which depends on choices of local parameters. See [50, Chap. IV].

Elliptic curves: There is a series of articles by Arason, Elman and Jacob, describing the Witt group of an elliptic curve with generators and relations. See Arason-Elman-Jacob [2] for an overview and for further references. See also the work of Parimala-Sujatha [65].

Real curves: For curves over $\mathbb{R}$, the story stretches from the original work of Knebusch [41, § V.4] to the most recent work of Monnier [53]. Note that the latter gives a systematic overview including singular curves, which were already considered in Dietel [23]. See also Rem. 1.2.15 below.

In dimension 2, we have:

Complex surfaces: Fernández-Carmena [24, Thm. 3.4] proved among other things the following result: if X is a smooth complex quasi-projective surface then $\mathrm{W}(X) \simeq (\mathbb{Z}/2)^{1+s+q+b}$ where s, q and b are the number of copies of $\mathbb{Z}/2$ in $\mathcal{O}_X(X)^*/(\mathcal{O}_X(X)^*)^2$, in ${}_2\mathrm{Pic}(X)$ and in ${}_2\mathrm{Br}(X)$ respectively.

Real surfaces: The main results are due to Sujatha [75, Thms. 3.1 and 3.2] and look as follows. See also Sujatha - van Hamel [76] for further developments.

Theorem 1.2.14 (Sujatha). *Let X be a smooth projective and integral surface over $\mathbb{R}$.*

(i) *Assume that $X(\mathbb{R}) \neq \varnothing$ and has s real connected components. Then*

$$\mathrm{W}(X) \simeq \mathbb{Z}^s \oplus (\mathbb{Z}/2)^m \oplus (\mathbb{Z}/4)^n.$$

(ii) *Assume that $X(\mathbb{R}) = \varnothing$. Then*

$$\mathrm{W}(X) \simeq (\mathbb{Z}/2)^m \oplus (\mathbb{Z}/4)^n \oplus (\mathbb{Z}/8)^t.$$

Moreover, the integers m, n and t can be described in terms of 2-torsion of the Picard and Brauer groups of X, and of the level of $\mathbb{R}(X)$ in case (ii).

Remark 1.2.15. For an algebraic variety X over $\mathbb{R}$, the formulas describing $\mathrm{W}(X)$, which can be found in the above literature, basically always look as follows: $\mathrm{W}(X) = \mathbb{Z}^s \oplus$ (2-primary torsion part), where s is the number of real connected components of $X(\mathbb{R})$ and where cohomological invariants are used to control the 2-primary torsion part. See Mahé's result 1.2.17.

Remark 1.2.16. Further results on Witt groups of schemes have been obtained by means of triangular Witt groups and are presented in Section 1.5. Even in low dimension, say up to 3, the situation is quite clarified by the corollaries of Thm. 1.5.15 below.

1.2.2 Motivation From Real Algebraic Geometry

There is a long lasting love-story between quadratic forms and real algebraic geometry, originating in their common passion for sums of squares. For a survey, see [18, Chap. 15]; early ideas are again in Knebusch [41, Chap. V].

A nice application of Witt groups to real geometry is the following problem, stated by Knebusch. Let X be an algebraic variety over $\mathbb{R}$. Consider the set of real points $X(\mathbb{R})$ with the real topology. Then its connected components are conjectured to be in one-to-one correspondence with *signatures* of $\mathrm{W}(X)$, that are ring homomorphisms $\mathrm{W}(X) \to \mathbb{Z}$. Basically, the construction goes as follows. Pick a closed point x in $X(\mathbb{R})$; its residue field $\mathbb{R}(x)$ is $\mathbb{R}$ and hence localization produces a homomorphism $\mathrm{W}(X) \to \mathrm{W}(\mathbb{R}(x)) = \mathbb{Z}$ and one can show that this homomorphism only depends on the connected component C_x of $X(\mathbb{R})$ where x was chosen. In this way, one obtains the following pairing, where $\mathrm{CC}(X(\mathbb{R}))$ denotes the above set of real connected components

$$\begin{array}{rcl} \lambda :\ \mathrm{CC}(X(\mathbb{R})) \times \mathrm{W}(X) & \longrightarrow & \mathbb{Z} \\ (C_x\,,\,\varphi) & \longmapsto & \varphi(x) \in \mathrm{W}(\mathbb{R}(x)) = \mathbb{Z}. \end{array}$$

This can be read either as a map $\Lambda : \mathrm{CC}(X(\mathbb{R})) \to \mathrm{Hom}_{\mathrm{rings}}(\mathrm{W}(X), \mathbb{Z})$ or as a ring homomorphism $\Lambda^* : \mathrm{W}(X) \to \mathrm{Cont}(X(\mathbb{R}), \mathbb{Z})$, the *total signature map.*

Theorem 1.2.17 (Mahé). *Let $X = \mathrm{Spec}(A)$ be an affine real algebraic variety. Then the map Λ is a bijection between the set of connected components of the real spectrum $\mathrm{Spec}_{\mathrm{r}}(A)$ and the set of signatures $\mathrm{W}(A) \to \mathbb{Z}$.*

See [47, Cor.3.3] for the above and see Houdebine-Mahé [31] for the extension to projective varieties. In fact, a key ingredient in the proof consists in showing that the cokernel of the total signature $\Lambda^* : \mathrm{W}(X) \to \mathrm{Cont}(X(\mathbb{R}), \mathbb{Z})$ is a 2-primary torsion group. Knowing this, it is interesting to try understanding the precise exponent of this 2-primary torsion group. Such exponents are obtained in another work of Mahé [48], and more recently by Monnier [52].

1.2.3 Rings with Involution, Polynomials and Laurent Rings

Definition 1.2.18. A *ring with involution* is a pair (R, σ) consisting of an associative ring R and an *involution* $\sigma : R \to R$, *i.e.* an additive homomorphism such that $\sigma(r \cdot s) = \sigma(s) \cdot \sigma(r)$, $\sigma(1) = 1$ and $\sigma^2 = \mathrm{id}_R$.

For a left R-module M we can define its dual $M^* = \mathrm{Hom}_R(M, R)$, which is naturally a right R-module via $(f \cdot r)(x) := f(x) \cdot r$ for all $x \in M$, $r \in R$ and $f \in M^*$. It inherits a left R-module structure via $r \cdot f := f \cdot \sigma(r)$, that is $(r \cdot f)(x) = f(x) \cdot \sigma(r)$. There is a natural R-homomorphism $\varpi_M : M \longrightarrow M^{**}$ given by $(\varpi_M(m))(f) := \sigma(f(m))$. When $P \in R\text{–Proj}$ is a finitely generated projective left R-module, this homomorphism ϖ_P is an isomorphism. Hence the category $(R\text{–Proj}, *, \varpi)$ is an additive category with duality. The same holds for $(R\text{–Proj}, *, \epsilon \cdot \varpi)$ for any central unit $\epsilon \in R^\times$ such that $\sigma(\epsilon) \cdot \epsilon = 1$, like for instance $\epsilon = -1$. The Witt group obtained this way is usually denoted

$$\mathrm{W}^\epsilon(R) := \mathrm{W}(R\text{–Proj}\,,\, *\,,\, \epsilon \cdot \varpi)$$

and is called the Witt group of ϵ-hermitian bilinear forms over R. This part of Witt group theory is of course quite important, and the reader is referred to the very complete Knus [42] for more information. We mention here two big K-theory like results.

Theorem 1.2.19 (Karoubi). *Let R be a ring with involution containing $\frac{1}{2}$. Then $\mathrm{W}^\epsilon(R[T]) = \mathrm{W}^\epsilon(R)$, where $R[T]$ has the obvious involution fixing T.*

See Karoubi [37, Part II]. An elementary proof is given in Ojanguren-Panin [59, Thm. 3.1], who also prove a general theorem for the Witt group of the ring of Laurent polynomials, giving in particular :

Theorem 1.2.20 (Ranicki). *Let R be a regular ring with involution containing $\frac{1}{2}$. Then the following homomorphism*

$$\mathrm{W}^\epsilon(R) \oplus \mathrm{W}^\epsilon(R) \longrightarrow \mathrm{W}^\epsilon(R[T, T^{-1}]) \qquad (\alpha, \beta) \mapsto \alpha + \beta \cdot \langle T \rangle$$

is an isomorphism, where the involution on $R[T, T^{-1}]$ fixes the variable T.

See Ranicki [69] where regularity is not required (neither is it in [59]) and where suitable Nil-groups are considered. Compare Thm. 1.5.27 below.

1.2.4 Semi-local Rings and Fields

Recall that a commutative ring R is *semi-local* if it has only finitely many maximal ideals. Local rings and fields are semi-local.

Theorem 1.2.21 (Witt Cancellation). *Let R be a commutative semi-local ring in which 2 is invertible. If (P_1, φ_1), (P_2, φ_2) and (Q, ψ) are symmetric spaces such that $(P_1, \varphi_1) \perp (Q, \psi)$ is isometric to $(P_2, \varphi_2) \perp (Q, \psi)$, then (P_1, φ_1) and (P_2, φ_2) are isometric.*

This was first proven for fields by Witt [86]. This result and much more information on these cancellation questions can be found in Knus [42, Chap. VI].

Remark 1.2.22. The above result is *wrong for non-commutative semi-local rings*, *i.e.* rings R such that $R/\operatorname{rad}(R)$ is semi-simple. Keller [39] gives a very explicit counter-example, constructed as follows : let k be a field of odd characteristic; let A_0 be the semi-localization of $k[X, Y]/(X^2 + Y^2 - 1)$ at the maximal ideals $\xi = (0, 1)$ and $\eta = (0, -1)$; let $B \subset A_0$ be the subring of those $f \in A_0$ such that $f(\xi) = f(\eta)$; finally define the non-commutative semi-local ring to be $A = \left\{ \left(\begin{smallmatrix} b & r \\ s & a_0 \end{smallmatrix}\right) \,\middle|\, b \in B\,,\ a_0 \in A_0\,,\ r, s \in \operatorname{rad}(A_0) \right\}$ with transposition as involution. Then there are two symmetric forms on the same projective right A-module $N := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \cdot A$ which are not isometric but become isometric after adding the rank one space $(A, \langle 1 \rangle)$. See more in [39] or in [42, VI.5.1].

Remark 1.2.23. Let R be a commutative semi-local ring containing $\frac{1}{2}$, with $\mathrm{Spec}(R)$ connected (otherwise do everything component by component). Then any finitely generated projective R-module if free. Using the sublagrangian reduction 1.1.32 and the above Witt cancellation, we know that any symmetric space (P, φ) over R can be written up to isometry as

$$(P, \varphi) \simeq (P_0, \varphi_0) \perp \mathrm{H}(R^m)$$

for $m \in \mathbb{N}$ and for (P_0, φ_0) without admissible sublagrangian – let us say that the space (P_0, φ_0) is *(admissibly) anisotropic* – and we know that the number m and the isometry class of (P_0, φ_0) are *unique*. Moreover, the spaces (P, φ) and (P_0, φ_0) are Witt equivalent and by Witt cancellation again, there is exactly one (admissibly) anisotropic space in one Witt class. This establishes the following result, the original motivation for studying Witt groups :

Corollary 1.2.24. *Let R be a commutative semi-local ring containing $\frac{1}{2}$ (for instance a field of characteristic not 2). The determination of the Witt group* $\mathrm{W}(R)$ *allows the classification up to isometry of all quadratic forms over R.*

Remark 1.2.25. Reading the above Corollary backwards, we avoid commenting the huge literature on Witt groups of fields, by referring the reader to the even bigger literature on quadratic forms at large. See in particular Lam [45], Scharlau [71] and Serre [74]. For instance, there exist so-called *structure theorems* for Witt groups of fields, due to Witt, Pfister, Scharlau and others, and revisited in Lewis [46], where further references can also be found. In fact, several results classically known for fields extend to (commutative) semi-local rings. See [41, Chap. II] again or Baeza [3].

1.3 A Glimpse at Other Theories

Our Chapter focusses on the internal theory of Witt groups but the reader might be interested in knowing which are the neighbor theories, more or less directly related to Witt groups. We give here a rapid overview with references.

Quadratic forms : When 2 is not a unit one must distinguish quadratic forms from the symmetric forms we mainly considered. See the classical references already given in Remark 1.2.25. The Witt group of quadratic forms can also be defined, see for instance Milnor-Husemoller [50, App. 1]. See also the recent Baeza [4] for quadratic forms over fields of characteristic two. The reader looking for a systematic treatise including quadratic forms and their connections with algebraic groups should consider *the* book [43].

Motivic approach : Techniques from algebraic geometry, Chow groups and motives, have been used to study quadratic forms over fields, by means of the corresponding quadrics. See the work of Izhboldin, Kahn, Karpenko, Merkurjev, Rost, Sujatha, Vishik and others, which is still in development and for which we only give here a sample of references: [32], [33], [34], [35], [38] among many more.

Topological Witt groups: Let M be a smooth paracompact manifold and let $R = \mathrm{C}^\infty(M, \mathbb{R})$ be the ring of smooth real-valued functions on M. It is legitimate, having Swan-Serre's equivalence in mind, to wonder if the Witt group of such a ring R can be interpreted in terms of the manifold M. The answer is that $\mathrm{W}(R)$ is isomorphic to $\mathrm{KO}(M)$ the real (topological) K^0 of M, that is the Grothendieck group of isomorphism classes of real vector bundles over M. This is due to Lusztig see [50, § V.2]. This should not be mistaken with the Witt group of real algebraic varieties discussed above.

Cohomological invariants: We already mentioned briefly in Rem. 1.2.15 the importance of cohomological invariants in the part dedicated to real algebraic geometry. For quadratic forms over fields, the relation between Witt groups and Galois cohomology groups is the essence of the famous Milnor Conjecture [49], now proven by Voevodsky, see *e.g.* Orlov, Vishik and Voevodsky [60]. See also Pfister's historical survey [66].

For a scheme X, there is a homomorphism $\mathrm{rk} : \mathrm{W}(X) \to \mathrm{Cont}(X, \mathbb{Z}/2)$, the *reduced rank*, to the continuous (hence locally constant) functions from X to $\mathbb{Z}/2$, which sends a symmetric space to its rank modulo 2 (metabolic spaces have even rank). The *fundamental ideal* $I(X)$ of $\mathrm{W}(X)$ is the kernel of this homomorphism.

Following [42, § VIII.1], we denote by $\mathrm{Disc}(X)$ the abelian group of isometry classes of symmetric line bundles, with $\otimes$ as product. We denote by $\delta : I(X) \to \mathrm{Disc}(X)$ the *signed discriminant*, which sends the class of an even-rank symmetric space (E, φ) of rank $2m$ to the symmetric bundle $\langle(-1)^m\rangle \cdot (\wedge^m E, \wedge^m \varphi)$.

One can define further the *Witt invariant*, which takes values in the Brauer group, see Knus [42, § IV.8] and which is defined by means of Clifford algebras. See also Barge-Ojanguren [15] for the lift of the latter to K-theory. Higher invariants are not known in this framework. One can try to define general invariants into subquotients of K-theory groups, for arbitrary exact categories or in more general frameworks. This was started by Szyjewski in [77] and remains "in progress" for higher ones.

Grothendieck-Witt groups: One often considers also GW the Grothendieck-Witt group, which is defined by the same generators as the Witt group but with less relations; namely if a space (P, φ) is metabolic with lagrangian L then one sets the relation $(P, \varphi) - \mathrm{H}(L) = 0$ in the Grothendieck-Witt group, instead of the relation $(P, \varphi) = 0$ $(= \mathrm{H}(L))$ in the Witt group. There is a group homomorphism $K_0 \to \mathrm{GW}$, induced by the hyperbolic functor $L \mapsto \mathrm{H}(L)$ and whose cokernel is the Witt group. We intentionally do not specify what sort of categories we define $\mathrm{GW}(-)$ for, because it applies whenever the Witt group is defined. For instance, in the triangular framework of the next two sections, it is also possible to define Grothendieck-Witt groups, as recently done by Walter [83].

Hermitian K-theory, Karoubi's Witt groups: The above Grothendieck-Witt group is equal to K_0^h, the 0-th group of Karoubi's hermitian K-theory. For a recent reference, see Hornbostel [29], where hermitian K-theory is extended to exact categories. There are higher and lower hermitian K-theory groups K_n^h and natural homomorphisms $Hyp : K_n \to K_n^h$ from K-theory towards hermitian K-theory, which fit in Karoubi's "Fundamental Theorem" long exact sequence. Karoubi's Witt groups are defined in a mixed way, namely as the cokernels of these homomorphisms $Hyp : K_n \to K_n^h$. For regular rings, these groups coincide with the triangular Witt groups, see more in [36] or in [30].

L-theory: We refer the reader to Williams [85] in this Handbook or to Ranicki [70] for the definition of the quadratic and symmetric L-theory groups of Wall-Mischenko-Ranicki and for further references. We shortly compare them to the triangular Witt groups to come. First, like triangular Witt groups, L-groups are algebraic, that is, their definition does not require the above hermitian K-theory. Secondly, *unlike* triangular Witt groups, L-groups also work when 2 is not assumed invertible and this is of central importance in surgery theory. Unfortunately, it does not seem unfair to say that the definition of these L-groups is rather *involved* and requires some heavy use of complexes.

The advantage of triangular Witt theory is two-fold: first, it applies to non-split exact categories and hence to schemes, and secondly, by its very definition, it factors via triangulated categories, freeing us from the burden of complexes.

Note that both theories coincide over split exact categories under the assumption that 2 is invertible and that even in the non-split case, the derived Witt groups of an exact category have a formation-like presentation by generators and relations (see Walter [83]). In the present stage of the author's understanding, the *triangular* theory of Witt groups, strictly speaking, does not exist without the "dividing by 2" assumption. Nevertheless, even when 2 is not assumed invertible, there are good reasons to believe that a sort of "L-theory of non-necessarily-split exact categories" should exist, unfolding the higher homotopies in a Waldhausen-category framework, using weak-equivalences, cofibrations and so on, but most probably renouncing the elegant simplicity of the triangular language...

1.4 Triangular Witt Groups: General Theory

The second half of this Chapter is dedicated to triangular Witt groups, *i.e.* Witt groups of triangulated categories with duality. The style is quite direct and a reader needing a more gentle introduction is referred to [10].

1.4.1 Basic Notions and Facts

All definitions and results of this Section are to be found in [6].

For the definition of a *triangulated category* we refer to Verdier's original source [81], to Weibel [84, Chap. 10], or to [6, § 1], where the reader can find Axiom (TR 4$^+$), the *enriched version of the Octahedron Axiom*, due to Beĭlinson, Bernstein and Deligne [17]. All known triangulated categories and all triangulated categories considered below satisfy this enriched axiom. Note that a triangulation is an additional structure, not intrinsic, on an additive category $\mathcal{K}$, which consists of a *translation* or *suspension* $T:\mathcal{K}\longrightarrow\mathcal{K}$ plus a collection of *distinguished* triangles satisfying some axioms. The fundamental idea is to replace admissible exact sequences by distinguished triangles.

Definition 1.4.1. Let $\delta = 1$ or -1. A *triangulated category with δ-duality* is an additive category with duality $(\mathcal{K},\#,\varpi)$ in the sense of Def. 1.1.6, where $\mathcal{K}$ is moreover triangulated, and satisfying the following conditions:

(a) The duality $\#$ is a δ-*exact* functor $\mathcal{K}^{\mathrm{op}}\longrightarrow\mathcal{K}$, which means that

$$T\circ\# \;\cong\; \#\circ T^{-1}$$

(we consider this isomorphism as an equality) and, more important, that for any distinguished triangle $A\xrightarrow{u}B\xrightarrow{v}C\xrightarrow{w}T(A)$ in $\mathcal{K}$, the following triangle is exact:

$$C^{\#}\xrightarrow{\quad v^{\#}\quad}B^{\#}\xrightarrow{\quad u^{\#}\quad}A^{\#}\xrightarrow{\;\delta\cdot T(w^{\#})\;}T(C^{\#})\;.$$

(b) The identification ϖ between the identity and the double dual is compatible with the triangulation, which means $\varpi_{T(M)} = T(\varpi_M)$ for all $M\in\mathcal{K}$.

Note that all "additive notions" presented in Subsection 1.1.1 also make sense in this framework, as for instance the monoid $\mathrm{MW}(\mathcal{K},\#,\varpi)$ of symmetric spaces (Def. 1.1.24). We now explain how the other classical notions which depended on the exact category structure (absent here) can be replaced.

Definition 1.4.2. A symmetric space (P,φ) is called *neutral* (or *metabolic* if no confusion occurs) when it admits a *lagrangian*, *i.e.* a triple (L,α,β) such that $\alpha: L\to P$ is a morphism, such that the following triangle is distinguished

$$L\xrightarrow{\quad\alpha\quad}P\xrightarrow{\quad\alpha^{\#}\varphi\quad}L^{\#}\xrightarrow{\quad\beta\quad}T(L)$$

and such that $\beta: L^{\#}\to T(L)$ is δ-symmetric, which here means:

$$\delta\cdot T(\beta^{\#}) = \varpi_{T(L)}\circ\beta.$$

In short, the symmetric short exact sequence $L \rightarrowtail P \twoheadrightarrow L^*$ is replaced by the above symmetric distinguished triangle. Note that we still have $\alpha^{\#}\varphi\alpha = 0$, that is (L, α) is a sublagrangian. (By the way, there is a triangular partial analogue of the sublagrangian reduction, called the *sublagrangian construction*, which can be found in [6, § 4] or, in a simpler case, in [5, § 3].)

Definition 1.4.3. Let $(\mathcal{K}, \#, \varpi)$ be a triangulated category with δ-duality. As before, its *Witt group* is the following quotient of abelian monoids:

$$\mathrm{W}(\mathcal{K}, \#, \varpi) := \frac{\mathrm{MW}(\mathcal{K}, \#, \varpi)}{\mathrm{NW}(\mathcal{K}, \#, \varpi)}\ ,$$

where $\mathrm{NW}(\mathcal{K}, \#, \varpi)$ is the submonoid of $\mathrm{MW}(\mathcal{K}, \#, \varpi)$ consisting of the classes of neutral spaces.

Definition 1.4.4. Let $(\mathcal{K}, \#, \varpi)$ be a triangulated category with δ-duality. Let $n \in \mathbb{Z}$ arbitrary. Then the square of the functor $T^n \circ \# : \mathcal{K}^{\mathrm{op}} \longrightarrow \mathcal{K}$ is again isomorphic to the identity, but this functor $T^n\,\#$ is only δ_n-exact, where $\delta_n := (-1)^n \cdot \delta$. We define the *n-th shifted duality on $\mathcal{K}$ to be*

$$T^n\big((\mathcal{K}, \#, \varpi)\big) := \big(\mathcal{K}\ ,\ T^n \circ \#\ ,\ \epsilon_n \cdot \varpi\big),$$

where $\epsilon_n := (-1)^{\frac{n(n+1)}{2}} \cdot \delta^n$.

It is easy to check that $T^m\big(T^n(\mathcal{K}, \#, \varpi)\big) = T^{m+n}(\mathcal{K}, \#, \varpi)$ for any $m, n \in \mathbb{Z}$, keeping in mind that the δ_n-exactness of $T^n\#$ is given by $\delta_n = (-1)^n \cdot \delta$.

Definition 1.4.5. The *n-th shifted Witt group* of $(\mathcal{K}, \#, \varpi)$, or simply of $\mathcal{K}$, is defined as the Witt group of $T^n(\mathcal{K}, \#, \varpi)$:

$$\mathrm{W}^n(\mathcal{K}, \#, \varpi) := \mathrm{W}\left(T^n(\mathcal{K}, \#, \varpi)\right).$$

Proposition 1.4.6. *For any $n \in \mathbb{Z}$ we have a natural isomorphism, induced by $T : \mathcal{K} \longrightarrow \mathcal{K}$, between $\mathrm{W}^n(\mathcal{K}, \#, \varpi)$ and $\mathrm{W}^{n+2}(\mathcal{K}, \#, -\varpi)$. In particular, we have the 4-periodicity: $\mathrm{W}^n(\mathcal{K}, \#, \varpi) \cong \mathrm{W}^{n+4}(\mathcal{K}, \#, \varpi)$.*

See [6, Prop. 2.14]. In fact, these isomorphisms are induced by equivalences of the underlying triangulated categories with duality.

Example 1.4.7. Assume that $(\mathcal{K}, \#, \varpi)$ is a triangulated category with exact duality (that is $\delta = +1$). Then so is $T^2(\mathcal{K}, \#, \varpi)$ and the latter is isomorphic to $(\mathcal{K}, \#, -\varpi)$. The other two $T^1(\mathcal{K}, \#, \varpi)$ and $T^3(\mathcal{K}, \#, \varpi)$ are both categories with skew-exact duality (that is $\delta_1 = \delta_3 = -1$), respectively isomorphic to $(\mathcal{K}\,,\, T\#\,,\, -\varpi)$ and $(\mathcal{K}\,,\, T\#\,,\, \varpi)$.

Definition 1.4.8. A *morphism of triangulated categories with duality* (F, η) is a morphism of categories with duality (Def. 1.1.5) such that F is an exact functor, *i.e.* F sends distinguished triangles to distinguished triangles. More precise definitions are available in [14] or in [27]. With this notion of morphism, all the groups $\mathrm{W}^n(-)$ constructed above become functorial.

The following very useful result [6, Thm. 3.5] contrasts with the classical framework (compare Ex. 1.2.6) :

Theorem 1.4.9. *Let $\mathcal{K}$ be a triangulated category with duality containing $\frac{1}{2}$ (see 1.4.10). Then a symmetric space (P,φ) which is Witt-equivalent to zero, i.e. such that $[P,\varphi]=0\in \mathrm{W}(\mathcal{K})$, is necessarily neutral.*

1.4.2 Agreement and Localization

Definition 1.4.10. Let $\mathcal{A}$ be an additive category (*e.g.* a triangulated category). We say that " $\frac{1}{2}\in\mathcal{A}$ " when the abelian groups $\mathrm{Hom}_{\mathcal{A}}(M,N)$ are uniquely 2-divisible for all objects $M,N\in\mathcal{A}$, *i.e.* if $\mathcal{A}$ is a $\mathbb{Z}[\frac{1}{2}]$-category.

The main result connecting usual Witt groups to the triangular Witt groups is the following.

Theorem 1.4.11. *Let $(\mathcal{E},*,\varpi)$ be an exact category with duality such that $\frac{1}{2}\in\mathcal{E}$. Equip the derived category $\mathrm{D^b}(\mathcal{E})$ with the duality $\#$ derived from $*$. Then the obvious functor $\mathcal{E}\longrightarrow \mathrm{D^b}(\mathcal{E})$, sending everything in degree 0, induces an isomorphism*

$$\mathrm{W}(\mathcal{E},*,\varpi)\xrightarrow{\simeq}\mathrm{W}(\mathrm{D^b}(\mathcal{E}),\#,\varpi)\,.$$

This is the main result of [7, Thm. 4.3], under the mild assumption that $\mathcal{E}$ is semi-saturated. The general case is deduced from this in [14, after Thm. 1.4].

Example 1.4.12. The above Theorem provides us with lots of (classical) examples : all those described in Section 1.2, the most important being schemes. So, if X is a scheme "containing $\frac{1}{2}$" (*i.e.* a scheme over $\mathbb{Z}[\frac{1}{2}]$) and if the bounded derived category $\mathcal{K}(X):=\mathrm{D^b}(\mathrm{VB}_X)$ of vector bundles over X is equipped with the derived duality twisted by a line bundle $\mathcal{L}$ (*e.g.* $\mathcal{L}=\mathcal{O}_X$), then $\mathrm{W}^0\big(\mathcal{K}(X)\big)$ is the usual Witt group of Knebusch $\mathrm{W}(X,\mathcal{L})$ and similarly $\mathrm{W}^2\big(\mathcal{K}(X)\big)$ is the usual Witt group of skew-symmetric forms $\mathrm{W}^-(X,\mathcal{L})$. The Witt groups

$$\mathrm{W}^n(\mathrm{D^b}(\mathrm{VB}_X))$$

are often called the n-th *derived Witt groups* of X. They are functorial (contravariant) for any morphism of scheme. Other triangulated categories with duality can be associated to a scheme X, see 1.5.2 below.

Remark 1.4.13. Let us stress that the definitions of Subsection 1.4.1 also make sense when 2 is not assumed invertible. The $\frac{1}{2}$-assumption is used to prove results, like Thm. 1.4.11 for instance. As already mentioned, in the case of the derived category $(\mathrm{D^b}(\mathcal{E}),\#,\varpi)$ of an exact category with duality $(\mathcal{E},*,\varpi)$, Walter has a description of W^1 and of W^3 in terms of formations, generalizing the "split" L-theoretic definitions. See [83].

The key computational device in the triangular Witt group theory is the following localization theorem.

Theorem 1.4.14. *Let $(\mathcal{K}, \#, \varpi)$ be a triangulated category with duality such that $\frac{1}{2} \in \mathcal{K}$. Consider a thick subcategory $\mathcal{J} \subset \mathcal{K}$ stable under the duality, meaning that $(\mathcal{J})^{\#} \subset \mathcal{J}$. Induce dualities from $\mathcal{K}$ to $\mathcal{J}$ and to $\mathcal{L} := \mathcal{K}/\mathcal{J}$. We have, so to speak,* an exact sequence of triangulated categories with duality *:*

$$\mathcal{J} \rightarrowtail \mathcal{K} \twoheadrightarrow \mathcal{L}.$$

Then, there is a 12-term periodic long exact sequence of Witt groups :

$$\cdots \to \mathrm{W}^{n-1}(\mathcal{L}) \xrightarrow{\partial} \mathrm{W}^{n}(\mathcal{J}) \longrightarrow \mathrm{W}^{n}(\mathcal{K}) \longrightarrow \mathrm{W}^{n}(\mathcal{L}) \xrightarrow{\partial} \mathrm{W}^{n+1}(\mathcal{J}) \to \cdots$$

where the connecting homomorphisms ∂ can be described explicitly.

This is [6, Thm. 6.2], with the easily removable extra hypothesis that $\mathcal{K}$ is "weakly cancellative" (see [14, Thm. 2.1] for how to remove it).

Remark 1.4.15. In applications, one often knows $\mathcal{K}$ and a localization $\mathcal{K} \twoheadrightarrow \mathcal{L}$, like in the case of the derived category of a regular scheme and of an open subscheme. Then the $\mathcal{J}$ is *defined* as the kernel of this localization and the *relative* Witt groups are defined to be the Witt groups of $\mathcal{J}$. See Subsection 1.5.1.

1.4.3 Products and Cofinality

The product structures on the groups W^n have been discussed in Gille-Nenashev [27]. Inspired by the situation of a triangulated category with duality *and compatible tensor product*, they consider the general notion of *(external) dualizing pairing* [27, Def. 1.11].

Theorem 1.4.16 (Gille-Nenashev). *Let $\boxtimes : \mathcal{K} \times \mathcal{L} \longrightarrow \mathcal{M}$ be a dualizing pairing. This induces naturally a left and a right pairing*

$$\mathrm{W}^{r}(\mathcal{K}) \times \mathrm{W}^{s}(\mathcal{L}) \xrightarrow[\star_r]{\star_l} \mathrm{W}^{r+s}(\mathcal{M})$$

differing by signs, having the following properties :

(i) *When $\mathcal{K} = \mathcal{L} = \mathcal{M}$, both products turn $\bigoplus_{n \in \mathbb{Z}} \mathrm{W}^n(\mathcal{K})$ into a graded ring.*
(ii) *The multiplicative structure is compatible with localization.*
(iii) *The multiplicative structure is compatible with 4-periodicity.*

Points (i) and (ii) are in [27, Thm. 2.9 and 2.11], for (iii) see [11, App. 1].

The behaviour of Witt groups with respect to idempotent completion can be controlled with the following result of [30], whose proof uses the technicalities (and not only the front results) of [6]. See [30, App. I].

Theorem 1.4.17 (Hornbostel-Schlichting). *Let $\mathcal{B}$ be a triangulated category with δ-duality ($\delta = \pm 1$) and $\mathcal{A}$ a full triangulated subcategory which is* cofinal (i.e. *any object $b \in \mathcal{B}$ is a direct summand of an object of $\mathcal{A}$). Then there is a 12-term periodic long exact sequence*

$$\cdots \to \mathrm{W}^n(\mathcal{A}) \to \mathrm{W}^n(\mathcal{B}) \to \hat{H}^n\big(\mathbb{Z}/2\mathbb{Z}\,,\, K_0(\mathcal{B})/K_0(\mathcal{A})\big) \to \mathrm{W}^{n+1}(\mathcal{A}) \to \cdots$$

involving Tate cohomology groups of $\mathbb{Z}/2\mathbb{Z}$ with coefficients in $K_0(\mathcal{B})/K_0(\mathcal{A})$, on which $\mathbb{Z}/2\mathbb{Z}$ acts via the duality, and where K_0 is the 0*-th K-theory group.*

1.5 Witt Groups of Schemes Revisited

1.5.1 Witt Cohomology Theories

Consider a scheme X containing $\frac{1}{2}$. Consider a presheaf $(\mathcal{K}, \#, \varpi)$ of triangulated categories with duality on the scheme X. That is: for each Zariski-open $U \hookrightarrow X$, we give a triangulated category with duality $\mathcal{K}(U)$ and a restriction $q_{V,U} : \mathcal{K}(U) \longrightarrow \mathcal{K}(V)$ for each inclusion $V \hookrightarrow U$, which is assumed to be a localization of triangulated categories, in a compatible way with the duality, and with the usual presheaf condition.

For each $U \subset X$ one can then consider the Witt groups of $\mathcal{K}(U)$, which we denote

$$\mathrm{W}^n(\mathcal{K}(U))$$

Here is a list of such presheaves, with their presheaves of Witt groups.

Example 1.5.1. Assume that X is *regular* (that is here: noetherian, separated and the local rings $\mathcal{O}_{X,x}$ are regular for all $x \in X$). For each open $U \subset X$, put $\mathcal{K}(U) := \mathrm{D^b}(\mathrm{VB}_U)$ the bounded derived category of vector bundles over U. Regularity is used to insure that the restriction $\mathcal{K}(X) \to \mathcal{K}(U)$ is a localization. By 1.4.12, the 0-th and 2-nd Witt groups of $\mathcal{K}(U)$ are the usual Witt groups of U of symmetric and skew-symmetric forms, respectively. The latter result remains true without regularity of course.

Example 1.5.2. Assume that X is Gorenstein of finite Krull dimension. For each open $U \subset X$, put $\mathcal{K}(U) := \mathrm{D^b_{Coh}}(\mathrm{QCoh}_U)$ the derived category of bounded complexes of quasi-coherent $\mathcal{O}_U$-modules with coherent homology. The duality is the derived functor of $\mathrm{Hom}_{\mathcal{O}_U}(-, \mathcal{O}_U)$. See details in Gille [25, § 2.5]. The Witt group obtained this way

$$\widetilde{\mathrm{W}}^n(U) := \mathrm{W}^n(\mathrm{D^b_{Coh}}(\mathrm{QCoh}_U))$$

is called the n-th *coherent Witt groups* of U. The groups $\widetilde{\mathrm{W}}^n(-)$ are only functorial for flat morphisms of schemes. They do not agree with derived Witt groups of 1.4.12 in general but do in the regular case, since the defining triangulated categories are equivalent.

Example 1.5.3. Let X be a scheme containing $\frac{1}{2}$. One can equip the category of perfect complexes over X with a duality, essentially as above. The Witt group obtained this way could be called the *perfect* Witt groups. Nevertheless, the presheaf of triangulated categories $U \mapsto \mathrm{D^{perf}}(U)$ would fit in the above approach only when $U \mapsto K_0(U)$ is flasque (it is not clear if this is really much more general than 1.5.1). Without this assumption, there will be a 2-torsion noise involved in the localization sequence below, by means of Thm. 1.4.17.

To any such data, we can associate relative Witt groups, as follows.

Definition 1.5.4. Let X be a scheme and $U \mapsto \mathcal{K}(U)$ a presheaf of triangulated categories with duality as above. Let $Z \subset X$ be a closed subset. Let us define W^n_Z, the *Witt groups with supports in* Z as the Witt groups of the kernel category $\mathcal{K}_Z(X) := \ker\big(\mathcal{K}(X) \twoheadrightarrow \mathcal{K}(X \setminus Z)\big)$

$$\mathrm{W}^n_Z(\mathcal{K}(X)) := \mathrm{W}^n\left(\mathcal{K}_Z(X)\right).$$

More generally, for any $U \subset X$, one defines $\mathrm{W}^n_Z(\mathcal{K}(U))$ as being $\mathrm{W}^n_{Z\cap U}(\mathcal{K}(U))$.

We have the following cohomological behaviour.

Theorem 1.5.5. *With the above notations, we have a 12-term periodic long exact sequence*

$$\cdots \to \mathrm{W}^{n-1}(U) \to \mathrm{W}^n_Z(X) \to \mathrm{W}^n(X) \to \mathrm{W}^n(U) \to \mathrm{W}^{n+1}_Z(X) \to \cdots$$

where $\mathrm{W}^n(-)$ *is a short for* $\mathrm{W}^n(\mathcal{K}(-))$*, when the triangulated categories* $\mathcal{K}(-)$ *are clear from the context and similarly for* $\mathrm{W}^n_Z(-)$*.*

This follows readily from the Localization Theorem 1.4.14. This was considered for derived Witt groups in [8, Thm. 1.6], and for coherent Witt groups in [25, Thm. 2.19]. We turn below to the question of identifying the groups $\mathrm{W}^*_Z(X)$ with some groups $\mathrm{W}^*(Z)$, but we first obtain as usual the following:

Corollary 1.5.6. *Assume that our presheaf* $\mathcal{K}$ *of triangulated categories is natural in* X *and* excisive *with respect to a class* $\mathcal{C}$ *of morphisms of schemes, i.e. for any morphism* $f : Y \to X$ *in* $\mathcal{C}$ *and for any closed subset* $Z \subset X$ *such that* $f^{-1}(Z) \xrightarrow{\sim} Z$ *(with reduced structures), then the induced functor*

$$f^* : \mathcal{K}_Z(X) \to \mathcal{K}_{f^{-1}(Z)}(Y)$$

is an equivalence. (This is the case for $\mathcal{K}(X) = \mathrm{D}^{\mathrm{b}}(\mathrm{VB}_X)$ *from Ex. 1.5.1 and* $\mathcal{C} =$ *flat morphisms of regular schemes; it is also the case for* $\mathcal{K}(X) = \mathrm{D}^{\mathrm{b}}_{\mathrm{Coh}}(\mathrm{QCoh}_X)$ *from Ex. 1.5.2 and* $\mathcal{C} =$ *flat morphisms of Gorenstein schemes.) Then, for any such morphism* $f : Y \to X$*, any* $Z \subset X$ *such that* $Z' := f^{-1}(Z) \xrightarrow{\sim} Z$*, there is a* Mayer-Vietoris *long exact sequence:*

$$\cdots \to \mathrm{W}^{n-1}(Y \setminus Z') \to \mathrm{W}^n(X) \to \mathrm{W}^n(Y) \oplus \mathrm{W}^n(X \setminus Z) \to \mathrm{W}^n(Y \setminus Z') \to \cdots$$

where $\mathrm{W}^n(-)$ *is a short for* $\mathrm{W}^n(\mathcal{K}(-))$*. (So this applies to derived Witt groups over regular schemes and to coherent Witt groups over Gorenstein schemes.)*

Remark 1.5.7. This holds in particular in the usual situation where $Y := U$ is an open subset, where $f : U \hookrightarrow X$ is the inclusion and where $Z \subset U$. In this case, putting $V := X \setminus Z$, we have $X = U \cup V$ and $Y \setminus f^{-1}(Z) = U \cap V$, recovering in this way the usual Mayer-Vietoris long exact sequence. The above generality is useful though, since it applies to elementary distinguished squares in the Nisnevich topology for instance. Observe that this result is a direct consequence of the following three things: first, the very *definition* of relative Witt groups via triangulated categories; secondly, the excision property of triangulated categories themselves; thirdly, of course, the localization theorem.

We now turn to *dévissage* (in the affine case).

Theorem 1.5.8 (Gille). *Let R be a Gorenstein $\mathbb{Z}[\frac{1}{2}]$-algebra of finite Krull dimension n and let $J \subset R$ be an ideal generated by a regular sequence of length $l \leq n$. Then the closed immersion $\iota : \operatorname{Spec}(R/J) \to \operatorname{Spec}(R)$ induces an isomorphism:*

$$\widetilde{\mathrm{W}}^i(R/J) \xrightarrow{\simeq} \widetilde{\mathrm{W}}^{i+l}_J(R)$$

where, of course, $\widetilde{\mathrm{W}}^j_J(R) := \widetilde{\mathrm{W}}^j_Z(R)$ is the j-th coherent Witt group of R with supports in the closed subset $Z = V(J)$ of $\operatorname{Spec}(R)$ defined by J.

This is [25, Thm. 4.1]. Since coherent and derived Witt groups agree in the regular case, one has the obvious and important:

Corollary 1.5.9 (Gille). *Let R be a regular $\mathbb{Z}[\frac{1}{2}]$-algebra of finite Krull dimension and let $J \subset R$ be an ideal generated by a regular sequence of length l. Assume moreover that R/J is itself regular. Then there is an isomorphism of derived Witt groups:*

$$\mathrm{W}^i(R/J) \xrightarrow{\simeq} \mathrm{W}^{i+l}_J(R).$$

It is natural to ask if the cohomology theory obtained by derived (and coherent) Witt groups is homotopy invariant. The following result is a generalization of Karoubi's Theorem 1.2.19.

Theorem 1.5.10. *Let X be a regular scheme containing $\frac{1}{2}$. Then the natural homomorphism of derived Witt groups $\mathrm{W}^i(X) \longrightarrow \mathrm{W}^i(\mathbb{A}^1_X)$ is an isomorphism for all $i \in \mathbb{Z}$. (In particular, for $i = 0$, this is an isomorphism of classical Witt groups.)*

This is [8, Cor. 3.3] and has then been generalized in [26] as follows (using coherent Witt group versions of the result):

Theorem 1.5.11 (Gille). *Let X be a regular scheme containing $\frac{1}{2}$ and let $\mathcal{E} \to X$ be a vector bundle of finite rank. Then the natural homomorphism $\mathrm{W}^i(X) \to \mathrm{W}^i(\mathcal{E})$ is an isomorphism for all $i \in \mathbb{Z}$.*

1.5.2 Local to Global

Recall our convention: *regular* means regular, noetherian and separated.

Consider an integral scheme X, for instance (the spectrum of) a domain R, and consider its function field Q (the field of fractions of R). It is natural to study the homomorphism

$$\mathrm{W}(X) \to \mathrm{W}(Q)$$

e.g. because the Witt groups of fields are better understood (see Cor. 1.2.24). It is immediate that for $R = \mathbb{R}[X,Y]/(X^2+Y^2)$, the map $\mathbb{Z} \simeq \mathrm{W}(\mathbb{R}) \to \mathrm{W}(R)$ is split injective but that $\mathrm{W}(Q)$ is 2-torsion, since -1 is a square in Q and hence that $2\cdot[P,\varphi] = [(P,\varphi)\perp(P,\varphi)] = [(P,\varphi)\perp(P,-\varphi)] = 0$. So the homomorphism $\mathrm{W}(R) \to \mathrm{W}(Q)$ is not injective in general.

For *regular* schemes of dimension up to 3, injectivity of $\mathrm{W}(X) \to \mathrm{W}(Q)$ holds: see Thm 1.5.19 below. It is well-known to fail in dimension 4 already, even for affine regular schemes. For an example of this, see Knus [42, Ex. VIII.2.5.3]. Nevertheless, injectivity remains true in the *affine complex* case, see Pardon [62] and Totaro [80]. In [9] it is proven that the kernel of $\mathrm{W}(X) \to \mathrm{W}(Q)$ is nilpotent with explicit exponent, generalizing earlier results of Craven-Rosenberg-Ware [22] and Knebusch [41].

Theorem 1.5.12. *Let X be a regular scheme containing $\frac{1}{2}$ and of finite Krull dimension d. Then there is an integer N, depending only on $\left[\frac{d}{4}\right]$ such that the N-th power of the kernel of* $\mathrm{W}(X) \to \mathrm{W}(Q)$ *is zero in* $\mathrm{W}(X)$.

One can always take $N = 2^{[\frac{d}{4}]}$ and one can take $N = \left[\frac{d}{4}\right] + 1$ if the conjectural injectivity $\mathrm{W}(\mathcal{O}_{X,x}) \hookrightarrow \mathrm{W}(Q)$ holds for all $x \in X$. This is indeed the case when X is defined over a field, as we discuss below. Moreover, Example [9, Cor. 5.3] show that $\left[\frac{d}{4}\right] + 1$ is the best exponent in all dimensions. We have alluded to the following conjecture of Knebusch, which is a special case of a general conjecture of Grothendieck :

Conjecture 1.5.13. Let R be a *regular (semi-) local* domain containing $\frac{1}{2}$ and let Q be its field of fractions. Then the natural homomorphism $\mathrm{W}(R) \to \mathrm{W}(Q)$ is injective.

The key result about this conjecture was obtained by Ojanguren in [56] and says that the conjecture holds if R is essential of finite type over some ground field. Conjecture 1.5.13 has been upgraded as follows by Pardon [61] :

Conjecture 1.5.14 (Gersten Conjecture for Witt groups). Let R be a *regular (semi-) local* ring containing $\frac{1}{2}$. There exists a complex

$$0 \to \mathrm{W}(R) \to \bigoplus_{x\in X^{(0)}} \mathrm{W}(\kappa(x)) \to \bigoplus_{x\in X^{(1)}} \mathrm{W}(\kappa(x)) \to \cdots \to \bigoplus_{x\in X^{(d)}} \mathrm{W}(\kappa(x)) \to 0$$

and it is *exact* ; where X is $\mathrm{Spec}(R)$, $X^{(p)}$ are the primes of height p and $d = \dim(X)$. The complex is now admitted to be the one of 1.5.15 below.

For a long time, it remained embarrassing not even to *know* a complex as above (call this *a Gersten-Witt complex*), which one would then conjecture to be exact. In the case of K-theory, the complex is directly obtained from the coniveau filtration. Analogously, by means of triangular Witt groups and of the localization theorem, it became possible to construct Gersten-Witt complexes for all regular schemes [14, Thm. 7.2]:

Theorem 1.5.15 (Balmer-Walter). *Let X be a regular scheme containing $\frac{1}{2}$ and of finite Krull dimension d. Then there is a convergent (cohomological) spectral sequence $E_1^{p,q} \Rightarrow \mathrm{W}^{p+q}(X)$ whose first page is isomorphic to copies of a Gersten-Witt complex for X in each line $q \equiv 0$ modulo 4 and whose other lines are zero. These isomorphisms involve local choices but a canonical description of the first page is:*

$$E_1^{p,q} := \mathrm{W}^{p+q}\left(\mathrm{D}^{(p)} \,/\, \mathrm{D}^{(p+1)}\right)$$

where $\mathrm{D}^{(p)} = \mathrm{D}^{(p)}(X)$ is the full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathrm{VB}_X)$ of those complexes having support of their homology of codimension $\geq p$.

This was reproven and adapted to coherent Witt groups with supports in Gille [25, Thm. 3.14]. Using Thm. 1.5.15, the following Corollaries are immediate:

Corollary 1.5.16. *Let X be regular integral $\mathbb{Z}[\frac{1}{2}]$-scheme of dimension 1. Let x_0 be the generic point and $Q = \kappa(x_0)$ be the function field of X. There is an exact sequence:*

$$0 \to \mathrm{W}(X) \longrightarrow \mathrm{W}(Q) \xrightarrow{\partial} \bigoplus_{x \in X \setminus \{x_0\}} \mathrm{W}(\kappa(x)) \longrightarrow \mathrm{W}^1(X) \to 0.$$

and we have $\mathrm{W}^2(X) = \mathrm{W}^3(X) = 0$.

Example 1.5.17. The above applies in particular to Dedekind domains containing $\frac{1}{2}$. For instance for $D := \mathbb{R}[X,Y]/(X^2+Y^2-1)$, it follows from [50, Ex. IV.3.5] that $\mathrm{W}^1(D) \simeq \mathbb{Z}$. Here, $\mathrm{W}(D) = \mathbb{Z} \oplus \mathbb{Z}/2$, see [42, VIII.6.1].

Corollary 1.5.18. *Let X be a regular scheme containing $\frac{1}{2}$ and of Krull dimension $d \leq 4$. Let $\mathrm{W}_{\mathrm{nr}}(X)$ be the unramified Witt group of X. The homomorphism $\mathrm{W}(X) \to \mathrm{W}_{\mathrm{nr}}(X)$ is surjective.*

Corollary 1.5.19. *Let X be a regular integral $\mathbb{Z}[\frac{1}{2}]$-scheme of Krull dimension 3 and of function field Q. Then, the above Gersten-Witt complex*

$$0 \to \mathrm{W}(X) \to \mathrm{W}(Q) \to \bigoplus_{x \in X^{(1)}} \mathrm{W}(\kappa(x)) \to \bigoplus_{x \in X^{(2)}} \mathrm{W}(\kappa(x)) \to \bigoplus_{x \in X^{(3)}} \mathrm{W}(\kappa(x)) \to 0$$

is exact at $\mathrm{W}(X)$ and at $\mathrm{W}(Q)$ and its homology in degree i (that is, where $X^{(i)}$ appears) is isomorphic to $\mathrm{W}^i(X)$ for $i = 1, 2, 3$.

See [14, § 10] for detailed results and definitions as well as for the following:

Corollary 1.5.20. *Let X be regular scheme containing $\frac{1}{2}$ and of dimension at most 7. Then, with the notations of Thm. 1.5.15, there is an exact sequence:*

$$\begin{tikzcd}
0 \arrow[r] & E^2_{4,0} \arrow[r] & \mathrm{W}^0(X) \arrow[r] & E^2_{0,0} \arrow[r] & E^2_{5,0} \arrow[r] & \mathrm{W}^1(X) \arrow[r] & E^2_{1,0} \arrow[d] \\
.\,0 & E^2_{3,0} \arrow[l] & \mathrm{W}^3(X) \arrow[l] & E^2_{7,0} \arrow[l] & E^2_{2,0} \arrow[l] & \mathrm{W}^2(X) \arrow[l] & E^2_{6,0} \arrow[l]
\end{tikzcd}$$

Note that $E^2_{p,0}$ is the p-th homology group of the Gersten-Witt complex of X.

Corollary 1.5.21. *The Gersten Conjecture holds in low dimension up to 4.*

This is [14, Thm. 10.4] in the local case. For *semi*-local, one needs [9, Cor. 3.6] plus the local vanishing of shifted Witt groups, which holds in any dimension:

Theorem 1.5.22 (Balmer-Preeti). *Let R be a semi-local commutative ring containing $\frac{1}{2}$. Then $\mathrm{W}^i(R) = 0$ for $i \not\equiv 0$ modulo 4.*

This is [7, Thm. 5.6] for local rings and will appear in [13] in general.

Totaro [80] used the above spectral sequence 1.5.15 in combination with the Bloch-Ogus and the Pardon spectral sequences (see [63] for the latter) to bring several interesting computations. He provides an example of a smooth *affine complex* 5-fold U such that $\mathrm{W}(U) \to \mathrm{W}(\mathbb{C}(U))$ is not injective and also gives a global complex version of Parimala's result 1.2.13 (note that being finitely generated over $\mathrm{W}(\mathbb{C}) = \mathbb{Z}/2$ means being finite), see [80, Thm. 1.4]:

Theorem 1.5.23 (Totaro). *Let X be a smooth complex 3-fold. Then the Witt group $\mathrm{W}(X)$ is finite if and only if the Chow group $\mathrm{CH}^2(X)/2$ is finite.*

We return to the Gersten Conjecture 1.5.14. In [58], Ojanguren and Panin established Purity, which is exactness of the complex at the first two places, for regular local rings containing a field. Using the general machinery of homotopy invariant excisive cohomology theories, as developed in Colliot-Thélène – Hoobler – Kahn [19], the author established the geometric case of the following result in [8]:

Theorem 1.5.24. *The Gersten Conjecture 1.5.14 holds for semi-local regular k-algebras over any field k of characteristic different from 2.*

Like for the original K-theoretic Gersten Conjecture, the geometric case [8, Thm. 4.3] is the crucial step. It can then be extended to regular local k-algebras via Popescu's Theorem, by adapting to Witt groups ideas that Panin introduced in K-theory. This is done in [12]. Now that we have the vanishing of odd-indexed Witt groups for *semi*-local rings as well, see Thm. 1.5.22, this Panin-Popescu extension also applies to *semi*-local regular k-algebras, as announced in the statement. Details of this last step have been checked in [51].

1.5.3 Computations

Here are some computations using triangular Witt groups:

Theorem 1.5.25 (Gille). *Let R be a Gorenstein $\mathbb{Z}[\frac{1}{2}]$-algebra of finite Krull dimension and $n \geq 1$. Consider the hyperbolic affine $(2n-1)$-sphere*

$$\Sigma_R^{2n-1} := \operatorname{Spec}\Big(R\,[\,T_1,\ldots,T_n\,,\,S_1,\ldots,S_n\,]\;\Big/\;\big(1-\textstyle\sum_{i=1}^{n} T_iS_i\,\big)\Big).$$

Then its coherent Witt groups are $\widetilde{\mathrm{W}}^i(\Sigma_R^{2n-1}) = \widetilde{\mathrm{W}}^i(R) \oplus \widetilde{\mathrm{W}}^{i+1-n}(R)$. In particular for R regular, these are derived Witt groups. In particular for $R = k$ a field, or a regular semi-local ring, the classical Witt groups of Σ_k^{2n-1} is

$$\mathrm{W}(\Sigma_k^{2n-1}) = \begin{cases} \mathrm{W}(k) \oplus \mathrm{W}(k) & \text{if } n \equiv 1 \text{ modulo } 4 \\ \mathrm{W}(k) & \text{if } n \not\equiv 1 \text{ modulo } 4\,. \end{cases}$$

Theorem 1.5.26 (Balmer-Gille). *Let X be a regular scheme containing $\frac{1}{2}$ and let $n \geq 2$. Consider the usual punctured affine space $\mathbb{U}_X^n \subset \mathbb{A}_X^n$ defined by $\mathbb{U}_X^n = \bigcup_{i=1}^n \{\,T_i \neq 0\,\}$. Then its total graded Witt ring $\mathrm{W}^{\mathrm{tot}} := \bigoplus_{i\in\mathbb{Z}/4} \mathrm{W}^i$ is:*

$$\mathrm{W}^{\mathrm{tot}}(\mathbb{U}_X^n) \simeq \mathrm{W}^{\mathrm{tot}}(X)\big[\,\varepsilon\,\big]\,/\,\varepsilon^2 = \mathrm{W}^{\mathrm{tot}}(X) \oplus \mathrm{W}^{\mathrm{tot}}(X)\cdot\varepsilon$$

where $\varepsilon \in \mathrm{W}^{n-1}(\mathbb{U}_X^n)$ is of degree $n-1$ and squares to zero: $\varepsilon^2 = 0$.

The element ε is given quite explicitly in [11] by means of Koszul complexes. The above hypothesis $n \geq 2$ is only needed for proving $\varepsilon^2 = 0$. For $n = 1$, the scheme $\mathbb{U}_X^1 = X \times \operatorname{Spec}\big(\mathbb{Z}\big[\,T, T^{-1}\big]\big)$ is the scheme of *Laurent "polynomials"* over X and one has the following generalization of Thm. 1.2.20:

Theorem 1.5.27. *Let X be a regular scheme containing $\frac{1}{2}$. Consider the scheme of Laurent polynomials $X[\,T, T^{-1}] = \mathbb{U}_X^1$. There is an isomorphism:*

$$\mathrm{W}^i(X) \oplus \mathrm{W}^i(X) \simeq \mathrm{W}^i\big(X[\,T, T^{-1}]\big)$$

given by $(\alpha, \beta) \mapsto \alpha + \beta \cdot \langle T\rangle$ where $\langle T\rangle$ is the rank one space with form T.

The most striking computation obtained by means of triangular Witt groups is probably the following generalization of Arason's Theorem 1.2.10:

Theorem 1.5.28 (Walter). *Let X be a scheme containing $\frac{1}{2}$ and $r \geq 1$. Let $\mathbb{P}_X^r$ be the projective space over X. Let $m \in \mathbb{Z}/2$. Consider $\mathcal{O}(m) \in \operatorname{Pic}(\mathbb{P}_X^r)/2$.*

For r even, $$\mathrm{W}^i\left(\mathbb{P}_X^r, \mathcal{O}(m)\right) = \begin{cases} \mathrm{W}^i(X) & \text{for } m \text{ even} \\ \mathrm{W}^{i-r}(X) & \text{for } m \text{ odd.} \end{cases}$$

For r odd, $$\mathrm{W}^i\left(\mathbb{P}_X^r, \mathcal{O}(m)\right) = \begin{cases} \mathrm{W}^i(X) \oplus \mathrm{W}^{i-r}(X) & \text{for } m \text{ even} \\ 0 & \text{for } m \text{ odd.} \end{cases}$$

This is indeed a special case of a general projective bundle theorem, for Witt and Grothendieck-Witt groups, which is to appear in [82]. Walter has also announced results for (Grothendieck-) Witt groups of quadratics, which are in preparation. The case of Grassmannians was started by Szyjewski in [78] and might also follow.

1.5.4 Witt Groups and $\mathbb{A}^1$-Homotopy Theory

Using the above cohomological behaviour of Witt groups, Hornbostel [28, Cor. 4.9 and Thm. 5.7] establishes the following representability result.

Theorem 1.5.29 (Hornbostel). *Witt groups are representable both in the unstable and the stable $\mathbb{A}^1$-homotopy categories of Morel and Voevodsky.*

This is one ingredient in Morel's announced proof of the following :

Theorem 1.5.30 (Morel). *Let k be a (perfect) field of characteristic not 2. Let $\mathcal{SH}_k$ be the stable $\mathbb{A}^1$-homotopy category over k. Then the graded ring*

$$\bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{SH}_k}(S^0, \mathbb{G}_{\mathrm{m}}{}^{\wedge n})$$

is isomorphic to the Milnor-Witt K-theory of k. In particular, $\mathrm{Hom}_{\mathcal{SH}_k}(S^0, S^0)$ *is isomorphic to the Grothendieck-Witt group of k and for all $n < 0$,* $\mathrm{Hom}_{\mathcal{SH}_k}(S^0, \mathbb{G}_{\mathrm{m}}{}^{\wedge n})$ *is isomorphic to the Witt group of k.*

Of course, this result requires further explanations (which can be found in [55, § 6] or in [54]) but the reader should at least close this Chapter remembering that Witt groups quite miraculously appear at the core of the stable homotopy category $\mathcal{SH}_k$, disguised as "motivic stable homotopy groups of spheres", objects which, at first sight, do not involve any quadratic form.

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