Lecture Notes in Mathematics

Edited by A. Dold, Heidelberg and B. Eckmann, Zürich

341

Algebraic K-Theory I

Battelle Institute Conference 1972

Higher K-Theories

Springer-Verlag
Berlin · Heidelberg · New York

Higher algebraic K-theory: I

Daniel Quillen*

The purpose of this paper is to develop a higher K-theory for additive categories with exact sequences which extends the existing theory of the Grothendieck group in a natural way. To describe the approach taken here, let $\underline{\underline{M}}$ be an additive category embedded as a full subcategory of an abelian category $\underline{\underline{A}}$, and assume $\underline{\underline{M}}$ is closed under extensions in $\underline{\underline{A}}$. Then one can form a new category $Q(\underline{\underline{M}})$ having the same objects as $\underline{\underline{M}}$, but in which a morphism from M' to M is taken to be an isomorphism of M' with a subquotient M_1/M_0 of M, where $M_0 \subset M_1$ are subobjects of M such that M_0 and M/M_1 are objects of $\underline{\underline{M}}$. Assuming the isomorphism classes of objects of $\underline{\underline{M}}$ form a set, the category $Q(\underline{\underline{M}})$ has a classifying space $BQ(\underline{\underline{M}})$ determined up to homotopy equivalence. One can show that the fundamental group of this classifying space is canonically isomorphic to the Grothendieck group of $\underline{\underline{M}}$, which motivates defining a sequence of K-groups by the formula

$$K_i(\underline{\underline{M}}) = \pi_{i+1}(BQ(\underline{\underline{M}}),0) .$$

It is the goal of the present paper to show that this definition leads to an interesting theory.

The first part of the paper is concerned with the general theory of these K-groups. Section 1 contains various tools for working with the classifying space of a small category. It concludes with an important result which identifies the homotopy-theoretic fibre of the map of classifying spaces induced by a functor. In K-theory this is used to obtain long exact sequences of K-groups from the exact homotopy sequence of a map.

Section 2 is devoted to the definition of the K-groups and their elementary properties. One notes that the category $Q(\underline{\underline{M}})$ depends only on $\underline{\underline{M}}$ and the family of those short sequences $0 \to M' \to M \to M'' \to 0$ in $\underline{\underline{M}}$ which are exact in the ambient abelian category. In order to have an intrinsic object of study, it is convenient to introduce the notion of an <u>exact category</u>, which is an additive category equipped with a family of short sequences satisfying some standard conditions (essentially those axiomatized in [Heller]). For an exact category $\underline{\underline{M}}$ with a set of isomorphism classes one has a sequence of K-groups $K_i(\underline{\underline{M}})$ varying functorially with respect to exact functors. Section 2 also contains the proof that $K_0(\underline{\underline{M}})$ is isomorphic to the Grothendieck group of $\underline{\underline{M}}$. It should be mentioned, however, that there are examples due to Gersten and Murthy showing that in general $K_1(\underline{\underline{M}})$ is not the same as the universal determinant group of Bass.

The next three sections contain four basic results which might be called the exactness, resolution, devissage, and localization theorems. Each of these generalizes a well-known result for the Grothendieck group ([Bass, Ch. VIII]), and, as will be apparent from the rest of the paper, they enable one to do a lot of K-theory.

The second part of the paper is concerned with applications of the general theory to rings and schemes. Given a ring (resp. a noetherian ring) A , one defines the groups

*Supported in part by the National Science Foundation.

$K_i(A)$ (resp. $K'_i(A)$) to be the K-groups of the category of finitely generated projective A-modules (resp. the abelian category of finitely generated A-modules). There is a canonical map $K_i(A) \to K'_i(A)$ which is an isomorphism for A regular by the resolution theorem. Because the devissage and localization theorems apply only to abelian categories, the interesting results concern the groups $K'_i(A)$. In section 6 we prove the formulas

$$K'_i(A) = K'_i(A[t]) \quad , \quad K'_i(A[t,t^{-1}]) = K'_i(A) \oplus K'_{i-1}(A)$$

for A noetherian, which entail the corresponding results for K-groups when A is regular. The first formula is proved more generally for a class of rings with increasing filtration, including some interesting non-commutative rings such as universal enveloping algebras. To illustrate the generality, the K-groups of certain skew fields are computed.

For a scheme (resp. noetherian) scheme X, the groups $K_i(X)$ (resp. $K'_i(X)$) are defined using the category of vector bundles (resp. coherent sheaves) on X, and there is a canonical map $K_i(X) \to K'_i(X)$ which is an isomorphism for X regular. Section 7 is devoted to the K'-theory. Especially interesting is a spectral sequence

$$E_1^{pq} = \coprod_{\mathrm{cod}(x)=p} K_{-p-q}(k(x)) \Longrightarrow K'_{-n}(X)$$

obtained by filtering the category of coherent sheaves according to the codimension of the support. In the case where X is regular and of finite type over a field, we carry out a program proposed by Gersten at this conference ([Gersten 3]), which leads to a proof of Bloch's formula

$$A^p(X) = H^p(X, \underline{K}_p(\underline{O}_X))$$

proved by Bloch in particular cases ([Bloch]), where $A^p(X)$ is the group of codimension p cycles modulo linear equivalence. One noteworthy feature of this formula is that the right side is clearly contravariant in X, which suggests rather strongly that higher K-theory might eventually provide a theory of the Chow ring for non-quasi-projective regular varieties.

Section 8 contains the computation of the K-groups of the projective bundle associated to a vector bundle over a scheme. This result generalizes the computation of the Grothendieck groups given in [SGA 6], and it may be viewed as a first step toward a higher K-theory for schemes, as opposed to the K'-theory of the preceding section. The proof, different from the one in [SGA 6], is based on the existence of canonical resolutions for regular sheaves on projective space, which may be of some independent interest. The method also permits one to determine the K-groups of a Severi-Brauer scheme in terms of the K-groups of the associated Azumaya algebra and its powers.

This paper contains proofs of all of the results announced in [Quillen 1], except for Theorem 1 of that paper, which asserts that the groups $K_i(A)$ here agree with those obtained by making BGL(A) into an H-space (see [Gersten 5]). From a logical point of view, this theorem should have preceded the second part of the present paper, since it is used there a few times. However, I recently discovered that the ideas involved its proof could be applied to prove the expected generalization of the localization theorem and

fundamental theorem for non-regular rings [Bass, p.494,663]. These results will appear in the next installment of this theory.

The proofs of Theorems A and B given in section 1 owe a great deal to conversations with Graeme Segal, to whom I am very grateful. One can derive these results in at least two other ways, using cohomology and the Whitehead theorem as in [Friedlander], and also by means of the theory of minimal fibrations of simplicial sets. The present approach, based on the Dold-Thom theory of quasi-fibrations, is quite a bit shorter than the others, although it is not as clear as I would have liked, since the main points are in the references. Someday these ideas will undoubtedly be incorporated into a general homotopy theory for topoi.

This paper was prepared with the editor's encouragement during the first two months of 1973. I mention this because the results in §7 on Gersten's conjecture and Bloch's formula, which were discovered at this time, directly affect the papers [Gersten 3, 4] and [Bloch] in this procedings, which were prepared earlier.

CONTENTS

§1. The classifying space of a small category

In the succeeding sections of this paper K-groups will be defined as the homotopy groups of the classifying space of a certain small category. In this rather long section we collect together the various facts about the classifying space functor we will need. All of these are fairly well-known, except for the important Theorem B which identifies the homotopy-fibre of the map of classifying spaces induced by a functor under suitable conditions. It will later be used to derive long exact sequences in K-theory from the homotopy exact sequence of a map.

Let $\underline{\underline{C}}$ be a small category. Its nerve, denoted $N\underline{\underline{C}}$, is the (semi-)simplicial set whose p-simplices are the diagrams in $\underline{\underline{C}}$ of the form

$$X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_p .$$

The i-th face (resp. degeneracy) of this simplex is obtained by deleting the object X_i (resp. replacing X_i by $\mathrm{id} : X_i \longrightarrow X_i$) in the evident way. The classifying space of $\underline{\underline{C}}$, denoted $B\underline{\underline{C}}$, is the geometric realization of $N\underline{\underline{C}}$. It is a CW complex whose p-cells are in one-one correspondence with the p-simplices of the nerve which are nondegenerate, i.e. such that none of the arrows is an identity map. (See [Segal 1], [Milnor 1].)

For example, let J be a (partially) ordered set regarded as a category in the usual way. Then BJ is the simplicial complex (with the weak topology) whose vertices are the elements of J and whose simplices are the totally ordered non-empty finite subsets of J. Conversely, if K is a simplicial complex and if J is the ordered set of simplices of K, then the simplicial complex BJ is the barycentric subdivision of K. Thus every simplicial complex (with the weak topology) is homeomorphic to the classifying space of some, and in fact many, ordered sets. Furthermore, since it is known that any CW complex is homotopy equivalent to a simplicial complex, it follows that any interesting homotopy type is realized as the classifying space of an ordered set. (I am grateful to Graeme Segal for bringing these remarks to my attention.)

As another example, let a group G be regarded as a category with one object in the usual way. Then BG is a classifying space for the discrete group G in the traditional sense. It is an Eilenberg-MacLane space of type K(G,1), so few homotopy types occur in this way.

Let X be an object of $\underline{\underline{C}}$. Using X to denote also the corresponding 0-cell of $B\underline{\underline{C}}$, we have a family of homotopy groups $\pi_i(B\underline{\underline{C}},X)$, $i \geq 0$, which will be called the homotopy groups of $\underline{\underline{C}}$ with basepoint X and denoted simply $\pi_i(\underline{\underline{C}},X)$. Of course, $\pi_0(\underline{\underline{C}},X)$ is not a group, but a pointed set, which can be described as the set $\pi_0\underline{\underline{C}}$ of components of the category $\underline{\underline{C}}$ pointed by the component containing X. In effect, connected components of $B\underline{\underline{C}}$ are in one-one correspondence with components of $\underline{\underline{C}}$.

We will see below that $\pi_1(\underline{\underline{C}},X)$ and also the homology groups of $B\underline{\underline{C}}$ can be defined "algebraically" without the use of spaces or some closely related machine such as semi-simplicial homotopy theory, or simplicial complexes and subdivision. The existence of similar descriptions of the higher homotopy groups seems to be unlikely, because so far nobody has produced an "algebraic" definition of the homotopy groups of a simplicial complex.

<u>Coverings of $B\mathbf{C}$ and the fundamental group.</u>

Let E be a covering space of $B\mathbf{C}$. For any object X of $\mathbf{C}$, let $E(X)$ denote the fibre of E over X considered as a 0-cell of $B\mathbf{C}$. If $u : X \to X'$ is a map in $\mathbf{C}$, it determines a path from X to X' in $B\mathbf{C}$, and hence gives rise to a bijection $E(u): E(X) \xrightarrow{\sim} E(X')$. It is easy to see that $E(fg) = E(f)E(g)$, hence in this way we obtain a functor $X \mapsto E(X)$ from $\mathbf{C}$ to Sets which is <u>morphism-inverting</u>, that is, it carries arrows into isomorphisms.

Conversely, given $F : \mathbf{C} \to \text{Sets}$, let $F\backslash\mathbf{C}$ denote the category of pairs (X,x) with X in $\mathbf{C}$ and $x \in F(X)$, in which a morphism $(X,x) \to (X',x')$ is a map $u : X \to X'$ such that $F(u)x = x'$. The forgetful functor $F\backslash\mathbf{C} \to \mathbf{C}$ induces a map of classifying spaces $B(F\backslash\mathbf{C}) \to B\mathbf{C}$ having the fibre $F(X)$ over X for each object X. Using [Gabriel-Zisman, App.I, 3.2] it is not difficult to see that when F is morphism-inverting, the map $B(F\backslash\mathbf{C}) \to B\mathbf{C}$ is locally trivial, and hence $B(F\backslash\mathbf{C})$ is a covering space of $B\mathbf{C}$. It is clear that the two procedures just described are inverse to each other, whence we have an equivalence of categories

$$(\text{Coverings of } B\mathbf{C}) \simeq (\text{Morph.-inv. } F : \mathbf{C} \to \text{Sets})$$

where the latter denotes the full subcategory of $\text{Funct}(\mathbf{C}, \text{Sets})$, the category of functors from $\mathbf{C}$ to Sets, consisting of the morphism-inverting functors.

Let $\mathbf{G} = \mathbf{C}[(\text{Ar}\mathbf{C})^{-1}]$ denote the groupoid obtained from $\mathbf{C}$ by formally adjoining the inverses of all the arrows [Gabriel-Zisman, I, 1.1] . The canonical functor from $\mathbf{C}$ to $\mathbf{G}$ induces an equivalence of categories

$$\text{Funct}(\mathbf{G}, \text{Sets}) \equiv (\text{Morph.-inv. } F : \mathbf{C} \to \text{Sets})$$

(<u>loc.cit.</u>, I, 1.2). Let X be an object of $\mathbf{C}$ and let G_X be the group of its automorphisms as an object of $\mathbf{G}$. When $\mathbf{C}$ is connected, the inclusion functor $G_X \to \mathbf{G}$ is an equivalence of categories, hence one has an equivalence

$$\text{Funct}(\mathbf{G}, \text{Sets}) \xrightarrow{\sim} \text{Funct}(G_X, \text{Sets}) = (G_X\text{-sets}).$$

Therefore by combining the above equivalences, we obtain an equivalence of categories of the category of coverings of $B\mathbf{C}$ with the category of G_X-sets given by the functor $E \mapsto E(X)$. By the theory of covering spaces this implies that there is a canonical isomorphism: $\pi_1(\mathbf{C},X) \simeq G_X$. The same conclusion holds when $\mathbf{C}$ is not connected, as both groups depend only on the component of $\mathbf{C}$ containing X. Thus we have established the following.

<u>Proposition 1.</u> <u>The category of covering spaces of $B\mathbf{C}$ is canonically equivalent to the category of morphism-inverting functors</u> $F : \mathbf{C} \to \text{Sets}$, <u>or what amounts to the same thing, the category</u> $\text{Funct}(\mathbf{G}, \text{Sets})$, <u>where</u> $\mathbf{G} = \mathbf{C}[(\text{Ar}\mathbf{C})^{-1}]$ <u>is the groupoid obtained by formally inverting the arrows of $\mathbf{C}$. The fundamental group</u> $\pi_1(\mathbf{C},X)$ <u>is canonically isomorphic to the group of automorphisms of X as an object of the groupoid $\mathbf{G}$.</u>

It follows in particular that a local coefficient system L of abelian groups on $B\mathbf{C}$ may be identified with the morphism-inverting functor $X \mapsto L(X)$ from $\mathbf{C}$ to abelian groups.

<u>The homology of $B\mathbf{C}$</u>

It is well-known that the homology and cohomology of the classifying space of a discrete group coincide with the homology and cohomology of the group in the sense of homological algebra. We now describe the generalization of this fact for an arbitrary small category.

Let A be a functor from $\mathbf{C}$ to Ab, the category of abelian groups, and let $H_p(\mathbf{C},A)$ denote the homology of the simplicial abelian group

$$C_p(\mathbf{C},A) = \coprod_{X_0 \to \cdots \to X_p} A(X_0)$$

of chains on $N\mathbf{C}$ with coefficients in A. (By the homology we mean the homology of the associated normalized chain complex.) Then there are canonical isomorphisms

$$H_p(\mathbf{C},A) = \varinjlim{}_p^{\mathbf{C}}(A)$$

where $\varinjlim{}_*^{\mathbf{C}}$ denotes the left derived functors of the right exact functor $\varinjlim$ from $\text{Funct}(\mathbf{C},\text{Ab})$ to Ab. This is proved by showing that $A \mapsto H_*(\mathbf{C},A)$ is an exact ∂-functor which coincides with $\varinjlim$ in degree zero and is effaceable in positive degrees. (See [Gabriel-Zisman, App.II, 3.3] .)

Let $H_*(B\mathbf{C},L)$ denote the singular homology of $B\mathbf{C}$ with coefficients in a local coefficient system L. Then there are canonical isomorphisms

$$H_p(B\mathbf{C},L) = H_p(\mathbf{C},L)$$

where we identify L with a morphism-inverting functor as above. This may be proved by filtering the CW complex $B\mathbf{C}$ by means of its skeleta and considering the associated spectral sequence. One has $E^1_{pq} = 0$ for $q \neq 0$ and E^1_{*0} = the normalized chain complex associated to $C_*(\mathbf{C},L)$. (Compare [Segal 1, 5.1].) The spectral sequence degenerates yielding the desired isomorphism.

Thus we have

$$(1) \qquad H_p(B\mathbf{C},L) = \varinjlim{}_p^{\mathbf{C}}(L)$$

and similarly we have a canonical isomorphism for cohomology

$$(2) \qquad H^p(B\mathbf{C},L) = \varprojlim{}^p_{\mathbf{C}}(L)$$

where $\varprojlim{}^*_{\mathbf{C}}$ denotes the right derived functors of the left exact functor $\varprojlim$ from $\text{Funct}(\mathbf{C},\text{Ab})$ to Ab.

<u>Properties of the classifying space functor.</u>

From now on we use the letters $\mathbf{C}$, $\mathbf{C}'$, etc. to denote small categories. If $f : \mathbf{C} \to \mathbf{C}'$ is a functor, it induces a cellular map $Bf : B\mathbf{C} \to B\mathbf{C}'$. In this way we obtain a faithful functor from the category of small categories to the category of CW complexes and cellular maps. This functor is of course not fully faithful. As a particularly interesting example, we note that there is an obvious canonical cellular homeomorphism

$$(3) \qquad B\mathbf{C} = B\mathbf{C}^o$$

where $\mathbf{C}^o$ is the dual category, which is not realized by a functor from $\mathbf{C}$ to $\mathbf{C}^o$ except in very special cases, e.g. groups.

By the compatibility of geometric realization with products [Milnor 1], one knows that the canonical map

(4) $$B(\underline{C} \times \underline{C}') \longrightarrow B\underline{C} \times B\underline{C}'$$

is a homeomorphism if either $B\underline{C}$ or $B\underline{C}'$ is a finite complex, and also if the product is given the compactly generated topology. As pointed out in [Segal 1], this implies the following.

Proposition 2. A natural transformation $\theta : f \to g$ of functors from $\underline{C}$ to $\underline{C}'$ induces a homotopy $B\underline{C} \times I \longrightarrow B\underline{C}'$ between Bf and Bg.

In effect, the triple (f,g,θ) can be viewed as a functor $\underline{C} \times \mathbf{1} \longrightarrow \underline{C}'$, where $\mathbf{1}$ is the ordered set $\{0<1\}$, and $B\mathbf{1}$ is the unit interval.

We will say that a functor is a homotopy equivalence if it induces a homotopy equivalence of classifying spaces, and that a category is contractible if its classifying space is.

Corollary 1. If a functor f has either a left or a right adjoint, then f is a homotopy equivalence.

For if f' is say left adjoint to f, then there are natural transformations $f'f \to id$, $id \to ff'$, whence Bf' is a homotopy inverse for Bf.

Corollary 2. A category having either an initial or a final object is contractible.

For then the functor from the category to the punctual category has an adjoint.

Let I be a small category which is filtering (= non-empty + directed [Bass, p.41]) and let $i \mapsto \underline{C}_i$ be a functor from I to small categories. Let $\underline{C}$ be the inductive limit of the $\underline{C}_i$; because filtered inductive limits commute with finite projective limits, we have $Ob\underline{C} = \varinjlim Ob\underline{C}_i$, $Ar\underline{C} = \varinjlim Ar\underline{C}_i$, and more generally $N\underline{C} = \varinjlim N\underline{C}_i$. Let $X_i \in Ob\underline{C}_i$ be a family of objects such that for every arrow $i \to i'$ in I, the induced functor $\underline{C}_i \to \underline{C}_{i'}$ carries X_i to $X_{i'}$, whence we have an inductive system $\pi_n(\underline{C}_i, X_i)$ indexed by I.

Proposition 3. If X is the common image of the X_i in $\underline{C}$, then

$$\varinjlim \pi_n(\underline{C}_i, X_i) = \pi_n(\underline{C}, X).$$

Proof. Because I is filtering and $N\underline{C} = \varinjlim N\underline{C}_i$, it follows that any simplicial subset of $N\underline{C}$ with a finite number of nondegenerate simplices lifts to $N\underline{C}_i$ for some i, and moreover the lifting is unique up to enlarging the index i in the evident sense. As every compact subset of a CW complex is contained in a finite subcomplex, we see that every compact subset of $B\underline{C}$ lifts to $B\underline{C}_i$ for some i, uniquely up to enlarging i. The proposition follows easily from this.

Corollary 1. Suppose in addition that for every arrow $i \to i'$ in I the induced functor $\underline{C}_i \to \underline{C}_{i'}$ is a homotopy equivalence. Then the functor $\underline{C}_i \to \underline{C}$ is a homotopy equivalence for each i.

Proof. Replacing I by the cofinal category $i\backslash I$ of objects under i, we can suppose i is the initial object of I. It then follows from the proposition that the map of CW complexes $B\underline{C}_i \to B\underline{C}$ induces isomorphisms on homotopy. Hence it is a homotopy equivalence by a well-known theorem of Whitehead.

Corollary 2. Any filtering category is contractible.

In effect, I is the inductive limit of the functor $i \mapsto I/i$, and the category I/i of objects over i has a final object, hence is contractible.

Sufficient conditions for a functor to be a homotopy equivalence.

Let $f : \underline{C} \to \underline{C}'$ be a functor and denote objects of $\underline{C}$ by X, X', etc. and objects of $\underline{C}'$ by Y, Y', etc. If Y is a fixed object of $\underline{C}'$, let $Y\backslash f$ denote the category consisting of pairs (X,v) with $v : Y \to fX$, in which a morphism from (X,v) to (X',v') is a map $w : X \to X'$ such that $f(w)v = v'$. In particular, when f is the identity functor of $\underline{C}'$, we obtain the category $Y\backslash\underline{C}'$ of objects under Y. Similarly one defines the category f/Y consisting of pairs (X,u) with $u : fX \to Y$.

Theorem A. If the category $Y\backslash f$ is contractible for every object Y of $\underline{C}'$, then the functor f is a homotopy equivalence.

In view of (3), this result admits a dual formulation to the effect that f is a homotopy equivalence when all of the categories f/Y are contractible.

Example. Let $g : K \to K'$ be a simplicial map of simplicial complexes, and let $f : J \to J'$ be the induced map of ordered sets of simplices in K and K', so that g is homeomorphic to Bf. If $\bar{\sigma}$ denotes the element of J' corresponding to a simplex σ of K', then $f/\bar{\sigma}$ is the ordered set of simplices in $g^{-1}(\sigma)$. In this situation the theorem says that a simplicial map is a homotopy equivalence when the inverse image of each (closed) simplex is contractible.

Before proving the theorem we derive a corollary. First we recall the definition of fibred and cofibred categories [SGA 1, Exp. VI] in a suitable form. Let $f^{-1}(Y)$ denote the fibre of f over Y, that is, the subcategory of $\underline{C}$ whose arrows are those mapped to the identity of Y by f. It is easily seen that f makes $\underline{C}$ a prefibred category over $\underline{C}'$ in the sense of loc.cit. if and only if for every object Y of $\underline{C}'$ the functor

$$f^{-1}(Y) \longrightarrow Y\backslash f \quad , \quad X \longmapsto (X, id_Y)$$

has a right adjoint. Denoting the adjoint by $(X,v) \longmapsto v^*X$, we obtain for any map $v : Y \to Y'$ a functor

$$v^* : f^{-1}(Y') \longrightarrow f^{-1}(Y)$$

determined up to canonical isomorphism, called base-change by v. The prefibred category $\underline{C}$ over $\underline{C}'$ is a fibred category if for every pair u,v of composable arrows in $\underline{C}'$, the canonical morphism of functors $u^*v^* \to (vu)^*$ is an isomorphism. We will call such functors f prefibred and fibred respectively.

Dually, f makes $\underline{C}$ into a precofibred category over $\underline{C}'$ when the functors $f^{-1}(Y) \to f/Y$ have left adjoints $(X,v) \mapsto v_*X$. In this case the functor $v_* : f^{-1}(Y) \to f^{-1}(Y')$ induced by $v : Y \to Y'$ is called cobase-change by v, and $\underline{C}$ is a cofibred category when $(vu)_* \xrightarrow{\sim} v_*u_*$ for all composable u,v. Such functors f will be called precofibred and cofibred respectively.

Corollary. Suppose that f is either prefibred or precofibred, and that $f^{-1}(Y)$ is contractible for every Y. Then f is a homotopy equivalence.

This follows from Prop. 2, Cor. 1.

Example. Let $S(\underline{\underline{C}})$ be the category whose objects are the arrows of $\underline{\underline{C}}$, and in which a morphism from $u : X \to Y$ to $u' : X' \to Y'$ is a pair $v : X' \to X$, $w : Y \to Y'$ such that $u' = wuv$. (Thus $S(\underline{\underline{C}})$ is the cofibred category over $\underline{\underline{C}}^{o} \times \underline{\underline{C}}$ with discrete fibres defined by the functor $(X,Y) \mapsto \mathrm{Hom}(X,Y)$.) One has functors

$$\underline{\underline{C}}^{o} \xleftarrow{\ s\ } S(\underline{\underline{C}}) \xrightarrow{\ t\ } \underline{\underline{C}}$$

given by source and target, and it is easy to see that these functors are cofibred. The categories $s^{-1}(X) = X \backslash \underline{\underline{C}}$ and $t^{-1}(Y) = (\underline{\underline{C}}/Y)^{o}$ have initial objects, hence are contractible. Therefore s and t are homotopy equivalences by the corollary. This construction provides the simplest way of realizing by means of functors the homotopy equivalence (3).

We now turn to the proof of Theorem A. We will need a standard fact about the realization of bisimplicial spaces which we now derive.

Let $\underline{\underline{\mathrm{Ord}}}$ be the category of ordered sets $\mathbf{p} = \{0 < 1 < \ldots < p\}$, $p \in \mathbb{N}$, so that by definition simplicial objects are functors with domain $\underline{\underline{\mathrm{Ord}}}^{o}$. The realization functor

$$(\mathbf{p} \mapsto X_p) \longmapsto |\mathbf{p} \mapsto X_p|$$

from simplicial spaces to spaces ([Segal 1]) may be defined as the functor left adjoint to the functor which associates to a space Y the simplicial space $\mathbf{p} \mapsto \underline{\mathrm{Hom}}(\Delta^{p}, Y)$, where $\underline{\mathrm{Hom}}$ denotes function space and Δ^{p} is the simplex having $\mathbf{p}$ as its set of vertices. In particular the realization functor commutes with inductive limits.

Let $T : \mathbf{p}, \mathbf{q} \mapsto T_{pq}$ be a bisimplicial space, i.e. a functor from $\underline{\underline{\mathrm{Ord}}}^{o} \times \underline{\underline{\mathrm{Ord}}}^{o}$ to spaces. Realizing with respect to $\mathbf{q}$ keeping $\mathbf{p}$ fixed, we obtain a simplicial space $\mathbf{p} \mapsto |\mathbf{q} \mapsto T_{pq}|$ which may then be realized with respect to $\mathbf{p}$. Also, we may realize first in the p-direction and then in the q-direction, or we may realize the diagonal simplicial space $\mathbf{p} \mapsto T_{pp}$. It is well-known (e.g. [Tornehave]) that these three procedures yield the same result:

Lemma. There are homeomorphisms

$$|\mathbf{p} \mapsto T_{pp}| \;=\; |\mathbf{p} \mapsto |\mathbf{q} \mapsto T_{pq}|| \;=\; |\mathbf{q} \mapsto |\mathbf{p} \mapsto T_{pq}||$$

which are functorial in the simplicial space T.

Proof. Suppose first that T is of the form

$$h^{rs} \times S : (\mathbf{p}, \mathbf{q}) \mapsto \mathrm{Hom}(\mathbf{p}, \mathbf{r}) \times \mathrm{Hom}(\mathbf{q}, \mathbf{s}) \times S$$

where S is a given space. Then

$$|\mathbf{p} \mapsto \mathrm{Hom}(\mathbf{p},\mathbf{r}) \times \mathrm{Hom}(\mathbf{p},\mathbf{s}) \times S| \;=\; \Delta^{r} \times \Delta^{s} \times S.$$

(This is the basic homeomorphism used to prove that geometric realization commutes with products [Milnor 1].) On the other hand, we have

$$|\mathbf{p} \mapsto |\mathbf{q} \mapsto \mathrm{Hom}(\mathbf{p},\mathbf{r}) \times \mathrm{Hom}(\mathbf{q},\mathbf{s}) \times S||$$
$$= |\mathbf{p} \mapsto \mathrm{Hom}(\mathbf{p},\mathbf{r}) \times \Delta^{s} \times S| \;=\; \Delta^{r} \times \Delta^{s} \times S$$

and similarly for the double realization taken in the other order. Thus the required functorial homeomorphisms exist on the full subcategory of bisimplicial spaces of this form.

But any T has a canonical presentation

$$\coprod_{(\mathbf{r},\mathbf{s}) \to (\mathbf{r}',\mathbf{s}')} h^{r's'} \times T_{rs} \rightrightarrows \coprod_{(\mathbf{r},\mathbf{s})} h^{rs} \times T_{rs} \longrightarrow T$$

which is exact in the sense that the right arrow is the cokernel of the pair of arrows. Since the three functors from bisimplicial spaces to spaces under consideration commute with inductive limits, the lemma follows.

Proof of Theorem A. Let $S(f)$ be the category whose objects are triples (X,Y,v) with X an object of $\underline{\underline{C}}$ and $v : Y \to fX$ a map in $\underline{\underline{C}}'$, and in which a morphism from (X,Y,v) to (X',Y',v') is a pair of arrows $u : X \to X'$, $w : Y' \to Y$ such that $v' = f(u)vw$. (Thus $S(f)$ is the cofibred category over $\underline{\underline{C}} \times \underline{\underline{C}}'^{o}$ defined by the functor $(X,Y) \mapsto \mathrm{Hom}(Y,fX)$.) We have functors

$$\underline{\underline{C}}'^{o} \xleftarrow{\ p_2\ } S(f) \xrightarrow{\ p_1\ } \underline{\underline{C}}$$

given by $p_1(X,Y,v) = X$, $p_2(X,Y,v) = Y$.

Let $T(f)$ be the bisimplicial set such that an element of $T(f)_{pq}$ is a pair of diagrams

$$(Y_p \to \ldots \to Y_0 \to fX_0 \,,\; X_0 \to \ldots \to X_q)$$

in $\underline{\underline{C}}'$ and $\underline{\underline{C}}$ respectively, and such that the i-th face in the p-(resp. q-)direction deletes the object Y_i (resp X_i) in the obvious way. Forgetting the first component gives a map of bisimplicial sets

$$(*) \qquad T(f)_{pq} \longrightarrow N\underline{\underline{C}}_q$$

where the latter is constant in the p-direction. Since the diagonal simplicial set of $T(f)$ is the nerve of the category $S(f)$, it is clear that the realization of $(*)$ is the map $Bp_1 : BS(f) \to B\underline{\underline{C}}$. (By the realization of a bisimplicial set we mean the space described in the above lemma, where the bisimplicial set is regarded as a bisimplicial space in the obvious way.) On the other hand, realizing $(*)$ with respect to p gives a map of simplicial spaces

$$\coprod_{X_0 \to \ldots \to X_q} B(\underline{\underline{C}}'/fX_0)^{o} \longrightarrow \coprod_{X_0 \to \ldots \to X_q} pt \;=\; N\underline{\underline{C}}_q$$

which is a homotopy equivalence for each q because the category $\underline{\underline{C}}'/fX_0$ has a final object. Applying a basic result of May and Tornehave ([Tornehave, A.3]), or the lemma below (Th. B), we see the realization of $(*)$ is a homotopy equivalence. Thus the functor p_1 is a homotopy equivalence.

Similarly there is a map of bisimplicial sets $T(f)_{pq} \to N(\underline{\underline{C}}'^{o})_p$ whose realization is the map $Bp_2 : BS(f) \to B\underline{\underline{C}}'^{o}$. Realizing with respect to q, we obtain a map of simplicial spaces

$$(**) \qquad \coprod_{Y_0 \leftarrow \ldots \leftarrow Y_p} B(Y_0 \backslash f) \longrightarrow \coprod_{Y_0 \leftarrow \ldots \leftarrow Y_p} pt \;=\; N(\underline{\underline{C}}'^{o})_p$$

which is a homotopy equivalence for each p, because the categories $Y \backslash f$ are contractible by hypothesis. Thus we conclude that the functor p_2 is a homotopy equivalence.

But we have a commutative diagram of categories

$$\begin{array}{ccccc} \underline{\underline{C}}'^{o} & \xleftarrow{p_2} & S(f) & \xrightarrow{p_1} & \underline{\underline{C}} \\ \| & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \underline{\underline{C}}'^{o} & \xleftarrow{p_2} & S(id_{\underline{\underline{C}}'}) & \xrightarrow{p_1} & \underline{\underline{C}}' \end{array}$$

where $f'(X,Y,v) = (fX,Y,v)$. The horizontal arrows are homotopy equivalences by what has been proved, (note that $Y\backslash id_{\underline{\underline{C}}'} = Y\backslash \underline{\underline{C}}'$ is contractible as it has an initial object). Thus f is a homotopy equivalence, whence the theorem.

<u>The exact homotopy sequence</u>.

Let $g : E \to B$ be a map of topological spaces and let b be a point of B. The <u>homotopy-fibre</u> of f over b is the space

$$F(g,b) = E \times_B B^I \times_B \{b\}$$

consisting of pairs (e,p) with e a point of E and p a path joining $g(e)$ and b. For any e in $g^{-1}(b)$ one has the <u>exact homotopy sequence</u> of g with basepoint e

$$\longrightarrow \pi_{i+1}(B,b) \longrightarrow \pi_i(F(g,b),\bar{e}) \longrightarrow \pi_i(E,e) \xrightarrow{g_*} \pi_i(B,b) \longrightarrow \cdots$$

where $\bar{e} = (e,\bar{b})$, $\bar{b}$ denoting the constant path at b.

Let $f : \underline{\underline{C}} \to \underline{\underline{C}}'$ be a functor and Y an object of $\underline{\underline{C}}'$. If $j : Y\backslash f \to \underline{\underline{C}}$ is the functor sending $(X, v : Y \to fX)$ to X, then $(X,v) \mapsto v : Y \to fX$ is a natural transformation from the constant functor with value Y to fj. Hence by Prop. 2 the composite $B(Y\backslash f) \to B\underline{\underline{C}} \to B\underline{\underline{C}}'$ contracts canonically to the constant map with image Y, and so we obtain a canonical map

$$B(Y\backslash f) \longrightarrow F(Bf, Y).$$

We want to know when this map is a homotopy equivalence, for then we have an exact sequence relating the homotopy groups of the categories $Y\backslash f$, $\underline{\underline{C}}$ and $\underline{\underline{C}}'$. Since the homotopy-fibres of a map over points connected by a path are homotopy equivalent, it is clearly necessary in order for the above map to be a homotopy equivalence for all Y, that the functor $Y'\backslash f \to Y\backslash f$, $(X,v) \mapsto (X,vu)$ induced by $u : Y \to Y'$ be a homotopy equivalence for every map u in $\underline{\underline{C}}'$. We are going to show the converse is true.

Because homotopy-fibres are not classifying spaces of categories, and hence are somewhat removed from what we ultimately will work with, it is convenient to formulate things in terms of homotopy-cartesian squares. Recall that a commutative square of spaces

$$\begin{array}{ccc} E' & \xrightarrow{h'} & E \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ B' & \xrightarrow{h} & B \end{array}$$

is called <u>homotopy-cartesian</u> if the map

$$E' \longrightarrow B' \times_B B^I \times_B E \;, \quad e' \mapsto (g'(e'), \overline{hg'(b')}, h'(e'))$$

from E' to the homotopy-fibre-product of h and g is a homotopy equivalence.

When B' is contractible, the map $F(g',b') \to E'$ is a homotopy equivalence for any b' in B', hence one has a map $E' \to F(g,h(b'))$ unique up to homotopy. In this case the square is easily seen to be homotopy-cartesian if and only if $E' \to F(g,h(b'))$ is a homotopy equivalence.

A commutative square of categories will be called homotopy-cartesian if the corresponding square of classifying spaces is. With this terminology we have the following generalization of Theorem A.

<u>Theorem B. Let $f : \underline{\underline{C}} \to \underline{\underline{C}}'$ be a functor such that for every arrow $Y \to Y'$ in $\underline{\underline{C}}'$, the induced functor $Y'\backslash f \to Y\backslash f$ is a homotopy equivalence. Then for any object Y of $\underline{\underline{C}}'$ the cartesian square of categories</u>

$$\begin{array}{ccc} Y\backslash f & \xrightarrow{j} & \underline{\underline{C}} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y\backslash \underline{\underline{C}}' & \xrightarrow{j'} & \underline{\underline{C}}' \end{array} \qquad \begin{array}{rcl} j(X,v) & = & X \\ f'(X,v) & = & (fX,v) \\ j'(Y',v) & = & Y' \end{array}$$

<u>is homotopy-cartesian. Consequently for any X in $f^{-1}(Y)$ we have an exact sequence</u>

$$\longrightarrow \pi_{i+1}(\underline{\underline{C}}',Y) \longrightarrow \pi_i(Y\backslash f, \bar{X}) \xrightarrow{j_*} \pi_i(\underline{\underline{C}},X) \xrightarrow{f_*} \pi_i(\underline{\underline{C}}',Y) \longrightarrow \cdots$$

<u>where</u> $\bar{X} = (X, id_Y)$.

As with Theorem A, this result admits a dual formulation with the categories f/Y.

<u>Corollary. Suppose $f : \underline{\underline{C}} \to \underline{\underline{C}}'$ is prefibred (resp. precofibred) and that for every arrow $u : Y \to Y'$ the base-change functor $u^* : f^{-1}(Y') \to f^{-1}(Y)$, (resp. the cobase-change functor $u_* : f^{-1}(Y) \to f^{-1}(Y')$) is a homotopy equivalence. Then for any Y in $\underline{\underline{C}}'$, the category $f^{-1}(Y)$ is homotopy equivalent to the homotopy-fibre of f over Y. (Precisely, the square</u>

$$\begin{array}{ccc} f^{-1}(Y) & \xrightarrow{i} & \underline{\underline{C}} \\ \downarrow & & \downarrow{\scriptstyle f} \\ pt & \xrightarrow{Y} & \underline{\underline{C}}' \end{array}$$

<u>where i is the inclusion functor, is homotopy-cartesian.) Consequently for any X in $f^{-1}(Y)$ we have an exact homotopy sequence</u>

$$\longrightarrow \pi_{i+1}(\underline{\underline{C}}',Y) \longrightarrow \pi_i(f^{-1}(Y),X) \xrightarrow{i_*} \pi_i(\underline{\underline{C}},X) \xrightarrow{f_*} \pi_i(\underline{\underline{C}}',Y) \longrightarrow .$$

This is clear, since $f^{-1}(Y) \to Y\backslash f$ is a homotopy equivalence for prefibred f.

For the proof of the theorem we will need a lemma based on the theory of quasi-fibrations [Dold-Lashof], which is a special case of a general result about the realization of a map of simplicial spaces [Segal 2]. A <u>quasi-fibration</u> is a map $g : E \to B$ of spaces such that the canonical map $g^{-1}(b) \to F(g,b)$ induces isomorphisms on homotopy for all b in B. When E, B are in the class $\underline{\underline{W}}$ of spaces having the homotopy type of a CW complex, one knows from [Milnor 2] that $F(g,b)$ is in $\underline{\underline{W}}$. Thus if $g^{-1}(b)$ is also in $\underline{\underline{W}}$, and g is a quasi-fibration, we have that $g^{-1}(b) \to F(g,b)$ is a homotopy equivalence, i.e. the square

$$\begin{array}{ccc} g^{-1}(b) & \xrightarrow{i} & E \\ \downarrow & & \downarrow g \\ pt & \xrightarrow{b} & B \end{array}$$

is homotopy-cartesian.

Lemma. Let $i \mapsto X_i$ be a functor from a small category I to topological spaces, and let $g : X_I \to BI$ be the space over BI obtained by realizing the simplicial space

$$p \mapsto \coprod_{i_0 \to \cdots \to i_p} X_{i_0} .$$

If $X_i \to X_{i'}$ is a homotopy equivalence for every arrow $i \to i'$ in I, then g is a quasi-fibration.

Proof. It suffices by Lemma 1.5 of [Dold-Lashof] to show that the restriction of g to the p-skeleton F_p of BI is a quasi-fibration for all p. We have a map of cocartesian squares

$$\begin{array}{ccc} \coprod \partial\Delta^p & \subset & \coprod \Delta^p \\ \downarrow & & \downarrow \\ F_{p-1} & \subset & F_p \end{array} \quad \xleftarrow{g} \quad \begin{array}{ccc} \coprod X_{i_0} \times \partial\Delta^p & \subset & \coprod X_{i_0} \times \Delta^p \\ \downarrow & & \downarrow \\ g^{-1}(F_{p-1}) & \subset & g^{-1}(F_p) \end{array}$$

where the disjoint unions are taken over the nondegenerate p-simplices $i_0 \to \cdots \to i_p$ of NI. Let U be the open set of F_p obtained by removing the barycenters of the p-cells, and let $V = F_p - F_{p-1}$. It suffices by Lemma 1.4 of loc. cit. to show the restrictions of g to U, V and $U \cap V$ are quasi-fibrations. This is clear for V and $U \cap V$, since over each p-cell g is a product map.

We will apply Lemma 1.3 of loc. cit. to $g|U$, assuming as we may by induction that $g|F_{p-1}$ is a quasi-fibration, and using the evident fibre-preserving deformation D of $g|U$ into $g|F_{p-1}$ provided by the radial deformation of Δ^p minus barycenter onto $\partial\Delta^p$. We have only to check that if D carries $x \in U$ into $x' \in F_{p-1}$, then the map $g^{-1}(x) \to g^{-1}(x')$ induced by D induces isomorphisms of homotopy groups. Supposing $x \notin F_{p-1}$ as we may, let x come from an interior point z of the copy of Δ^p corresponding to the simplex $s = (i_0 \to \cdots \to i_p)$, and let the radial deformation push z into the open face of Δ^p with vertices $j_0 < \cdots < j_q$. Then it is easy to see that $g^{-1}(x) = X_{i_0}$ and $g^{-1}(x') = X_k$, $k = i_{j_0}$, and that the map in question is the one $X_{i_0} \to X_k$ induced by the face $i_0 \to k$ of s. As these induced maps are homotopy equivalences by hypothesis, the proof of the lemma is complete.

Proof of Theorem B. We return to the proof of Theorem A. The functor $p_1 : S(f) \to \underline{\underline{C}}$ is a homotopy equivalence as before, but not necessarily the functor p_2. The map $Bp_2 : BS(f) \to B(\underline{\underline{C}}'^{o})$ is the realization of the map (**). Thus applying the preceding lemma to the functor $Y \mapsto B(Y\backslash f)$ from $\underline{\underline{C}}'^{o}$ to spaces, we see that Bp_2 is a quasi-fibration, and hence the cartesian square

$$\begin{array}{ccc} Y\backslash f & \longrightarrow & S(f) \\ \downarrow & & \downarrow p_2 \\ pt & \xrightarrow{Y} & \underline{\underline{C}}'^{o} \end{array}$$

is homotopy-cartesian. Consider now the diagram

$$\begin{array}{ccccc} Y\backslash f & \longrightarrow & S(f) & \xrightarrow{\sim} & \underline{\underline{C}} \\ \downarrow & (1) & \downarrow f' & (2) & \downarrow f \\ Y\backslash \underline{\underline{C}}' & \longrightarrow & S(id_{\underline{\underline{C}}'}) & \xrightarrow{\sim} & \underline{\underline{C}}' \\ \wr\downarrow & (3) & \wr\downarrow & & \\ pt & \xrightarrow{Y} & \underline{\underline{C}}'^{o} & & \end{array}$$

in which the squares are cartesian, and in which the sign '$\sim$' denotes a homotopy equivalence. Since the square (1) + (3) is homotopy-cartesian, it follows that (1) is homotopy-cartesian, hence (1) + (2) is also, whence the theorem.

§2. The K-groups of an exact category

Exact categories. Let $\underline{\underline{M}}$ be an additive category which is embedded as a full subcategory of an abelian category $\underline{\underline{A}}$, and suppose that $\underline{\underline{M}}$ is closed under extensions in $\underline{\underline{A}}$ in the sense that if an object A of $\underline{\underline{A}}$ has a subobject A' such that A' and A/A' are isomorphic to objects of $\underline{\underline{M}}$, then A is isomorphic to an object of $\underline{\underline{M}}$. Let $\underline{\underline{E}}$ be the class of sequences

(1) $$0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{j} M'' \longrightarrow 0$$

in $\underline{\underline{M}}$ which are exact in the abelian category $\underline{\underline{A}}$. We call a map in $\underline{\underline{M}}$ an admissible monomorphism (resp. admissible epimorphism) if it occurs as the map i (resp. j) of some member (1) of $\underline{\underline{E}}$. Admissible monomorphisms and epimorphisms will sometimes be denoted $M' \rightarrowtail M$ and $M \twoheadrightarrow M''$, respectively.

The class $\underline{\underline{E}}$ clearly enjoys the following properties:

a) Any sequence in $\underline{\underline{M}}$ isomorphic to a sequence in $\underline{\underline{E}}$ is in $\underline{\underline{E}}$. For any M', M'' in $\underline{\underline{M}}$, the sequence

(2) $$0 \longrightarrow M' \xrightarrow{(id,0)} M' \oplus M'' \xrightarrow{pr_2} M'' \longrightarrow 0$$

is in $\underline{\underline{E}}$. For any sequence (1) in $\underline{\underline{E}}$, i is a kernel for j and j is a cokernel for i in the additive category $\underline{\underline{M}}$.

b) The class of admissible epimorphisms is closed under composition and under base-change by arbitrary maps in $\underline{\underline{M}}$. Dually, the class of admissible monomorphisms is closed under composition and under cobase-change by arbitrary maps in $\underline{\underline{M}}$.

c) Let $M \to M''$ be a map possessing a kernel in $\underline{\underline{M}}$. If there exists a map $N \to M$ in $\underline{\underline{M}}$ such that $N \to M \to M''$ is an admissible epimorphism, then $M \to M''$ is an admissible epimorphism. Dually for admissible monomorphisms.

For example, suppose given a sequence (1) in $\underline{\underline{E}}$ and a map $f : N \to M''$ in $\underline{\underline{M}}$. Form the diagram in $\underline{\underline{A}}$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \xrightarrow{j} & M'' & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow f & & \\ 0 & \longrightarrow & M' & \longrightarrow & P & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

where P is a fibre product of f and j in $\underline{\underline{A}}$. Because $\underline{\underline{M}}$ is closed under extensions in $\underline{\underline{A}}$, we can suppose P is an object of $\underline{\underline{M}}$. Hence the basechange of j by f exists in $\underline{\underline{M}}$ and it is an admissible epimorphism.

Definition. An exact category is an additive category $\underline{\underline{M}}$ equipped with a family $\underline{\underline{E}}$ of sequences of the form (1), called the (short) exact sequences of $\underline{\underline{M}}$, such that the properties a), b), c) hold. An exact functor $F : \underline{\underline{M}} \longrightarrow \underline{\underline{M}}'$ between exact categories is an additive functor carrying exact sequences in $\underline{\underline{M}}$ into exact sequences in $\underline{\underline{M}}'$.

Examples. Any abelian category is an exact category in an evident way. Any additive category can be made into an exact category in at least one way by taking $\underline{\underline{E}}$ to be the family of split exact sequences (2). A category which is 'abelian' in the sense of [Heller] is an exact category which is Karoubian (i.e. every projector has an image), and conversely.

Now suppose given an exact category $\underline{\underline{M}}$. Let $\underline{\underline{A}}$ be the additive category of additive contravariant functors from $\underline{\underline{M}}$ to abelian groups which are left exact, i.e. carry (1) to an exact sequence

$$0 \longrightarrow F(M'') \longrightarrow F(M) \longrightarrow F(M') .$$

(Precisely, choose a universe containing $\underline{\underline{M}}$, and let $\underline{\underline{A}}$ be the category of left exact functors whose values are abelian groups in the universe.) Following well-known ideas (e.g. [Gabriel]), one can prove $\underline{\underline{A}}$ is an abelian category, that the Yoneda functor h embeds $\underline{\underline{M}}$ as a full subcategory of $\underline{\underline{A}}$ closed under extensions, and finally that a sequence (1) is in $\underline{\underline{E}}$ if and only if h carries it into an exact sequence in $\underline{\underline{A}}$. The details will be omitted, as they are not really important for the sequel.

The category $Q\underline{\underline{M}}$.

If $\underline{\underline{M}}$ is an exact category, we form a new category $Q\underline{\underline{M}}$ having the same objects as $\underline{\underline{M}}$ but with morphisms defined in the following way. Let M and M' be objects in $\underline{\underline{M}}$ and consider all diagrams

(3) $$M \overset{j}{\twoheadleftarrow} N \overset{i}{\rightarrowtail} M'$$

where j is an admissible epimorphism and i is an admissible monomorphism. We consider isomorphisms of these diagrams which induce the identity on M and M', such isomorphisms being unique when they exist. A morphism from M to M' in the category $Q\underline{\underline{M}}$ is by definition an isomorphism class of these diagrams. Given a morphism from M' to M'' represented by the diagram

$$M' \overset{j'}{\twoheadleftarrow} N' \overset{i'}{\rightarrowtail} M''$$

the composition of this morphism with the morphism from M to M' represented by (3) is the morphism represented by the pair $j \cdot pr_1$, $i' \cdot pr_2$ in the diagram

$$\begin{array}{ccccc} N \times_{M'} N' & \overset{pr_2}{\rightarrowtail} & N' & \overset{i'}{\rightarrowtail} & M'' \\ pr_1 \downarrow & & \downarrow j' & & \\ N & \overset{i}{\rightarrowtail} & M' & & \\ j \downarrow & & & & \\ M & & & & \end{array}$$

It is clear that composition is well-defined and associative. Thus when the isomorphism classes of diagrams (3) form a set (e.g. if every object of $\underline{\underline{M}}$ has a set of subobjects) then $Q\underline{\underline{M}}$ is a well-defined category. We assume this to be the case from now on.

It is useful to describe the preceding construction using admissible sub- and quotient objects. By an admissible subobject of M we will mean an isomorphism class of admissible monomorphisms $M' \rightarrowtail M$, isomorphism being understood as isomorphism of objects over M. Admissible subobjects are in one-one correspondence with admissible quotient objects defined in the analogous way. The admissible subobjects of M form an ordered set with the ordering: $M_1 \leq M_2$ if the unique map $M_1 \rightarrow M_2$ over M is an admissible monomorphism. When $M_1 \leq M_2$, we call (M_1, M_2) an admissible layer of M, and we call the cokernel M_2/M_1 an admissible subquotient of M.

With this terminology, it is clear that a morphism from M to M' in $Q\underline{\underline{M}}$ may be identified with a pair $((M_1, M_2), \theta)$ consisting of an admissible layer in M' and an isomorphism $\theta : M \xrightarrow{\sim} M_2/M_1$. Composition is the obvious way of combining an isomorphism of M with an admissible subquotient of M' and an isomorphism of M' with an admissible subquotient of M'' to get an isomorphism of M with an admissible subquotient of M''.

For example, the morphisms from 0 to M in $Q\underline{\underline{M}}$ are in one-one correspondence with the admissible subobjects of M. Isomorphisms from M to M' in $Q\underline{\underline{M}}$ are the same as isomorphisms from M to M' in $\underline{\underline{M}}$.

If $i : M' \rightarrowtail M$ is an admissible monomorphism, then it gives rise to a morphism from M' to M in $Q\underline{\underline{M}}$ which will be denoted

$$i_! : M' \longrightarrow M.$$

Such morphisms will be called injective. Similarly, an admissible epimorphism $j : M \twoheadrightarrow M''$ gives rise to a morphism

$$j^! : M'' \longrightarrow M$$

and these morphisms will be called surjective. By definition, any morphism u in $Q\underline{\underline{M}}$ can be factored $u = i_! j^!$, and this factorization is unique up to unique isomorphism. If we form the bicartesian square

(4) $$\begin{array}{ccc} N & \overset{i}{\rightarrowtail} & M' \\ j \downarrow & & \downarrow j' \\ M & \overset{i'}{\rightarrowtail} & N' \end{array}$$

then $u = j'^! i'_!$, and this injective-followed-by-surjective factorization is also unique up to unique isomorphism. A map which is both injective and surjective is an isomorphism, and it is of the form $\theta_! = (\theta^{-1})^!$ for a unique isomorphism θ in $\underline{\underline{M}}$.

Injective and surjective maps in $Q\underline{\underline{M}}$ should not be confused with monomorphisms and epimorphisms in the categorical sense. Indeed, every morphism in $Q\underline{\underline{M}}$ is a monomorphism. In fact, the category $Q\underline{\underline{M}}/M$ is easily seen to be equivalent to the ordered set of admissible layers in M with the ordering: $(M_0,M_1) \le (M_0',M_1')$ if $M_0' \le M_0 \le M_1 \le M_1'$.

We can use the operations $i \mapsto i_!$ and $j \mapsto j^!$ to characterize the category $Q\underline{\underline{M}}$ by a <u>universal property</u>. First we note that these operations have the following properties:

a) If i and i' are composable admissible monomorphisms, then $(i'i)_! = i'_! i_!$. Dually, if j and j' are composable admissible epimorphisms then $(jj')^! = j'^! j^!$. Also $(id_M)_! = (id_M)^! = id_M$.

b) If (4) is a bicartesian square in which the horizontal (resp. vertical) maps are admissible monomorphisms (resp. epimorphisms), then $i_! j^! = j'^! i'_!$.

Now suppose given a category $\underline{\underline{C}}$ and for each object M of $\underline{\underline{M}}$ an object hM of $\underline{\underline{C}}$, and for each $i : M' \rightarrowtail M$ (resp. $j : M \twoheadrightarrow M''$) a map $i_! : hM' \to hM$ (resp. $j^! : hM'' \to hM$) such that the properties a), b) hold. Then it is clear that this data induces a unique functor $Q\underline{\underline{M}} \to \underline{\underline{C}}$, $M \mapsto hM$ compatible with the operations $i \mapsto i_!$ and $j \mapsto j^!$ in the two categories.

In particular, an exact functor $F : \underline{\underline{M}} \to \underline{\underline{M}}'$ between exact categories induces a functor $Q\underline{\underline{M}} \to Q\underline{\underline{M}}'$, $M \mapsto FM$, $i_! \mapsto (Fi)_!$, $j^! \mapsto (Fj)^!$. We note also that if $\underline{\underline{M}}^o$ is the dual exact category, then we have an isomorphism of categories

(5) $$Q(\underline{\underline{M}}^o) = Q\underline{\underline{M}}$$

such that the injective arrows in the former correspond to surjective arrows in the latter and conversely.

<u>The fundamental group of</u> $Q\underline{\underline{M}}$. Suppose now that $\underline{\underline{M}}$ is a small exact category, so that the classifying space $B(Q\underline{\underline{M}})$ is defined. Let 0 be a given zero object of $\underline{\underline{M}}$.

<u>Theorem 1.</u> <u>The fundamental group</u> $\pi_1(B(Q\underline{\underline{M}}), 0)$ <u>is canonically isomorphic to the Grothendieck group</u> $K_0\underline{\underline{M}}$.

Proof. The Grothendieck group is by definition the abelian group with one generator $[M]$ for each object M of $\underline{\underline{M}}$ and one relation $[M] = [M'][M'']$ for each exact sequence (1) in $\underline{\underline{M}}$. We note that it could also be defined as the not-necessarily-abelian group with the same generators and relations, because the relations $[M'][M''] = [M' \oplus M''] = [M''][M']$ force the group to be abelian.

According to Prop. 1, the category of covering spaces of $B(Q\underline{\underline{M}})$ is equivalent to the category $\underline{\underline{F}}$ of morphism-inverting functors $F : Q\underline{\underline{M}} \to \text{Sets}$. It suffices therefore to show the group $K_0\underline{\underline{M}}$ acts naturally on $F(0)$ for F in $\underline{\underline{F}}$, and that the resulting functor from $\underline{\underline{F}}$ to $K_0\underline{\underline{M}}$ - sets is an equivalence of categories.

Let $i_M : 0 \rightarrowtail M$ and $j_M : M \twoheadrightarrow 0$ denote the obvious maps, and let $\underline{\underline{F}}'$ be the full subcategory of $\underline{\underline{F}}$ consisting of F such that $F(M) = F(0)$ and $F(i_{M!}) = id_{F(0)}$ for all M. Clearly any F is isomorphic to an object of $\underline{\underline{F}}'$, so it suffices to show

$\underline{\underline{F}}'$ is equivalent to $K_0\underline{\underline{M}}$ - sets.

Given a $K_0\underline{\underline{M}}$ - set S, let $F_S : Q\underline{\underline{M}} \to \text{Sets}$ be the functor defined by
$$F_S(M) = S,\ F_S(i_!) = id_S,\ F_S(j^!) = \text{multiplication by } [\text{Ker } j] \text{ on } S,$$
using the universal property of $Q\underline{\underline{M}}$. Clearly $S \mapsto F_S$ is a functor from $K_0\underline{\underline{M}}$ - sets to $\underline{\underline{F}}'$. On the other hand if $F \in \underline{\underline{F}}'$, then given $i : M' \rightarrowtail M$ we have $i \cdot i_{M'} = i_M$, hence $F(i_!) = id_{F(0)}$. Given the exact sequence
$$0 \longrightarrow M' \xrightarrow{\ i\ } M \xrightarrow{\ j\ } M'' \longrightarrow 0$$
we have $j^! i_{M''!} = i_! j_{M'}^!$, hence $F(j^!) = F(j_{M'}^!) \in \text{Aut}(F(0))$. Also
$$F(j_M^!) = F(j^! j_{M''}^!) = F(j_{M'}^!) F(j_{M''}^!)$$
so by the universal property of $K_0\underline{\underline{M}}$, there is a unique group homomorphism from $K_0\underline{\underline{M}}$ to $\text{Aut}(F(0))$ such that $[M] \mapsto F(j_M^!)$. Thus we have a natural action of $K_0\underline{\underline{M}}$ on $F(0)$ for any F in $\underline{\underline{F}}'$. In fact, it is clear that the resulting functor $F \mapsto F(0)$ from $\underline{\underline{F}}'$ to $K_0\underline{\underline{M}}$ - sets is an isomorphism of categories with inverse $S \mapsto F_S$, so the proof of the theorem is complete.

<u>Higher</u> <u>K-groups</u>. The above theorem offers some motivation for the following definition of K-groups for a small exact category $\underline{\underline{M}}$.

<u>Definition</u>. $K_i\underline{\underline{M}} = \pi_{i+1}(B(Q\underline{\underline{M}}), 0)$.

Note first of all that the K-groups are independent of the choice of the zero object 0. Indeed, given another zero object $0'$, there is a unique map $0 \to 0'$ in $Q\underline{\underline{M}}$, hence there is a canonical path from 0 to $0'$ in the classifying space.

Secondly we note that <u>the preceding definition extends to exact categories having a set of isomorphism classes of objects</u>. We define $K_i\underline{\underline{M}}$ to be $K_i\underline{\underline{M}}'$, where $\underline{\underline{M}}'$ is a small subcategory equivalent to $\underline{\underline{M}}$, the choice of $\underline{\underline{M}}'$ being irrelevant by Prop. 2. From now on we will only consider exact categories whose isomorphism classes form a set, except when mentioned otherwise. In addition, when we apply the results of §1, it will be tacitly assumed that we have replaced any large exact category by an equivalent small one.

<u>Elementary properties of</u> <u>K-groups</u>. An exact functor $f : \underline{\underline{M}} \to \underline{\underline{M}}'$ induces a functor $Q\underline{\underline{M}} \to Q\underline{\underline{M}}'$, and hence a homomorphism of K-groups which will be denoted

(6) $$f_* : K_i\underline{\underline{M}} \longrightarrow K_i\underline{\underline{M}}'.$$

In this way K_i becomes a functor from exact categories and exact functors to abelian groups. Moreover, isomorphic functors induce the same map on K-groups by Prop. 2. From (5) we have

(7) $$K_i(\underline{\underline{M}}^o) = K_i\underline{\underline{M}}.$$

The product $\underline{\underline{M}} \times \underline{\underline{M}}'$ of two exact categories is an exact category in which a sequence is exact when its projections in $\underline{\underline{M}}$ and $\underline{\underline{M}}'$ are. Clearly $Q(\underline{\underline{M}} \times \underline{\underline{M}}') = Q\underline{\underline{M}} \times Q\underline{\underline{M}}'$. Since the classifying space functor is compatible with products (§1, (4)), we have

(8) $$K_i(\underline{\underline{M}} \times \underline{\underline{M}}') \simeq K_i\underline{\underline{M}} \oplus K_i\underline{\underline{M}}', \quad x \mapsto pr_{1*}(x) + pr_{2*}(x).$$

The functor $\oplus : \underline{\underline{M}} \times \underline{\underline{M}} \to \underline{\underline{M}}$, $(M,M') \mapsto M \oplus M'$ is exact, so it induces a homomorphism

$$K_i\underline{\underline{M}} \oplus K_i\underline{\underline{M}} = K_i(\underline{\underline{M}} \times \underline{\underline{M}}) \xrightarrow{\oplus_*} K_i\underline{\underline{M}} .$$

This map coincides with the sum in the abelian group $K_i\underline{\underline{M}}$ because the functors $M \mapsto 0 \oplus M$, $M \mapsto M \oplus 0$ are isomorphic to the identity.

Let $j \mapsto \underline{\underline{M}}_j$ be a functor from a small filtering category to exact categories and functors, and let $\varinjlim \underline{\underline{M}}_j$ be the inductive limit of the $\underline{\underline{M}}_j$ in the sense of Prop. 3. Then $\varinjlim \underline{\underline{M}}_j$ is an exact category in a natural way, and $Q(\varinjlim \underline{\underline{M}}_j) = \varinjlim Q\underline{\underline{M}}_j$, hence from Prop. 3 we obtain an isomorphism

(9) $$K_i(\varinjlim \underline{\underline{M}}_j) = \varinjlim K_i\underline{\underline{M}}_j .$$

<u>Example</u>. Let A be a ring with 1 and let $\underline{\underline{P}}(A)$ denote the additive category of finitely generated projective (left) A-modules. We regard $\underline{\underline{P}}(A)$ as an exact category in which the exact sequences are those sequences which are exact in the category of all A-modules, and we define the K-groups of the ring A by

$$K_iA = K_i(\underline{\underline{P}}(A)).$$

A ring homomorphism $A \to A'$ induces an exact functor $A' \otimes_A ? : \underline{\underline{P}}(A) \to \underline{\underline{P}}(A')$ which is defined up to canonical isomorphism, hence it induces a well-defined homomorphism

(10) $$(A' \otimes_A ?)_* : K_iA \longrightarrow K_iA'.$$

making K_iA a covariant functor of A. From (8) we have

(11) $$K_i(A \times A') = K_iA \oplus K_iA'.$$

If $j \mapsto A_j$ is a filtered inductive system of rings, we have from (9) an isomorphism

(12) $$K_i(\varinjlim A_j) = \varinjlim K_iA_j .$$

(To apply (9), one replaces $\underline{\underline{P}}(A_j)$ by the equivalent category $\underline{\underline{P}}(A_j)'$ whose objects are the idempotent matrices over A_j , so that $\underline{\underline{P}}(\varinjlim A_j)' = \varinjlim \underline{\underline{P}}(A_j)'$.) Finally we note that $P \mapsto \mathrm{Hom}_A(P,A)$ is an equivalence of $\underline{\underline{P}}(A)$ with the dual category to $\underline{\underline{P}}(A^{op})$, where A^{op} is the opposed ring to A, hence from (7) we get a canonical isomorphism

(13) $$K_i(A) = K_i(A^{op}) .$$

<u>Remarks</u>. It can be proved that the groups K_iA defined here agree with those defined by making $BGL(A)$ into an H-space and taking homotopy groups (see for example [Gersten 5]). In particular, they coincide for $i = 1 , 2$ with the groups defined by by Bass and Milnor, and with the K-groups computed for a finite field in [Quillen 2]. On the other hand, for a general exact category $\underline{\underline{M}}$, the group $K_1(\underline{\underline{M}})$ is not the same as the universal determinant group defined in [Bass, p.389]. There is a canonical homomorphism from the universal determinant group to $K_1(\underline{\underline{M}})$, but Gersten and Murthy have produced examples showing that it is neither surjective nor injective in general.

§3. <u>Characteristic exact sequences and filtrations</u>

Let $\underline{\underline{M}}$ be an exact category and regard the family $\underline{\underline{E}}$ of short exact sequences in $\underline{\underline{M}}$ as an additive category in the obvious way. We denote objects of $\underline{\underline{E}}$ by E, E', etc. and let sE, tE, qE denote the sub-, total, and quotient objects of E, whence we have an exact sequence

$$0 \longrightarrow sE \longrightarrow tE \longrightarrow qE \longrightarrow 0$$

in $\underline{\underline{M}}$ associated to each object E of $\underline{\underline{E}}$. A sequence in $\underline{\underline{E}}$ will be called exact if it gives rise to three exact sequences in $\underline{\underline{M}}$ on applying s, t, and q. With this notion of exactness, it is clear that $\underline{\underline{E}}$ is an exact category, and that s, t, and q are exact functors from $\underline{\underline{E}}$ to $\underline{\underline{M}}$.

<u>Theorem</u> 2. <u>The functor</u> $(s,q) : Q\underline{\underline{E}} \to Q\underline{\underline{M}} \times Q\underline{\underline{M}}$ <u>is a homotopy equivalence</u>.

Proof. It suffices by Theorem A to show the category $(s,q)/(M,N)$ is contractible for any given pair M,N of objects of $\underline{\underline{M}}$. Put $\underline{\underline{C}} = (s,q)/(M,N)$; it is the fibred category over $Q\underline{\underline{E}}$ consisting of triples (E,u,v), where $u : sE \to M$, $v : qE \to N$ are maps in $Q\underline{\underline{M}}$. Let $\underline{\underline{C}}'$ be the full subcategory of $\underline{\underline{C}}$ consisting of the triples (E,u,v) such that u is surjective, and let $\underline{\underline{C}}''$ be the full subcategory of triples such that u is surjective and v is injective.

<u>Lemma</u>. <u>The inclusion functors</u> $\underline{\underline{C}}' \to \underline{\underline{C}}$ <u>and</u> $\underline{\underline{C}}'' \to \underline{\underline{C}}'$ <u>have left adjoints</u>.

Consider first the inclusion of $\underline{\underline{C}}'$ in $\underline{\underline{C}}$. Let $X = (E,u,v) \in \underline{\underline{C}}$; it suffices to show that there is a universal arrow $X \to \bar{X}$ in $\underline{\underline{C}}$ with $\bar{X}$ in $\underline{\underline{C}}'$.

Let $u = j^! i_!$ where $i : sE \rightarrowtail M'$, $j : M \twoheadrightarrow M'$, and define the exact sequence i_*E by 'pushout':

$$\begin{array}{lccccccccc} E : & 0 & \longrightarrow & sE & \longrightarrow & tE & \longrightarrow & qE & \longrightarrow & 0 \\ & & & \downarrow i & & \downarrow & & \| & & \\ i_*E : & 0 & \longrightarrow & M' & \longrightarrow & T & \longrightarrow & qE & \longrightarrow & 0 . \end{array}$$

Let $\bar{X} = (i_*E, j^!, v)$; it belongs to $\underline{\underline{C}}'$ and there is a canonical arrow $X \to \bar{X}$ given by the evident injective map $E \to i_*E$.

Now suppose given $X \to X'$ with $X' = (E', j'^!, v')$ in $\underline{\underline{C}}'$. Represent the map $E \to E'$ by the pair $E \rightarrowtail E_0$, $E' \twoheadrightarrow E_0$. Since

$$sE \rightarrowtail sE_0 \twoheadleftarrow sE' \overset{j'}{\twoheadleftarrow} M$$

represents u, we can suppose E_0 chosen so that $sE \rightarrowtail sE_0$ is the map i, and $M \twoheadrightarrow sE_0$ is j. By the universal property of pushouts, the map $E \rightarrowtail E_0$ factors uniquely $E \rightarrowtail i_*E \rightarrowtail E_0$, so it is clear that we have a map $\bar{X} \to X'$ in $\underline{\underline{C}}'$ such that $X \to \bar{X} \to X'$ is the given map $X \to X'$.

It remains to show the uniqueness of the map $\bar{X} \to X'$. Consider factorizations $X \to X'' \to X'$ of $X \to X'$ such that X'' is in $\underline{\underline{C}}'$. Note that $\underline{\underline{C}}/X' = Q\underline{\underline{E}}/E'$ is equivalent to the ordered set of admissible layers in E'. Let (E_0 , E_1) be the layer corresponding to $X \to X'$ and (E''_0 , E''_1) the layer corresponding to $X'' \to X'$ so that

$(E_0, E_1) \leq (E''_0, E''_1)$ and $sE''_1 = sE'$. There is a least such layer (E''_0, E''_1) given by $tE''_0 = tE_0$, $tE''_1 = sE' + tE_1$, which is characterized by the fact that the map $E_1/E_0 \to E''_1/E''_0$ is injective and induces an isomorphism on quotient objects. Thus among the factorizations $X \to X'' \to X'$ there is a least one, unique up to canonical isomorphism, and characterized by the condition that $E \to E''$ should be injective and induce an isomorphism $qE \xrightarrow{\sim} qE''$. Since the factorization $X \to \bar{X} \to X'$ has this property, it is clear that the map $\bar{X} \to X'$ is uniquely determined. Thus $\underline{\underline{C}}' \to \underline{\underline{C}}$ has the left adjoint $X \mapsto \bar{X}$.

Next consider the inclusion of $\underline{\underline{C}}''$ in $\underline{\underline{C}}'$, and let $(E,u,v) \in \underline{\underline{C}}'$. Represent $v : qE \twoheadrightarrow N$ by the pair $j : N' \twoheadrightarrow qE$, $i : N' \rightarrowtail N$, and define j^*E by pull-back:

$$\begin{array}{ccccccccc} 0 & \to & sE & \to & T & \to & N' & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & sE & \to & tE & \to & qE & \to & 0 . \end{array}$$

One verifies by an argument essentially dual to the preceding one that $(E,u,v) \mapsto (j^*E, u, i_!)$ is left adjoint to the inclusion of $\underline{\underline{C}}''$ in $\underline{\underline{C}}'$. This finishes the lemma.

By Prop. 2, Cor. 1, the categories $\underline{\underline{C}}$ and $\underline{\underline{C}}''$ are homotopy equivalent. Let $(E, j^!, i_!) \in \underline{\underline{C}}''$, and let $j_M : M \twoheadrightarrow 0$ and $i_N : 0 \rightarrowtail N$ be the obvious maps. A map from $(0, j_M{}^!, i_{N!})$ to $(E, j^!, i_!)$ may be identified with an admissible subobject E' of E such that $sE' = sE$ and $qE' = 0$. Clearly E' is unique, so $(0, j_M{}^!, i_{N!})$ is an initial object of $\underline{\underline{C}}''$. Thus $\underline{\underline{C}}''$, and hence $\underline{\underline{C}}$ is contractible, which finishes the proof of the theorem.

Corollary 1. Let $\underline{\underline{M}}'$ and $\underline{\underline{M}}$ be exact categories and let

$$0 \to F' \to F \to F'' \to 0$$

be an exact sequence of exact functors from $\underline{\underline{M}}'$ to $\underline{\underline{M}}$. Then

$$F_* = F'_* + F''_* : K_i\underline{\underline{M}}' \to K_i\underline{\underline{M}} .$$

Proof. It clearly suffices to treat the case of the exact sequence

$$0 \to s \to t \to q \to 0$$

of functors from $\underline{\underline{E}}$ to $\underline{\underline{M}}$. Let $f : \underline{\underline{M}} \times \underline{\underline{M}} \to \underline{\underline{E}}$ be the exact functor sending (M', M'') to the split exact sequence

$$0 \to M' \to M' \oplus M'' \to M'' \to 0 .$$

The functors tf and $\oplus(s,q)f$ are isomorphic, hence

$$t_* f_* = \oplus_*(s_*, q_*)f_* = (s_* + q_*)f_* : (K_i\underline{\underline{M}})^2 \to K_i\underline{\underline{M}} .$$

But f_* is a section of $(s_*, q_*) : K_i\underline{\underline{E}} \to (K_i\underline{\underline{M}})^2$ which is an isomorphism by the theorem. Thus $t_* = s_* + q_*$, proving the corollary.

Note that the category of functors from a category $\underline{\underline{C}}$ to an exact category $\underline{\underline{M}}$ is an exact category in which a sequence of functors is exact if it is pointwise exact. We thus have the notion of an admissible filtration $0 = F_0 \subset F_1 \subset \ldots \subset F_n = F$ of a functor F. This means that $F_{p-1}(X) \to F_p(X)$ is an admissible monomorphism in $\underline{\underline{M}}$ for every X

in $\underline{\underline{C}}$, and it implies that there exist quotient functors F_p/F_q for $q \leq p$, determined up to canonical isomorphism. It is easily seen that if $\underline{\underline{C}}$ is an exact category, and if the functors F_p/F_{p-1} are exact for $1 \leq p \leq n$, then all the quotients F_p/F_q are exact.

Corollary 2. (Additivity for 'characteristic' filtrations) Let $F : \underline{\underline{M}}' \to \underline{\underline{M}}$ be an exact functor between exact categories equipped with an admissible filtration $0 = F_0 \subset \ldots \subset F_n = F$ such that the quotient functors F_p/F_{p-1} are exact for $1 \leq p \leq n$. Then

$$F_* = \sum_{p=1}^{n} (F_p/F_{p-1})_* : K_i\underline{\underline{M}}' \to K_i\underline{\underline{M}} .$$

Corollary 3. (Additivity for 'characteristic' exact sequences) If

$$0 \to F_0 \to \ldots \to F_n \to 0$$

is an exact sequence of exact functors from $\underline{\underline{M}}'$ to $\underline{\underline{M}}$, then

$$\sum_{p=0}^{n} (-1)^p (F_p)_* = 0 : K_i\underline{\underline{M}}' \to K_i\underline{\underline{M}} .$$

These result from Cor. 1 by induction.

Applications. We give two simple examples to illustrate the preceding results.

Let X be a ringed space, and put $K_iX = K_i\underline{\underline{P}}(X)$, where $\underline{\underline{P}}(X)$ is the category of vector bundles on X, (i.e. sheaves of $\underline{\underline{O}}_X$-modules which are locally direct factors of $\underline{\underline{O}}_X^n$) equipped with the usual notion of exact sequence. Given E in $\underline{\underline{P}}(X)$, we have an exact functor $E \otimes ? : \underline{\underline{P}}(X) \to \underline{\underline{P}}(X)$ which induces a homomorphism of K-groups $(E \otimes ?)_* : K_iX \to K_iX$. If $0 \to E' \to E \to E'' \to 0$ is an exact sequence of vector bundles, then Cor. 1 implies $(E \otimes ?)_* = (E' \otimes ?)_* + (E'' \otimes ?)_*$. Thus we obtain products

(1) $$K_0X \otimes_{\mathbb{Z}} K_iX \to K_iX , \quad [E] \otimes x \mapsto (E \otimes ?)_* x$$

which clearly make K_iX into a module over K_0X. (Products $K_iX \otimes K_jX \to K_{i+j}X$ can also be defined, but this requires more machinery.)

Graded rings. Let $A = A_0 \oplus A_1 \oplus \ldots$ be a graded ring and denote by $\underline{\underline{P}}gr(A)$ the category of graded finitely generated projective A-modules $P = \oplus P_n$, $n \in \mathbb{Z}$. The group $K_i(\underline{\underline{P}}gr(A))$ is a $\mathbb{Z}[t,t^{-1}]$-module, where multiplication by t is the automorphism induced by the translation functor $P \mapsto P(-1)$, $P(-1)_n = P_{n-1}$.

Proposition. There is a $\mathbb{Z}[t,t^{-1}]$-module isomorphism

$$\mathbb{Z}[t,t^{-1}] \otimes_{\mathbb{Z}} K_iA_0 \xrightarrow{\sim} K_i(\underline{\underline{P}}gr(A)) , \quad 1 \otimes x \mapsto (A \otimes_{A_0} ?)_* x .$$

Proof. Given P in $\underline{\underline{P}}gr(A)$, let F_kP be the A-submodule of P generated by P_n for $n \leq k$, and let $\underline{\underline{P}}_q$ be the full subcategory of $\underline{\underline{P}}gr(A)$ consisting of those P for which $F_{-q-1}P = 0$ and $F_qP = P$. We have an exact functor

$$T : \underline{\underline{P}}gr(A) \to \underline{\underline{P}}gr(A_0) , \quad T(P) = A_0 \otimes_A P$$

where A_0 is considered as a graded ring concentrated in degree zero. It is known ([Bass], p.637) that P is non-canonically isomorphic to

$$A \otimes_{A_0} T(P) = \coprod_n A(-n) \otimes_{A_0} T(P)_n .$$