Algebraic K-theory

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Contents

Introduction

Algebraic K-theory is the use of homotopy theory to study matrices.

The main reference for the field of higher algebraic K-theory is Quillen's foundational paper [13]. For lower algebraic K-theory, including mainly K_2, we refer to [11], and for lower algebraic K-theory, including mainly K_1, to [1]. For background in algebraic topology we refer to [6].

Many modern K-theory papers can be found in the K-theory preprint archives, online at `http://www.math.uiuc.edu/K-theory/`.

These notes are updated frequently during Spring Semester, 2003, as I teach Math 416, so may occasionally contain inaccuracies and mistakes. The most recent revision was April 30, 2003.

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Here are the homework assignments:

- 1) 1.2.22
- 2) 1.2.23
- 3) 2.4.9
- 4) 2.7.4
- 5) 4.2.9
- 6) 6.2.1
- 7) 3.1.11
- 8) 3.3.3
- 9) 3.3.5

CHAPTER 1

Grothendieck groups

1.1. Direct sum Grothendieck groups

We begin by introducing the direct sum version of the Grothendieck group.

DEFINITION 1.1.1. For a small additive category $\mathcal{C}$ we define $K_0^{\oplus}\mathcal{C}$ to be the abelian group given by the following generators and relations. For each object $C \in \mathcal{C}$ there is a generator called $[C]$. For each isomorphism $C \cong C'$ of objects of $\mathcal{C}$ there is a relation $[C] = [C']$. Finally, for each direct sum $C \oplus C'$ of objects C and C' in $\mathcal{C}$ there is a relation $[C \oplus C'] = [C] + [C']$.

Since $C \oplus C' \cong C' \oplus C$, we could have omitted the word "abelian" in the definition above.

An equivalence $F : \mathcal{B} \to \mathcal{C}$ of small additive categories induces an isomorphism $K_0^{\oplus}\mathcal{B} \xrightarrow{\cong} K_0^{\oplus}\mathcal{C}$. Thus if $\mathcal{D}$ is an additive category that is not small, and if it has a set $S \subset \operatorname{Obj}\mathcal{D}$ containing at least one representative from each isomorphism class of objects, then we may let $\mathcal{C}$ be the full subcategory of $\mathcal{D}$ with $\operatorname{Obj}\mathcal{C} = S$, thereby obtaining a small additive subcategory of $\mathcal{D}$ equivalent to it. We may then define $K_0^{\oplus}\mathcal{D} := K_0^{\oplus}\mathcal{C}$. A different choice of S would give a different abelian group $K_0^{\oplus}\mathcal{D}$, but it would be canonically isomorphic to the other one, so one choice of S is just as good as another.

NOTATION 1.1.2. Let R be a ring (with 1) and let $\mathcal{M}_R$ denote the additive category of finitely generated left R-modules.

The category $\mathcal{M}_R$ is not small. Indeed, the class of R-modules isomorphic to 0 isn't even a set, because there are so many different things we could use for the single element of an R-module. Fortunately, $\mathcal{M}_R$ has a set of objects containing at least one representative from each isomorphism class, namely, the set of modules which are quotients of R^n for some $n \geq 0$. (Every quotient module of R^n is explicitly constructed as the set of cosets of some submodule in R^n, so the collection of quotient modules of R^n is in one-to-one correspondence with the set of submodules of R^n.) Hence $K_0^{\oplus}\mathcal{M}_R$ is defined.

EXAMPLE 1.1.3. Let $\mathcal{C}$ be the category of countably generated R-modules. Then by the same argument as above, $K_0^{\oplus}\mathcal{C}$ is defined, but we can prove that $K_0^{\oplus}\mathcal{C} = 0$. For, if $C \in \mathcal{C}$, then so is $D := C \oplus C \oplus C \oplus \cdots$. But the isomorphism $C \oplus D \cong D$ yields the equation $[C] + [D] = [D]$, and thus $[C] = 0$. (The same argument works if we replace countability by any infinite cardinality.)

Suppose $F : \operatorname{Obj}\mathcal{C} \to A$ is a function assigning to each object $C \in \mathcal{C}$ an element $F(C) \in A$ of an abelian group A, and suppose that F is *additive* in the sense that the following identities hold: $F(C) = F(C')$ whenever $C \cong C'$ and $F(C \oplus C') =$

$F(C) + F(C')$. Then there is a unique group homomorphism $F_* : K_0^{\oplus}\mathcal{C} \to A$ such that $F_*[C] = F(C)$ for each $C \in \mathcal{C}$.

EXAMPLE 1.1.4. Let $\Bbbk$ be a field, and let $\dim : \operatorname{Obj} \mathcal{M}_{\Bbbk} \to \mathbb{Z}$ be the additive function which associates to a vector space its dimension. Then $\dim_* : K_0^{\oplus}\mathcal{M}_{\Bbbk} \xrightarrow{\cong} \mathbb{Z}$ sends the generator $[\Bbbk^1]$ of the cyclic group $K_0^{\oplus}\mathcal{M}_{\Bbbk}$ to 1, and thus is an isomorphism.

EXAMPLE 1.1.5. Let G be a finite group, and let $\Bbbk$ be a field of characteristic not dividing the order of G. Let $R = \Bbbk[G]$ be the group ring. A module $M \in \mathcal{M}_R$ is essentially just a finite dimensional representation of G over $\Bbbk$, and by the Wedderburn theorem, it is a direct sum of simple modules (irreducible representations). Let $V_1, \ldots, V_s$ be a complete set of inequivalent representatives of the isomorphism classes of simple modules, and define additive functions $F_i : \operatorname{Obj} \mathcal{M}_R \to \mathbb{Z}$ by setting $F_i(M) = e_i$ if $M \cong V_1^{e_1} \oplus \cdots \oplus V_s^{e_s}$. The resulting maps $F_{i*} : K_0^{\oplus}\mathcal{M}_R \to \mathbb{Z}$ can be assembled into an isomorphism $F_* : K_0^{\oplus}\mathcal{M}_R \xrightarrow{\cong} \mathbb{Z}^s$.

EXAMPLE 1.1.6. Consider the additive category $\mathcal{M}_{\mathbb{Z}}$. Every $M \in \mathcal{M}_{\mathbb{Z}}$ can be written in the form $M \cong \mathbb{Z}^r \oplus \mathbb{Z}/p_1^{e_1} \oplus \cdots \oplus \mathbb{Z}/p_t^{e_t}$ in essentially one way. Hence $K_0^{\oplus}\mathcal{M}_{\mathbb{Z}}$ is a free abelian group with one generator $[\mathbb{Z}]$ and one generator $[\mathbb{Z}/p^e]$ for each prime power p^e with $e \geq 1$.

1.2. Exact sequence Grothendieck groups

We introduce now the exact sequence version of the Grothendieck group. First we abstract the most frequent source for the notion of short exact sequence by presenting the following definition.

DEFINITION 1.2.1. Let $\mathcal{A}$ be a small abelian category, and we let $\mathcal{C}$ be a full additive subcategory of $\mathcal{A}$ closed under extension in the sense that whenever $0 \to C' \to A \to C'' \to 0$ is an exact sequence of $\mathcal{A}$ with $C' \in \mathcal{C}$ and $C'' \in \mathcal{C}$, then $A \in \mathcal{C}$. A sequence $0 \to C' \to C \to C'' \to 0$ in $\mathcal{C}$ is called a *short exact sequence* of $\mathcal{C}$ if it is exact in $\mathcal{A}$. We call $\mathcal{C}$, equipped with the collection of its short exact sequences, an *exact* category.

A more intrinsic characterization of exact categories will be given later.

A functor $F : \mathcal{C} \to \mathcal{D}$ between exact categories will be called *exact* if for every short exact sequence $0 \to C' \to C \to C'' \to 0$ of $\mathcal{C}$ the sequence $0 \to FC' \to FC \to FC'' \to 0$ is an exact sequence of $\mathcal{D}$.

EXAMPLE 1.2.2. The category $\mathcal{M}_R$, with the usual notion of short exact sequence, is an exact category, for it is closed under extension in the abelian category of all R-modules. It is abelian if and only if R is left noetherian.

DEFINITION 1.2.3. For a small exact category $\mathcal{C}$ we define $K_0\mathcal{C}$ to be the abelian group given by the following generators and relations. For each object $C \in \mathcal{C}$ there is a generator called $[C]$, and for each short exact sequence $0 \to C' \to C \to C'' \to 0$ in $\mathcal{C}$ there is a relation $[C] = [C'] + [C'']$.

As before, for categories such as $\mathcal{M}_R$ which are not quite small, we may define $K_0\mathcal{M}_R$ by choosing a small exact category equivalent to it.

DEFINITION 1.2.4. For an exact functor $F : \mathcal{C} \to \mathcal{D}$ of small exact categories, we define $K_0F : K_0\mathcal{C} \to K_0\mathcal{D}$ to be the unique homomorphism defined on generators by $F[C] = [FC]$.

LEMMA 1.2.5. *If $\mathcal{C}$ is a small exact category, if $C' \cong C$ in $\mathcal{C}$, then $[C'] = [C]$ in $K_0\mathcal{M}$.*

PROOF. The exact sequence $0 \to C' \to C \to 0 \to 0$ shows us that $[C] = [C'] + [0]$. The exact sequence $0 \to 0 \to 0 \to 0 \to 0$ shows us that $[0] = [0] + [0]$, and thus $0 = [0]$. The result follows. □

LEMMA 1.2.6. *If $\mathcal{C}$ is a small exact category, $[C' \oplus C''] = [C'] + [C'']$ holds in $K_0\mathcal{C}$ for any objects C' and C''.*

PROOF. Use the exact sequence $0 \to C' \to C' \oplus C'' \to C'' \to 0$. □

Since $C' \oplus C'' \cong C'' \oplus C'$, we could have omitted the word "abelian" in definition 1.2.3 and proved instead that the group given by the generators and relations above is abelian.

LEMMA 1.2.7. *If $\mathcal{C}$ is a small exact category, there is a natural map $K_0^{\oplus}\mathcal{C} \to K_0\mathcal{C}$ sending each $[C]$ to $[C]$.*

PROOF. Apply 1.2.5 and 1.2.6 to show that the defining relations of $K_0^{\oplus}\mathcal{C}$ hold in $K_0\mathcal{C}$. □

LEMMA 1.2.8. *If $\mathcal{C}$ is a small exact category in which every exact sequence of $\mathcal{C}$ splits, then the natural map $K_0^{\oplus}\mathcal{C} \to K_0\mathcal{C}$ is an isomorphism.*

PROOF. If the exact sequence $0 \to C' \to C \to C'' \to 0$ splits, then $C \cong C' \oplus C''$, and the defining relation $[C] = [C'] + [C'']$ holds already in $K_0^{\oplus}\mathcal{C}$. □

Let R be a ring (with 1). The following category is an example of a category in which every short exact sequence splits.

DEFINITION 1.2.9. Let $\mathcal{P}_R$ denote the category of finitely generated projective left R-modules. It is an exact category, because it is closed under extension in the abelian category of all R-modules.

Now we define the Grothendieck group of a ring R.

DEFINITION 1.2.10. Let $K_0(R) = K_0(\mathcal{P}_R)$.

EXAMPLE 1.2.11. For another example of a category in which every short exact sequence splits, consider a paracompact topological space X and let $\mathcal{P}_X$ denote the category of real vector bundles (of finite rank) over X. Using partitions of unity one can show that any vector bundle E can be equipped with Hermitian inner product on its fibers that vary continuously from one fiber to another. Given a subbundle $F \subseteq E$, orthogonal projection with respect to the inner product provides a splitting map $E \to F$.

Suppose $F : \operatorname{Obj}\mathcal{C} \to A$ is a function assigning to each object $C \in \mathcal{C}$ an element $F(C) \in A$ of an abelian group A, and suppose that F is *additive* in the sense that the following identity always holds: $F(C) = F(C') + F(C'')$ for every exact sequence $0 \to C' \to C \to C'' \to 0$ in $\mathcal{M}$. Then there is a unique group homomorphism $F_* : K_0\mathcal{C} \to A$ such that $F_*[C] = F(C)$ for each $C \in \mathcal{C}$. The group $K_0\mathcal{C}$ is the natural way to record all the additive functions.

EXAMPLE 1.2.12. The group $K_0\mathcal{M}_{\mathbb{Z}}$ is an infinite cyclic group generated by $[\mathbb{Z}]$: the analogues of the generators found in (1.1.6) are mostly redundant, for the short exact sequence $0 \to \mathbb{Z} \xrightarrow{p^e} \mathbb{Z} \to \mathbb{Z}/p^e \to 0$ shows that $[\mathbb{Z}] = [\mathbb{Z}] + [\mathbb{Z}/p^e]$, and hence that $[\mathbb{Z}/p^e] = 0$ in $K_0\mathcal{M}_{\mathbb{Z}}$. Hence $K_0\mathcal{M}_{\mathbb{Z}}$ is cyclic, generated by $[\mathbb{Z}]$. The map $\mathrm{rank}_* : K_0\mathcal{M}_{\mathbb{Z}} \to \mathbb{Z}$ that measures the rank of an abelian group sends $[\mathbb{Z}]$ to 1, so must be an isomorphism.

DEFINITION 1.2.13. Let $\mathcal{M}^1_{\mathbb{Z}}$ denote the (abelian) category of finite abelian groups. (The superscript 1 indicates that the finite abelian grops are those $\mathbb{Z}$-modules with support of codimension 1 or more.)

EXAMPLE 1.2.14. Let's compute $K_0\mathcal{M}^1_{\mathbb{Z}}$. Any finite abelian group isomorphic to a direct sum of groups of the form $\mathbb{Z}/p^e$, where p is prime. The exact sequence $0 \to \mathbb{Z}/p^{e-1} \to \mathbb{Z}/p^e \to \mathbb{Z}/p \to 0$ shows that $[\mathbb{Z}/p^e] = [\mathbb{Z}/p^{e-1}] + [\mathbb{Z}/p]$ in $K_0\mathcal{M}^1_{\mathbb{Z}}$. By induction we see that $[\mathbb{Z}/p^e] = e[\mathbb{Z}/p]$, and thus the set $\{[\mathbb{Z}/p] \mid p \text{ prime}\}$ generates $K_0\mathcal{M}^1_{\mathbb{Z}}$. The function $F_p : \mathrm{Obj}\,\mathcal{M}^1_{\mathbb{Z}} \to \mathbb{Z}$ that assigns to an abelian group the number of factors in its composition series isomorphic to $\mathbb{Z}/p$ is an additive function. The homomorphisms $F_{p*} : K_0\mathcal{M}^1_{\mathbb{Z}} \to \mathbb{Z}$ assemble to provide an isomorphism $F_* : K_0\mathcal{M}^1_{\mathbb{Z}} \xrightarrow{\cong} \coprod_P \mathbb{Z}$, where P is the set of primes.

We can rephrase that computation slightly, for a convenient way to compute the number $F_p(M)$ is to take the p-part of the prime factorization of the order $\#M$: it's $p^{F_p(M)}$. Hence the vector $F_*[M]$ is completely encoded by the number $\#M$. Indeed, $\# : \mathrm{Obj}\,\mathcal{M}^1_{\mathbb{Z}} \to \mathbb{Q}^\times$ is an additive function. (Here $R^\times$ is our notation for the multiplicative group of units in a ring R.) The corresponding homomorphism $\#_* : K_0\mathcal{M}^1_{\mathbb{Z}} \to \mathbb{Q}^\times$ is injective, and the image is the set of positive rational numbers.

It's tempting to record that calculation as a short exact sequence $0 \to K_0\mathcal{M}^1_{\mathbb{Z}} \xrightarrow{\#_*} \mathbb{Q}^\times \xrightarrow{\mathrm{sign}} \mathbb{Z}^\times \to 0$. But the sequence splits naturally, so we may turn it around and write it like this.

exZ (1.2.1)
$$0 \to \mathbb{Z}^\times \to \mathbb{Q}^\times \xrightarrow{\partial} K_0\mathcal{M}^1_{\mathbb{Z}} \to 0$$

That turns out to be the right way to view it, for it allows us to motivate the introduction of the higher K-groups as functors derived from K_0 in a certain sense. Thinking of K_0 as a functor from the category of small exact categories to the category of abelian groups, we may ask what sort of functor it is. Evidently, if $F : \mathcal{B} \to \mathcal{C}$ is an exact functor that is surjective on isomorphism classes of objects, then K_0F is surjective, too. So, by analogy with lines of development that have proved fruitful in homological algebra, we may ask whether K_0 is something like a right exact functor or an exact functor. Of course, for that, we would need some sort of notion of "short exact sequence" of exact categories. Fortunately, such a notion, at least for abelian categories, has been developed.

DEFINITION 1.2.15. Let $F : \mathcal{B} \to \mathcal{C}$ be an exact functor of abelian categories such that for any object $C \in \mathcal{C}$ there is an object $B \in \mathcal{B}$ with $FB \cong C$. Assume also that every arrow of $\mathcal{C}$ can be written in the form $F(g)^{-1} \circ F(f)$ for suitable arrows f and g in $\mathcal{B}$ such that $F(g)$ is an isomorphism. Let $\ker F$ denote the full subcategory of $\mathcal{B}$ whose objects are those objects $B \in \mathcal{B}$ with $FB \cong 0$. Let $\mathcal{A} \to \ker F$ be an equivalence of categories. We will call $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ a *short exact sequence* of abelian categories.

EXAMPLE 1.2.16. The sequence $\mathcal{M}_{\mathbb{Z}}^1 \to \mathcal{M}_{\mathbb{Z}} \to \mathcal{M}_{\mathbb{Q}}$ is a short exact sequence of abelian categories.

If we apply K_0 and recall the computations above we get a right exact sequence $K_0\mathcal{M}_{\mathbb{Z}}^1 \to K_0\mathcal{M}_{\mathbb{Z}} \to K_0\mathcal{M}_{\mathbb{Q}} \to 0$ in which the first map is zero and the second map is an isomorphism. Since $K_0\mathcal{M}_{\mathbb{Z}}^1$ is not zero, the sequence $0 \to K_0\mathcal{M}_{\mathbb{Z}}^1 \to K_0\mathcal{M}_{\mathbb{Z}} \to K_0\mathcal{M}_{\mathbb{Q}} \to 0$ is not exact, but splicing it with (1.2.1) we may get a longer exact sequence that looks like this.

ZQseq

$$0 \to \mathbb{Z}^\times \to \mathbb{Q}^\times \xrightarrow{\partial} K_0\mathcal{M}_{\mathbb{Z}}^1 \to K_0\mathcal{M}_{\mathbb{Z}} \to K_0\mathcal{M}_{\mathbb{Q}} \to 0 \tag{1.2.2}$$

This sequence is highly suggestive, even though it is not very long, because in it $\mathbb{Z}^\times$ occurs three spaces to the left of $K_0\mathcal{M}_{\mathbb{Z}}$, and $\mathbb{Q}^\times$ occurs three spaces to the left of $K_0\mathcal{M}_{\mathbb{Q}}$. Recall that the long exact sequences of homological algebra involving derived functors (such as $\operatorname{Tor}_n^R(M,N)$, $\operatorname{Ext}_R^n(M,N)$, or $H^n(X,\mathcal{F})$) follow the same sort of pattern. That suggests that we look for some sort of derived functors K_n of K_0 which would fit into a long exact sequence

Kles

$$\cdots \to K_2\mathcal{C} \to K_1\mathcal{A} \to K_1\mathcal{B} \to K_1\mathcal{C} \to K_0\mathcal{A} \to K_0\mathcal{B} \to K_0\mathcal{C} \to 0 \tag{1.2.3}$$

whenever $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ is a short exact sequence of abelian categories. This expectation turns out to be fulfilled, and proving it is one of our goals. The wonderful surprise is that the construction requires homotopy theory, rather than homological algebra!

Let's pause for a moment to take a closer look at the boundary map in (1.2.2). The map $\partial : \mathbb{Q}^\times \to K_0\mathcal{M}_{\mathbb{Z}}^1$ should split $\#_*$, so should be defined so that $\partial(n) = [\mathbb{Z}/n\mathbb{Z}]$ for $n \in \mathbb{Z} - \{0\}$. One simple definition that works for all $n/m \in \mathbb{Q}^\times$ is to set $\partial(n/m) = [\mathbb{Z}/n\mathbb{Z}] - [\mathbb{Z}/m\mathbb{Z}]$. Check that it is well defined by using the following short exact sequences.

$$0 \to \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/tn\mathbb{Z} \to \mathbb{Z}/t\mathbb{Z} \to 0$$
$$0 \to \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/tm\mathbb{Z} \to \mathbb{Z}/t\mathbb{Z} \to 0$$

Now compute $\partial(tn/tm) = [\mathbb{Z}/tn\mathbb{Z}] - [\mathbb{Z}/tm\mathbb{Z}] = ([\mathbb{Z}/t\mathbb{Z}] + [\mathbb{Z}/n\mathbb{Z}]) - ([\mathbb{Z}/t\mathbb{Z}] + [\mathbb{Z}/m\mathbb{Z}]) = [\mathbb{Z}/n\mathbb{Z}] - [\mathbb{Z}/m\mathbb{Z}] = \partial(n/m)$. Observe also that the equivalence relation defining fractions is generated by requiring $tn/tm = n/m$. If we replace $\mathbb{Z}$ by the coordinate ring R of a nonsingular affine algebraic curve, then $K_0\mathcal{M}_R^1$ is the group of divisors on the curve, ∂ is the function that assigns to a nonzero rational function its divisor, and $\operatorname{coker}\partial$ is the divisor class group.

The computation above suggests the following exercises, in which we show how to define the Grothendieck group as a set of equivalence classes.

By analogy with standard terminology for groups, we introduce the notion of a *set defined by generators and relations.*

DEFINITION 1.2.17. Given a set Y and a set $R \subseteq Y \times Y$ the *set defined by the generators Y and the relations R* will be the set X of equivalence classes in Y for the equivalence relation generated by R. We may refer to a relation $(y, y') \in R$ by writing the equation $y = y'$ instead, anticipating its truth in X.

Since X is a quotient set of Y, the generators didn't generate any new elements that weren't already there, so the relations are the important part of the concept.

EXERCISE 1.2.18. Let $S \subseteq R$ be a multiplicative subset of a commutative ring R, and consider the set $L(R,S)$ defined by the following generators and relations.

There is a generator for each $[r, s]$ for each $r \in R$ and each $s \in S$, and there is a relation $[tr, ts] = [r, s]$ for all $r \in R$ and $s, t \in S$. Show that defining $[r, s] \cdot [r', s'] := [rr', ss']$ and $[r, s] + [r', s'] := [rs' + r's, ss']$ gives two well defined binary operations that make $L(R, S)$ into a ring which is naturally isomorphic to the ring of fractions $S^{-1}R$.

EXERCISE 1.2.19. Let M be a commutative additive monoid, and consider the set $L(M, M)$ defined by the following generators and relations. There is a generator $[m, n]$ for each $m, n \in M$, and there is a relations $[t + m, t + n] = [m, n]$ for all $m, n, t \in M$. Show that defining $[m, n] + [m', n'] := [m + m', n + n']$ gives a well defined binary operation that makes $L(M, M)$ into an abelian group. Show that the function $M \to L(M, M)$ defined by $n \mapsto [n, 0]$ is a universal map of monoids from M to a group.

EXERCISE 1.2.20. Let $\mathcal{C}$ be a small additive category, and consider the set $L^{\oplus}(\mathcal{C}, \mathcal{C})$ defined by the following generators and relations. There is a generator $[C, D]$ for each pair of objects C, D in $\mathcal{C}$, and there is a relation $[C, D] = [C \oplus E, D \oplus E]$ for all objects C, D, and E in $\mathcal{C}$. Show that defining $[C, D] + [C', D'] := [C \oplus C', D \oplus D']$ gives a well defined binary operation that makes $L^{\oplus}(\mathcal{C}, \mathcal{C})$ into an abelian group. Produce a natural isomorphism $K_0^{\oplus}\mathcal{C} \cong L^{\oplus}(\mathcal{C}, \mathcal{C})$.

DEFINITION 1.2.21. Two objects C and D of an additive category $\mathcal{C}$ are called *stably isomorphic* if there is an object $E \in \mathcal{C}$ such that $C \oplus E \cong D \oplus E$.

EXERCISE 1.2.22. Let $\mathcal{C}$ be a small additive category. Show that two objects C and D of $\mathcal{C}$ are stably isomorphic if and only if $[C] = [D]$ in $K_0^{\oplus}\mathcal{C}$.

EXERCISE 1.2.23. Let $\mathcal{C}$ be a small exact category, and consider the set $L(\mathcal{C}, \mathcal{C})$ defined by the following generators and relations. There is a generator $[C, D]$ for each pair of objects C, D in $\mathcal{C}$, and there is a relation $[C', D'] = [C, D]$ whenever there is a pair of short exact sequences $0 \to C' \to C \to E \to 0$ and $0 \to D' \to D \to E \to 0$ in $\mathcal{C}$ (sharing the same quotient object E). Show that defining $[C, D] + [C', D'] := [C \oplus C', D \oplus D']$ gives a well defined binary operation that makes $L(\mathcal{C}, \mathcal{C})$ into an abelian group. Produce a natural isomorphism $K_0\mathcal{C} \cong L(\mathcal{C}, \mathcal{C})$.

CHAPTER 2

Constructing topological spaces combinatorially

2.1. Making spaces from simplices

Given a space, a good way to compute its homology is to find an explicit cell-decomposition or triangulation of the space. The reverse is true as well: if one wants to construct interesting spaces, one way to proceed is to build up the space in stages by attaching cells (disks) of increasing dimension. First we set out some collection of 0-cells (points), obtaining a discrete space. Then we add some 1-cells (edges, intervals) by attaching each endpoint to a previously existing 0-cell, obtaining a graph. Then we add some 2-cells (planar disks, triangles) by specifying loops in the previously existing graph which will support the boundaries of our 2-cells. We continue in this way through cells of higher and higher dimension. To finish off the procedure in case we desire to add cells of arbitrarily high dimension we take the union of the sequence of previously constructed spaces.

The attachment of 1-cells described above is *combinatorial* in the sense that only a finite amount of data is needed to describe the attachment, namely the pair of 0-cells to which to attach the endpoints. For cells of higher dimension, the attachments could be any continuous map from the boundary of new cell to the previously constructed space, but that level of generality is not useful; we prefer to simplify the situation by using only attachment maps which can be described combinatorially. For example, we may choose some standard cell decomposition for the boundary of the k-cell, and require the attachment maps to be suitably linear on the cells of the boundary.

The choice which seems to work best is to identify the k-cell with the standard simplex of dimension k, namely the set

sim (2.1.1)
$$|\Delta^k| = \big\{(a_0, \dots, a_k) \in [0,1]^{k+1} \mid \sum_{i=0}^{k} a_i = 1\big\}.$$

(The use of such composite notation in a definition is an abuse, for later we will attach separate meanings to Δ^n and to $|X|$.) With this definition, $|\Delta^0|$ is a point, $|\Delta^1|$ is a line segment, $|\Delta^2|$ is a triangle, $|\Delta^3|$ is a tetrahedron, and so on. The boundary of $|\Delta^k|$ is the union of $k+1$ simplices (faces) of dimension $k-1$ each of which corresponds to one of the boundary conditions $a_i = 0$ for $i = 0, \dots, k$.

The attachment maps will presumably be affine maps from one simplex to another that send each vertex to a vertex. Let's clarify that terminology. We define $e_i = (0, \dots, 0, 1, 0, \dots, 0) \in |\Delta^k|$, where the 1 is in the i-th spot. The *vertices* of $|\Delta^k|$ are the points $\{e_0, \dots, e_n\}$. Any point $a \in |\Delta^k|$ can be written in the form $a = \sum a_i e_i$ with $\sum a_i = 1$. An *affine* map $\phi : |\Delta^k| \to |\Delta^\ell|$ is a function satisfying $\phi(\sum a_i e_i) = \sum a_i \phi(e_i)$.

We order the vertices by declaring that $e_i \leq e_j$ when $i \leq j$. By means of this ordering we regard the standard simplices as *oriented* or *directed*.

We envision a sort of package of combinatorial data, called X, say, which encodes a recipe for construction of a particular space by cell attachment. There will be a procedure, called *geometric realization*, for following the recipe and producing a topological space $|X|$. It was John Milnor [10] who prescribed a successful format for the package X, namely, that it should be what is known today as a *simplicial set*. Governing his choice was the desire that geometric realization be a functor, i.e., that it be useful not only for constructing spaces, but also for constructing maps between spaces and homotopies between maps; this feature turns out to be very important. By a bit of additional good luck, the notion of simplicial set supports a broad array of homotopy theory techniques, which in turn allow strong theorems about the algebraic K-groups to be proved.

2.2. Constructing directed graphs by gluing

Let's look at the simplest possible example, the problem of specifying a directed graph. Here is an example of one, where we have given names to the vertices, and names (not colors) to the edges.

graph1 (2.2.1)

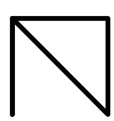
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[d, "h"] \\ Z & W \arrow[ul, "e"'] \end{tikzcd}$$

We can think of it as being obtained by gluing a collection of disjoint edges together by identifying various vertices.

$$\begin{tikzcd} X \arrow[d, "g"] & X \arrow[r, "f"] & Y & Y & & Y \arrow[d, "h"] \\ Z & & & & W \arrow[ul, "e"'] & W \end{tikzcd}$$

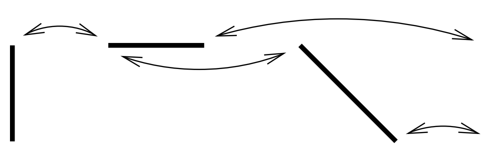
A topological space realizing the graph (2.2.1) geometrically would look like this.

graph1-space (2.2.2)



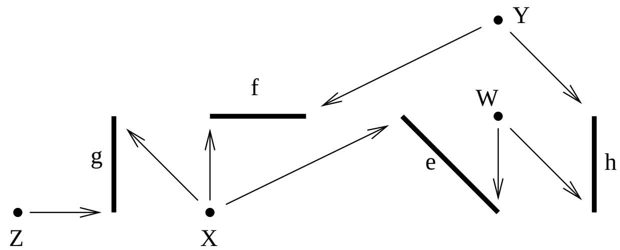
It could be obtained by gluing a collection of disjoint edges together at various endpoints.

(2.2.3)



The space obtained by identification of those vertices can be described as the colimit of the following diagram of spaces, each of which is a copy of $|\Delta^0|$ or $|\Delta^1|$. The arrows in the diagram are copies of one of the inclusions $|\Delta^0| \rightrightarrows |\Delta^1|$ that send the single point of $|\Delta^0|$ to one of the endpoints of $|\Delta^1|$.

graph3-diagram (2.2.4)



The same combinatorial data will suffice to construct either the abstract directed graph (2.2.1) or the space (2.2.2) that realizes it geometrically. The combinatorial data required consists of a set $V = \{X, Y, Z, W\}$ of vertices, a set $E = \{f, g, h, e\}$ of edges, and a pair of functions $\mathrm{src}, \mathrm{tar} : E \rightrightarrows V$ describing the incidence relations. The function src provides the source vertex for an edge, and tar provides the target vertex, so that in our example $X = \mathrm{src}\, g = \mathrm{src}\, f = \mathrm{tar}\, e$.

The alert reader has already noticed that we haven't defined the term *abstract directed graph*; a precise definition might be that an abstract directed graph *is* nothing more than a pair of functions $E \rightrightarrows V$. We are accustomed to attaching geometric significance to the elements of E and V by familiarity with illustrations or geometric realizations of abstract directed graphs, and we will learn to do the same with the simplices of an abstract simplicial set.

A diagram such as $E \rightrightarrows V$ in the category of sets is most conveniently encoded as a functor $\mathcal{I} \to \mathrm{Set}$ from an abstract index category $\mathcal{I} = \langle \bullet \rightrightarrows \bullet \rangle$. The objects and arrows of $\mathcal{I}$ may be regarded as names labeling the objects and arrows of the diagram. A less abstract choice of index category would be the category of spaces $\mathcal{O} = \langle |\Delta^0| \rightrightarrows |\Delta^1| \rangle$, where the arrows shown are the endpoint inclusion maps, because the arrows of $\mathcal{O}$ are the arrows in diagram (2.2.4). The logical choice for a functor G from $\mathcal{O}$ to Set, at least on objects, would be to set $G(|\Delta^0|) = V$ and $G(|\Delta^1|) = E$.

$$\begin{tikzcd}
|\Delta^0| \arrow[r, shift left] \arrow[r, shift right] \arrow[d, mapsto, dotted, "G"] & |\Delta^1| \arrow[d, mapsto, dotted, "G"] \\
V & E \arrow[l, shift left] \arrow[l, shift right]
\end{tikzcd}$$

Judging by the directions of the horizontal arrows in the diagram above, the functor G should be a contravariant one, i.e., it should be a functor $G : \mathcal{O}^{\mathrm{op}} \to \mathrm{Set}$. It might be startling at first that G should reverse the direction of the arrows, but that's the way it has to be, because each edge has a unique source endpoint and a unique target endpoint, but a vertex might be an endpoint of many edges.

Let's agree that an abstract directed graph is defined to be a functor $G : \mathcal{O}^{\mathrm{op}} \to \mathrm{Set}$. Given two abstract directed graphs G and G', a map $\eta : G \to G'$ should be a natural transformation. Unraveling the definitions, we see that such a map amounts to a pair of functions $\eta_0 : V \to V'$ and $\eta_1 : E \to E'$ between the vertex and edge sets of G and G', such that $\mathrm{src}(\eta_1(e)) = \eta_0(\mathrm{src}(e))$ and $\mathrm{tar}(\eta_1(e)) = \eta_0(\mathrm{tar}(e))$ for all edges $e \in E$ of G.

Recall that the category $\mathcal{O}$ currently contains the arrows of Top we wish to use in colimit constructions of spaces. What about other plausible choices for $\mathcal{O}$? A larger choice for $\mathcal{O}$ would presumably allow us to construct more interesting spaces.

For example, we may wish to include in $\mathcal{O}$ the reversal map $R : |\Delta^1| \to |\Delta^1|$ that flips the interval end for end. We would then be able to construct the colimit of the diagram $1, R : |\Delta^1| \rightrightarrows |\Delta^1|$, which is homeomorphic to $|\Delta^1|$ and thus doesn't provide us a new space.

A more interesting alternative would be to enlarge $\mathcal{O}$ by including the unique surjective map $|\Delta^1| \to |\Delta^0|$, together with the various composite arrows needed to make $\mathcal{O}$ a category. The reader can verify that now a functor $F : \mathcal{O}^{\mathrm{op}} \to \mathrm{Set}$ would be a new kind of directed graph where each vertex X is provided with a distinguished edge 1_X starting and ending at that vertex.

graph4 (2.2.5)
$$\begin{tikzcd} X \arrow[loop left, "1_X"] \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[loop right, "1_Y"] \arrow[d, "h"] \\ Z \arrow[loop left, "1_Z"] & W \arrow[loop right, "1_W"] \arrow[ul, "e"'] \end{tikzcd}$$

If we realize
$$\begin{tikzcd} A \arrow[loop left] \end{tikzcd}$$
geometrically as $|\Delta^0|$ and
$$\begin{tikzcd} A \arrow[loop left] \arrow[r] & B \arrow[loop right] \end{tikzcd}$$
as $|\Delta^1|$, and express the directed graph (2.2.5) as a colimit of a diagram involving graphs isomorphic to these two, then it would be realized geometrically as (2.2.2), which is a space we've seen before, but there are some new maps between such spaces that can be described combinatorially, such as those that collapse an edge to a vertex X by wrapping it around 1_X.

2.3. Presheaves and geometric realization

A *presheaf* on a small category $\mathcal{O}$ is a contravariant functor $X : \mathcal{O}^{\mathrm{op}} \to \mathrm{Set}$. The presheafs on $\mathcal{O}$ form a category we'll call $\mathrm{Pre}\,\mathcal{O}$, where the arrows are the natural transformations. In the previous section we discovered that presheaves on $\mathcal{O}$ have something to do with describing spaces that can be obtained as colimits of diagrams in $\mathcal{O}$. In this section we examine presheaves to see how they might be constructed and used.

Recall that one may define groups, abelian groups, rings, modules, and even sets (1.2.17) by generators and relations. Let's consider what it might mean to define a presheaf by generators and relations. The generators would end up being "elements" of the presheaf, whatever that means, and the "relations" would be accidental (or planned) coincidences (or equations) between expressions derivable from the generators by combining them with the available "operations" on the elements.

For the purposes of this discussion let's define an *element* of X to be a pair (A, x) where A is an object of $\mathcal{O}$ and $x \in X(A)$. We may also write simply $x \in X$ and trust the reader to remember that A must be specified as well. The only available operations on elements of X are those obtained by applying functions like $X(f)$, where $f : B \to A$ is an arrow of $\mathcal{O}$. For brevity of notation, let's define $xf = (X(f))(x)$, so that if $g : C \to B$ is another arrow, we may express the statement that X is a functor by the identity $(xf)g = x(fg)$, which looks pleasingly like a statement of associativity. From that identity it follows that applying several such operations to x is equivalent to applying one. We see that a set $\mathcal{G}$ of elements could be said to *generate* X if every element (B, y) has the form $y = xf$ for some $(A, x) \in \mathcal{G}$ and some arrow $f : B \to A$. The conceivable *relations* between two generators (A, x) and (A', x') could all be written in the form

xrelation
$$xf = x'f' \tag{2.3.1}$$

for some arrows $f : B \to A$ and $f' : B \to A'$ of $\mathcal{O}$.

Let's explore how to take a list of generators and relations and produce a presheaf X. Consider first the case where we have just one generator (A, x) and no relations. Here the symbol x is nothing more than our proposed name for the generator, and conveys no information about the structure of X, whereas A is a particular object of $\mathcal{O}$. For $B \in \mathcal{O}$ the elements of $X(B)$ would be formal expressions of the form xf, for any $f : B \to A$. Since there are no relations, those elements should all be different, and thus there would be a bijection $\mathrm{Hom}(B, A) \to X(B)$ defined by $f \mapsto xf$.

Let's discard the irrelevant symbol x and define a new presheaf called $\dot{A}$ to play the role of X by setting $\dot{A}(B) = \mathrm{Hom}_{\mathcal{O}}(B, A)$. (Since Hom is contravariant in the first variable, $\dot{A}$ is a presheaf.) If we wish to bring the symbol x back into play, we may write $X = x\dot{A} \xrightarrow{\cong} \dot{A}$.

The elements of $\dot{A}$ are the arrows of $\mathcal{O}$ whose target is A. The single generator of $\dot{A}$ is 1_{A}. In category theory we often identify an object with its identity arrow, so we may also say that $\dot{A}$ is generated by A.

A presheaf X given by generators and relations, if there are no relations imposed, ought to be *free*, which means that a map $\eta : X \to Y$ (where Y is another presheaf on $\mathcal{O}$) may be specified by telling where the generators ought to go. The statement that $\dot{A}$ is free is the content of the following lemma, because the single generator 1_{A} is sent by η to some element of $Y(A)$.

LEMMA 2.3.1 (Yoneda's lemma). *Let $\mathcal{O}$ be a small category, A an object of $\mathcal{O}$ and Y a presheaf on $\mathcal{O}$. There is a natural isomorphism* $\mathrm{Hom}(\dot{A}, Y) \cong Y(A)$ *defined by sending η to $\eta(1_{\mathrm{A}})$.*

PROOF. Given $y \in Y(A)$ a map η with $y = \eta(1_{\mathrm{A}})$ must be defined by setting $\eta(f) = yf$ for $f : B \to A$. Naturality of η amounts to the equation $(xf)g = x(fg)$ for $g : C \to B$. $\square$

COROLLARY 2.3.2. *The functor $h : \mathcal{O} \to \mathrm{Pre}\,\mathcal{O}$ defined by $A \mapsto \dot{A}$ is a fully faithful embedding. Morover, if $h(A) \cong h(B)$, then $A \cong B$.*

PROOF. For an arrow $f : B \to A$ in $\mathcal{O}$, we see that $\eta = h(f) : \dot{B} \to \dot{A}$ is the map defined by $g \mapsto fg$. The Yoneda isomorphism $\mathrm{Hom}(\dot{B}, \dot{A}) \cong \dot{A}(B) = \mathrm{Hom}(B, A)$ applied to η yields $\eta(1_{\mathrm{B}}) = f\,1_{\mathrm{B}} = f$, thus providing an inverse for $h : \mathrm{Hom}(B, A) \to \mathrm{Hom}(\dot{B}, \dot{A})$.

The second statement is an easy consequence of the first, applied to the four possible Hom-sets involving the two objects A and B. $\square$

The Yoneda embedding h is so natural that it is tempting to identify A with $\dot{A}$ and to write $\mathrm{Hom}(A, Y) = Y(A)$. In algebraic geometry this usage is especially common, where $\mathcal{O}$ is a small category of schemes or varieties. Fortunately, eliding the small dot is painless.

A presheaf isomorphic to $\dot{A}$ for some $A \in \mathcal{O}$ is called *representable*.

Now let's address the question of presenting an arbitrary presheaf X by generators and relations. It is logical to take all the elements of X as the generators and all the equations as the relations. An equation of the form $xf = x'f'$, as envisioned in (2.3.1), is a consequence of the equations $xf = y$ and $y = x'f'$ if y is defined to be xf, so relations of the form $xf = y$ are the only ones needed. The relation $xf = y$ is equivalent to commutativity of the following diagram.

$$\begin{tikzcd} B \arrow[d, "f"] & & \dot{B} \arrow[dl, "y"'] \arrow[d, "\dot{f}"] \\ A & X & \dot{A} \arrow[l, "x"] \end{tikzcd} \tag{2.3.2}$$

Commutativity of all such triangles would mean that X receives a map from the colimit of a certain diagram that has one object for each element of X. We construct that diagram now.

DEFINITION 2.3.3. Given a presheaf X on a small category $\mathcal{O}$, the category $\mathrm{Elem}(X)$ is the category whose objects are the elements (A, x) of X, with $A \in \mathcal{O}$ and $x \in X(A)$. An arrow $(B, y) \to (A, x)$ is an arrow $f : B \to A$ such that $xf = y$.

We have practically proved the following lemma.

LEMMA 2.3.4. *Given a presheaf X on a small category $\mathcal{O}$, let $F : \mathrm{Elem}(X) \to \mathrm{Pre}\,\mathcal{O}$ be the functor defined by $(A, x) \mapsto \dot{A}$, and let $\eta_{(A,x)} : \dot{A} \to X$ be the arrow defined by x. The resulting map $\eta : \mathrm{colim}\, F \to X$ is an isomorphism.*

PROOF. This follows mostly from the discussion above. The map is surjective because we used every element of X as a generator. The map is injective because we used all the equations as relations.

Here is a second proof. Letting Y be another presheaf on $\mathcal{O}$ we observe that $\mathrm{Hom}(X, Y) \cong \lim_{x:\dot{A}\to X} Y(A) \cong \lim_{x:\dot{A}\to X} \mathrm{Hom}(\dot{A}, Y) \cong \mathrm{Hom}(\mathrm{colim}_{x:\dot{A}\to X} \dot{A}, Y)$. These natural isomorphisms show that the covariant functors on $\mathrm{Pre}\,\mathcal{O}$ represented by X and $\mathrm{colim}_{x:\dot{A}\to X} \dot{A}$ are isomorphic, so by Yoneda's lemma, so are X and $\mathrm{colim}_{x:\dot{A}\to X} \dot{A}$. $\square$

COROLLARY 2.3.5. *Every presheaf on a small category $\mathcal{O}$ is an inductive limit of representable presheaves.*

THEOREM 2.3.6. *The category $\mathrm{Pre}\,\mathcal{O}$ is cocomplete. Suppose $G : \mathcal{O} \to \mathcal{C}$ is a functor to a cocomplete category $\mathcal{C}$. Then there is a functor $R : \mathrm{Pre}\,\mathcal{O} \to \mathcal{C}$ that preserves colimits, unique up to natural isomorphism, which makes the diagram*

$$\begin{tikzcd} \mathcal{O} \arrow[r, "h"] \arrow[dr, "G"'] & \mathrm{Pre}\,\mathcal{O} \arrow[d, "R"] \\ & \mathcal{C} \end{tikzcd}$$

commute up to natural isomorphism.

PROOF. Given an inductive system X_i of presheaves, we may construct $X = \mathrm{colim}_i X_i$ by defining $X(A) = \mathrm{colim}_i X_i(A)$ for each $A \in \mathcal{O}$. The universal property of colimits is used to define X on arrows, and to check that X is the colimit. Hence $\mathrm{Pre}\,\mathcal{O}$ is cocomplete.

Since R is required to preserve colimits, and every presheaf X on $\mathcal{O}$ is an inductive limit of a diagram that comes from $\mathcal{O}$ via h, our only choice is to define R on objects by $X \mapsto \operatorname{colim}_{x:\dot{A}\to X} G(A)$, where the colimit is indexed by the category $\operatorname{Elem}(X)$, as above. Functoriality of colimits (see Lemma A.31) is used to define R on arrows; a map $X \to Y$ of presheaves induces a functor $\operatorname{Elem}(X) \to \operatorname{Elem}(Y)$ on indexing categories, and one takes the natural transformation required from identity maps $1 : G(A) \to G(A)$.

Define a functor $S : \mathcal{C} \to \operatorname{Pre}\mathcal{O}$ by $C \mapsto (A \mapsto \operatorname{Hom}(G(A), C))$. We see that S is a right adjoint for R from the following chain of natural isomorphisms:

$$\begin{aligned}
\operatorname{Hom}(R(X), C) &= \operatorname{Hom}(\operatorname*{colim}_{x:\dot{A}\to X} G(A), C) \\
&\cong \lim_{x:\dot{A}\to X} \operatorname{Hom}(G(A), C) \\
&= \lim_{x:\dot{A}\to X} (S(C))(A) \\
&\cong \lim_{x:\dot{A}\to X} \operatorname{Hom}(\dot{A}, S(C)) \\
&\cong \operatorname{Hom}(\operatorname*{colim}_{x:\dot{A}\to X} \dot{A}, S(C)) \\
&\cong \operatorname{Hom}(X, S(C))
\end{aligned}$$

Any left adjoint preserves colimits (see A.45), and hence R does. □

We apply theorem 2.3.6 in the following examples.

EXAMPLE 2.3.7. Let T be a topological space and let $\mathcal{O}$ be the category of open subsets U of T. The arrows are the inclusions $U \hookrightarrow U'$ of one subset into a larger one. The presheaves on $\mathcal{O}$ are called *presheaves on T*, and the theorem extends the forgetful functor $\mathcal{O} \to \mathrm{Top}$ defined by $U \mapsto U$ to a functor $\operatorname{Pre}\mathcal{O} \to \mathrm{Top}$ that forms the *étalé space* S associated to a presheaf F. The space S comes with a map $S \to T$ which is a local homeomorphism, and its presheaf of sections is the *sheaf associated to F*.

EXAMPLE 2.3.8. Let Simp be the subcategory of Top whose objects are the standard simplices $|\Delta^k|$ for $k \geq 0$, and whose arrows are the affine maps which send vertices to vertices and preserve the ordering of the vertices. Let $\mathcal{O} := \mathrm{Simp}$. Theorem 2.3.6 extends the inclusion $G : \mathrm{Simp} \hookrightarrow \mathrm{Top}$ to a functor $\operatorname{Pre}\mathrm{Simp} \to \mathrm{Top}$.

Now we replace Simp by an equivalent category which is completely combinatorial.

EXAMPLE 2.3.9. Define Ord to be the category of finite nonempty ordered sets of the form $\underline{n} := \{0 < 1 < \cdots < n\}$ and let $\mathcal{O} := \mathrm{Ord}$. The arrows are the functions $f : \underline{m} \to \underline{n}$ that preserve the ordering, i.e., $i \leq j \implies f(i) \leq f(j)$. The functor $\Delta : \mathrm{Ord} \to \mathrm{Simp}$ defined on objects by $n \mapsto |\Delta^n|$ and on arrows by $f \mapsto \left(\sum a_i e_i \mapsto \sum a_i e_{f(i)}\right)$ is an equivalence of categories. Theorem 2.3.6 extends that functor to a functor $\operatorname{Pre}\mathrm{Ord} \to \mathrm{Top}$ called *geometric realization* that preserves colimits. An object X of $\operatorname{Pre}\mathrm{Ord}$ is called a *simplicial set*, we'll write $|X|$ for its geometric realization, and we let $\mathrm{SSet} := \operatorname{Pre}\mathrm{Ord}$ be the category of simplicial sets. We point out that $|\dot{\underline{n}}| \cong |\Delta^n|$, so we may introduce the notation $\Delta^n := \dot{\underline{n}}$ for the representable simplicial sets, as an abuse of notation. We'll write $X_n := X(\underline{n})$ and refer to an element of X_n as an n-simplex of X.

Here is an illustration of $\underline{2}$. It looks like a triangle, and the geometric realization of the simplicial set Δ^2 it represents *is* a triangle, by definition.

$$\begin{tikzcd} 0 \arrow[r] \arrow[dr] & 1 \arrow[d] \\ & 2 \end{tikzcd}$$

REMARK 2.3.10. We can easily write the underlying set of $|X|$ as a set with generators and relations, because the forgetful functor Top $\to$ Set preserves colimits, and any colimit of a diagram of sets can be written in terms of generators and relations. The generators are pairs $[x, a]$ where x is an n-simplex of X and $a \in |\Delta^n|$. (The element x is the index for an object $|\Delta^n|$ in the diagram whose colimit is $|X|$, and the corresponding map $|\Delta^n| \to |X|$ sends $a \in |\Delta^n|$ to $[x, a] \in |X|$.) The relations are those of the form $[x, a] = [y, b]$ where y is an m-simplex of X and $a \in |\Delta^m|$, $f : \underline{m} \to \underline{n}$, $y = xf$, and $a = fb := (\Delta(f))(b)$. The equations amount to the commutativity of the following diagram.

(2.3.3)
$$\begin{tikzcd} & \underline{\dot{m}} \arrow[dl, "y"'] \arrow[d, "\dot{f}"] \\ X & \underline{\dot{n}} \arrow[l, "x"] \end{tikzcd} \qquad \begin{tikzcd} \underline{m} \arrow[d, "f"] \\ \underline{n} \end{tikzcd} \qquad \begin{tikzcd} & {|\Delta^m|} \arrow[dl, "y"'] \arrow[d, "f"] & {*} \arrow[l, "b"'] \arrow[dl, "a"] \\ {|X|} & {|\Delta^n|} \arrow[l, "x"] \end{tikzcd}$$

The relation can be written in the form $[x, fb] = [xf, b]$, which is reminiscent of the defining relation in the tensor product of two modules. For more information about that analogy, see the section on "coends" in [8, IX.6].

Simplicial sets will be our main objects of study in this book. As we have seen, any space obtained by gluing simplices using arrows from Simp can also be obtained as the geometric realization of a simplicial set. For example, if $T = \operatorname{colim} F$ where $F : \mathcal{I} \to$ Simp is a functor from a small index category $\mathcal{I}$, then we can define $X := \operatorname{colim}(h \circ \Delta^{-1} \circ F) \in$ SSet to obtain a simplicial set with $|X| \cong T$.

For example, the circle, as a topological space, can be obtained as the coequalizer of the pair of maps $|\Delta^0| \rightrightarrows |\Delta^1|$ that send the point of $|\Delta^0|$ to the endpoints of $|\Delta^1|$. The coequalizer S^1 of the analogous diagram $\Delta^0 \rightrightarrows \Delta^1$ is a simplicial set whose geometric realization $|S|^1$ is homeomorphic to a circle. The n-simplices of S^1 are the maps $\underline{n} \to \underline{1}$ with the two constant maps identified; thus $\#S^1_n = n + 1$.

The sphere of dimension k can be constructed as a geometric realization in a similar way, by collapsing the boundary of Δ^n; the n-simplices would be the maps $\underline{n} \to \underline{k}$ with the nonsurjective maps identified. Alternatively, one could collapse the boundary of $(\Delta^1)^k$; the n-simplices would be the maps $\underline{n} \to \underline{1}^k$ with those not surjective on each coordinate identified.

DEFINITION 2.3.11. Suppose X and Y are simplicial sets. The product $X \times Y$ is the simplicial set defined by $\underline{n} \mapsto X(\underline{n}) \times Y(\underline{n})$. It serves as the product in the category SSet.

DEFINITION 2.3.12. Suppose S is a set and X is a simplicial set. The product $S \times X$ is the simplicial set defined by $\underline{n} \mapsto S \times X(\underline{n})$.

DEFINITION 2.3.13. Suppose S is a set. The *constant* simplicial set corresponding to S is $S \times \Delta^0$. An isomorphic simplicial set can be defined by $\underline{n} \to S$.

REMARK 2.3.14. Any small category equivalent to Ord would serve just as well as Ord in the definition of simplicial set. For example, in the direction of realism, we could use Simp. Or in the direction of versatility we could pick a big set U to serve as our *universe*, and let Ord be the category of finite ordered nonempty sets whose underlying sets are subsets of U. Such a choice might make more natural certain operations on ordered sets, such as taking the image of a map, or concatenation of two ordered sets into one.

EXAMPLE 2.3.15. Let $\mathcal{O} := \mathrm{Ord}^k$, and consider the functor $G : \mathrm{Ord}^k \to \mathrm{Top}$ defined by $(\underline{m}_1, \dots, \underline{m}_k) \mapsto |\Delta^{m_1}| \times \cdots \times |\Delta^{m_k}|$. An object X of $\mathrm{Pre}\,(\mathrm{Ord}^k)$ is called a *k-fold multisimplicial set*, and we let $\mathrm{SSet}_k := \mathrm{Pre}\,(\mathrm{Ord}^k)$ be the category of simplicial sets. (A 2-fold multisimplicial set is called a *bisimplicial set*.) Theorem 2.3.6 extends G to a functor $\mathrm{Pre}\,(\mathrm{Ord}^k) \to \mathrm{Top}$ (also) called *geometric realization* that preserves colimits. We'll write $|X|$ for the geometric realization of X; it is a space obtained by gluing k-fold products of simplices. Let $\Delta^{m_1,\dots,m_k}$ denote the presheaf on Ord^k represented by $(\underline{m}_1, \dots, \underline{m}_k)$, so that $|\Delta^{m_1,\dots,m_k}| \cong |\Delta^{m_1}| \times \cdots \times |\Delta^{m_k}|$. We'll write $X_{m_1,\dots,m_k} := X(\underline{m}_1, \dots, \underline{m}_k)$ and refer to an element of $X_{m_1,\dots,m_k}$ as an $(m_1, \dots, m_k)$-simplex of X.

2.4. Geometric realization of partially ordered sets

Suppose T is a partially ordered set. We introduce a simplicial set NT, called the *nerve* of T, defined by $\underline{n} \mapsto \mathrm{Hom}_{\mathrm{Poset}}(\underline{n}, T)$. An n-simplex of NT is a chain $t_0 \le \cdots \le t_n \in T$ of length n. The functor $N : \mathrm{Poset} \to \mathrm{SSet}$ is a fully faithful embedding, so we may often omit the N. The geometric realization of a partially ordered set is defined by $|T| := |NT|$. Evidently, if T' is another partially ordered set, $N(T \times T') = NT \times NT'$.

Another plausible choice of coordinates on $|\Delta^k|$ is obtained by considering the partial sums $s_j = \sum_{i=0}^{j-1} a_i$, yielding the following alternative representation of the standard simplex.

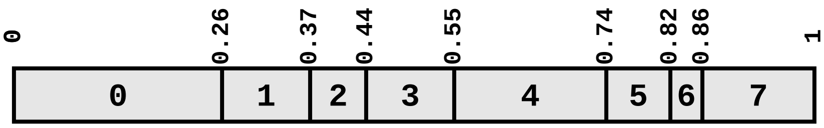
simop

$$|\Delta^k| \cong D_k := \left\{ (s_0, \dots, s_{k+1}) \in [0,1]^{k+2} \mid 0 = s_0 \le s_1 \le \cdots \le s_{k+1} = 1 \right\}. \tag{2.4.1}$$

We can get the back to the original coordinates with the formula $a_i = s_{i+1} - s_i$.

REMARK 2.4.1. There is another way to present the elements of D_k, which makes the maps between the simplices easy to understand. From an element $s \in D_k$, construct a function $t : [0,1] \to \underline{k}$, defined except at a finite number of points, by letting $t(x) = i$ if $s_i < x < s_{i+1}$. Filling in the remaining values in some arbitrary but order-preserving way, we agree to regard two functions $[0,1] \rightrightarrows \underline{k}$ as equivalent if they differ at only a finite number of points in $[0,1]$. Let's use the notation $[0,1]^* \to \underline{k}$ to depict such an equivalence class of order preserving functions $[0,1] \to \underline{k}$. Thus we have a bijection $D_k \cong \|\underline{k}\|$, where $\|\underline{k}\|$ is the set of maps $[0,1]^* \to \underline{k}$. Evidently, composition is well defined and makes $\underline{k} \mapsto \|\underline{k}\|$ into a functor $\mathrm{Ord} \to \mathrm{Top}$.

We may illustrate a map $t : [0,1]^* \to \underline{k}$ by drawing a rectangle of length 1 to represent the unit interval $[0,1]$ and by labelling each subinterval $t^{-1}(i)$ with i, as in the following figure, where $s = (0, .26, .37, .44, .55, .74, .82, .86, 1)$.



REMARK 2.4.2. We can make the use of the symbol $[0,1]^*$ in Remark 2.4.1 totally legitimate by constructing a new category Poset* to hold it and the “arrows” we used. An object will be a pair consisting of a partially ordered set T and a family of *essential* subsets of T. The complement of an essential subset will be called *negligible*. We require a finite intersection of essential subsets to be essential, and any subset containing an essential set to be essential. Consideration of the empty intersection shows that T is essential. Two functions $T \rightrightarrows U$ will be called *essentially* equal if they agree on the elements of some essential subset of T. Essential equality is is an equivalence relation. An arrow $T \to U$ in the new category will be an essential equality class of order preserving maps $p : T \to U$ such that the preimage $p^{-1}(G)$ of any essential subset $G \subseteq U$ is essential. It is easy to check that composition is well defined. We embed Poset into Poset* by declaring T to be the only essential subset of T; when necessary, we identity T with its image in Poset*. For a partially ordered set T we define an object $T^* \in$ Poset* to be T with the finite subsets as the negligible subsets.

EXERCISE 2.4.3. Check that the isomorphism $|\Delta^k| \cong \|\underline{k}\|$ is a natural isomorphism of functors Ord $\rightrightarrows$ Top.

REMARK 2.4.4. If T is a partially ordered set, then by combining Remark 2.3.10 and Remark 2.4.1, we see that the set underlying $|T|$ may be defined by generators $[x, a]$ where $x : \underline{n} \to T$ and $a : [0,1]^* \to \underline{n}$, and by relations $[x, fb] = [xf, b]$, as illustrated in the following diagram.

$$
\begin{tikzcd}
 & \underline{m} \arrow[dl, "y"'] \arrow[d, "f"] & {[0,1]^*} \arrow[l, "b"'] \arrow[dl, "a"] \\
T & \underline{n} \arrow[l, "x"] &
\end{tikzcd}
\tag{2.4.2}
$$

Now let $\|T\|$ be the set of equivalence classes of order-preserving functions $[0,1] \to T$ with finite image, where two functions are declared to be equivalent if they differ at only a finite number of points of $[0,1]$.[1] As before, let's depict such an equivalence class as a map $[0,1]^* \to T$. The map $\eta : |T| \to \|T\|$ defined by $[x, a] \mapsto xa$ is well-defined, because $\eta[x, fb] = xfb = \eta[xf, b]$.

Define the *image* of a map $[0,1]^* \to T$ to be the intersection of the images of the functions in the equivalence class. Alternatively, the image consists of those elements of T whose preimage under any function in the equivalence class is infinite. We say that a map $[0,1]^* \to T$ is surjective if its image is T.

The map η is surjective because the image of any map $[0,1]^* \to T$ is isomorphic to $\underline{n}$ for some n.

[1] If we wanted to give $\|T\|$ a topology, we would use the weak topology for the union $\|T\| = \bigcup_U \|U\|$, where U runs over the finite chains $U \subseteq T$, giving each $\|U\|$ the topology of the standard simplex.

DEFINITION 2.4.5. We say that a simplex $x \in T_n$ is *nondegenerate* if the function $x : \underline{n} \to T$ is injective.

DEFINITION 2.4.6. We say that a point $a \in |\Delta^n|$ is an interior point if all of its coordinates a_i are nonzero. That is equivalent to requiring that the corresponding map $[0,1]^* \to \underline{n}$ is surjective.

DEFINITION 2.4.7. We say that a generator $[x, a]$ is in *normal form* if $x : \underline{n} \to T$ is nondegenerate and a is an *interior point* of $|\Delta^n|$.

If $[x, a]$ is in normal form, then x and a are determined by $\eta[x, a]$, so injectivity of η can be established by showing that every generator can be reduced to one in normal form using the relations. Let's proceed to do that, starting with a generator $[x, a]$. First, if a is not an interior point, we may select an injective map $f : \underline{m} \to \underline{n}$ whose image is $\{i \in \underline{n} \mid a_i > 0\}$ and a point $b \in |\Delta^m|$ with $fb = a$; the point b is obtained from a by removing the coordinates that are zero. Using one of the relations we see that $[x, a] = [x, fb] = [xf, b] = [y, b]$. Now consider $y : \underline{m} \to T$. If y is not injective, it factors through its image as $y = zg$, where $g : \underline{m} \twoheadrightarrow \underline{p}$ is a surjection and $z : \underline{p} \rightarrowtail T$ is injective. Letting $c = gb$ we use another relation to see that $[y, b] = [zg, b] = [z, gb] = [z, c]$. The point c is an interior point because b was an interior point and g is surjective, so $[z, c]$ is in normal form. This is illustrated in the following commutative diagram, which clarifies how the image $\underline{p}$ of xa relates to x and a.

(2.4.3)
$$\begin{tikzcd}
 & \underline{p} \arrow[ddl, tail, "z"'] & {[0,1]^*} \arrow[l, two heads, "c"'] \arrow[dl, "b"] \arrow[ddl, "a"] \\
 & \underline{m} \arrow[u, two heads, "g"] \arrow[dl, "y"] \arrow[d, tail, "f"] & \\
T & \underline{n} \arrow[l, "x"] &
\end{tikzcd}$$

We have shown that $\eta : |T| \xrightarrow{\cong} \|T\|$ is a bijection.

An alternative approach to this result would have been to show that the index category $\mathrm{Elem}(NT)$ in the definition of $|T|$ can be replaced by the subcategory of nondegenerate simplices of T. One essentially replaces a simplex $\underline{n} \to T$ by its image, which is a functorial operation. Another way to phrase that is that we could replace Ord by its subcategory of injective arrows. Thus $|T|$ is a colimit of a diagram of standard simplices involving only injective maps.

LEMMA 2.4.8. *Suppose T and U are partially ordered sets. Then the natural map $|T \times U| \to |T| \times |U|$ is a bijection.*

PROOF. It is enough to show that the natural map $\|T \times U\| \to \|T\| \times \|U\|$ is a bijection, in other words, that the natural map

$$\mathrm{Hom}([0,1]^*, T \times U) \to \mathrm{Hom}([0,1]^*, T) \times \mathrm{Hom}([0,1]^*, U)$$

is a bijection. But this follows from the universal property of products in the category of partially ordered sets, together with the observation that two maps $[0,1] \rightrightarrows T \times U$ differ at only a finite number of points if and only if the projections $[0,1] \rightrightarrows T$ and $[0,1] \rightrightarrows U$ both do. $\square$

Similar proofs of 2.4.8 can be found in [4] and in [2].

The bijection of the lemma is not always a homeomorphism. We'll treat that issue in detail in section 2.5.

EXERCISE 2.4.9. Let Int be the category of finite ordered *endpointed* sets of the form $\langle n\rangle := \{-\infty < 1 < 2 < \cdots < n < \infty\}$ for $n \geq 0$. The arrows are the functions that preserve the ordering, send $-\infty$ to $-\infty$, and send ∞ to ∞. For example, there are exactly two arrows $\langle 1\rangle \to \langle 0\rangle$, one that sends 1 to ∞ and one that sends 1 to $-\infty$. Produce an isomorphism of categories $\mathrm{Int}^{\mathrm{op}} \cong \mathrm{Ord}$. Invent some simplicial sets by finding some interesting functors $\mathrm{Int} \to \mathrm{Set}$.

2.5. Geometric realization of products

Our goal in this section is to investigate compatibility of geometric realization of simplicial sets with products. Since every simplicial set is a colimit of a diagram consisting of representable simplicial sets, we investigate them first. We begin by introducing some general facts about the compatibility of products with colimits of topological spaces.

LEMMA 2.5.1. [3, 3.5.8] *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

COROLLARY 2.5.2. *The natural map $|\Delta^m \times \Delta^n| \to |\Delta^m| \times |\Delta^n|$ is a homeomorphism.*

PROOF. By Lemma 2.4.8 the map is a bijection. The partially ordered set $\underline{m}\times\underline{n}$ is finite, so $\Delta^m \times \Delta^n$ has only a finite number number of nondegenerate simplices, and thus $|\Delta^m \times \Delta^n|$ is a finite union of compact subsets, hence is compact. Now apply Lemma 2.5.1. $\square$

LEMMA 2.5.3. [3, 3.6.1] *If B is a compact Hausdorff space, then it is locally compact.*

LEMMA 2.5.4. [3, 4.3.2] *If X and Y are topological spaces, $f : X \to Y$ is an identification map, and B is a locally compact space, the $f \times 1_{\mathrm{B}} : X \times B \to Y \times B$ is an identification map.*

COROLLARY 2.5.5. *If B is a locally compact space, then the functor $X \mapsto X \times B$ preserves colimits.*

PROOF. It preserves coproducts, and by Lemma 2.5.4 it preserves coequalizers. Since colimits can be expressed in terms of coequalizers and coproducts, it preserves them, too. $\square$

LEMMA 2.5.6. *If X is a simplicial set, and $m \geq 0$, then the natural map $|X \times \Delta^m| \to |X| \times |\Delta^m|$ is a homeomorphism.*

PROOF. As in Lemma 2.3.4 we write $X \cong \mathrm{colim}_{x:\Delta^n\to X}\, \Delta^n$ and use compatibility of products with colimits of sets and presheaves together with Lemmas 2.5.4 and 2.5.3 for the space $B = |\Delta^m| \cong |\Delta^m|$, which is a compact Hausdorff space. We also use Corollary 2.5.2.

$$\begin{aligned}
|X \times \Delta^m| &\cong |(\operatorname{colim} \Delta^n) \times \Delta^m| \\
&\cong |\operatorname{colim}(\Delta^n \times \Delta^m)| \\
&\cong \operatorname{colim} |\Delta^n \times \Delta^m| \\
&\xrightarrow{\cong} \operatorname{colim}(|\Delta^n| \times |\Delta^m|) \\
&\cong (\operatorname{colim} |\Delta^n|) \times |\Delta^m| \\
&\cong |X| \times |\Delta^m|
\end{aligned}$$

□

DEFINITION 2.5.7. A *simplicial homotopy* between two maps $f, g : X \rightrightarrows Y$ of simplicial sets is a map $h : X \times \Delta^1 \to Y$ such that the following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, "i_0"] \arrow[dr, "f"'] & X \times \Delta^1 \arrow[d, "h"] & X \arrow[l, "i_1"'] \arrow[dl, "g"] \\
& Y &
\end{tikzcd}$$

Here i_0 and i_1 are derived from the two maps $\Delta^0 \to \Delta^1$. The two maps are called *simplicially homotopic*, or just *homotopic*.

COROLLARY 2.5.8. *The geometric realizations of homotopic maps between simplicial sets are homotopic.*

PROOF. Apply lemma 2.5.6 to the geometric realization of the diagram in the definition above, and we get the following diagram, hence the result.

$$\begin{tikzcd}
|X| \arrow[r, "i_0"] \arrow[dr, "|f|"'] & |X| \times |\Delta^1| \arrow[d, "|h|"] & |X| \arrow[l, "i_1"'] \arrow[dl, "|g|"] \\
& |Y| &
\end{tikzcd}$$

□

LEMMA 2.5.9. *Suppose X and Y are simplicial sets, and $|Y|$ is locally compact. Then the natural map $|X \times Y| \to |X| \times |Y|$ is a homeomorphism.*

PROOF. We use Lemma 2.5.6 and argue as in its proof.

$$\begin{aligned}
|X \times Y| &\cong |(\operatorname{colim} \Delta^n) \times Y| \\
&\cong |\operatorname{colim}(\Delta^n \times Y)| \\
&\cong \operatorname{colim} |\Delta^n \times Y| \\
&\xrightarrow{\cong} \operatorname{colim}(|\Delta^n| \times |Y|) \\
&\cong (\operatorname{colim} |\Delta^n|) \times |Y| \\
&\cong |X| \times |Y|
\end{aligned}$$

□

2.6. Geometric realization of categories

Suppose T is a partially ordered set. We can regard it as a category whose objects are the elements of T and whose arrows $i \to j$ are the pairs (i, j) such that $i \le j$. Thus $\mathrm{Hom}_T(i, j)$ has one element if $i \le j$, and otherwise has no elements. This conversion procedure yields a fully faithful functor Poset $\to$ Cat, which is compatible with products. Applying this procedure to $\underline{n}$ yields a category some of whose arrows are $0 \to 1 \to 2 \to \cdots \to n$.

Now suppose $\mathcal{C}$ is a small category. Interpreting ordered sets as categories as above, we introduce a simplicial set $N\mathcal{C}$, called the *nerve* of $\mathcal{C}$, defined by $\underline{n} \mapsto \mathrm{Hom}_{\mathrm{Cat}}(\underline{n}, \mathcal{C})$. Thus an n-simplex of $N\mathcal{C}$ is a chain of arrows $C_0 \to C_1 \to \cdots \to C_n$. The functor $N : \mathrm{Cat} \to \mathrm{SSet}$ is a fully faithful embedding, so we may often omit the N. It also preserves products. The *geometric realization* (or *classifying space*) of a small category is defined by $B\mathcal{C} := |\mathcal{C}| := |N\mathcal{C}|$.

EXERCISE 2.6.1. Let $\mathbb{L}^n \subset \Delta^n$ be the union of the edges joining vertices whose numbers differ by 1. Show that a simplicial set X is isomorphic to the nerve of a category $\mathcal{C}$ if and only if for every n the restriction map $\mathrm{Hom}(\mathbb{L}^n, X) \to \mathrm{Hom}(\Delta^n, X)$ is an isomorphism.

PROPOSITION 2.6.2. *The geometric realization of a natural transformation η : $F \to G$ between two functors $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ provides a homotopy $|F| \sim |G|$.*

PROOF. Interpreting the ordered set $\underline{1}$ as the category $0 \to 1$ allows us to regard η as a functor $\mathcal{C} \times \underline{1} \to \mathcal{D}$ which makes the following diagram commute.

$$\begin{tikzcd}
\mathcal{C} \arrow[dr, "0"'] \arrow[drr, "F"] & & \\
& \mathcal{C} \times \underline{1} \arrow[r] & \mathcal{D} \\
\mathcal{C} \arrow[ur, "1"] \arrow[urr, "G"'] & &
\end{tikzcd}$$

Applying geometric realization and using the homeomorphisms $|\mathcal{C} \times \underline{1}| \cong |\mathcal{C}| \times |\underline{1}| \cong |\mathcal{C}| \times |\Delta^1| \cong |\mathcal{C}| \times [0, 1]$ (see Lemma 2.5.6) we get the following diagram, which provides the desired homotopy.

$$\begin{tikzcd}
|\mathcal{C}| \arrow[dr, "0"'] \arrow[drr, "|F|"] & & \\
& |\mathcal{C}| \times [0,1] \arrow[r] & |\mathcal{D}| \\
|\mathcal{C}| \arrow[ur, "1"] \arrow[urr, "|G|"'] & &
\end{tikzcd}$$

□

PROPOSITION 2.6.3. *If $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are adjoint functors between a pair of small categories, then their geometric realizations $|F| : |\mathcal{C}| \to |\mathcal{D}|$ and $|G| : |\mathcal{D}| \to |\mathcal{C}|$ are inverse homotopy equivalences.*

PROOF. Apply Proposition 2.6.2 to the unit and the counit of the adjunction, defined in Remark A.42. □

COROLLARY 2.6.4. *If $\mathcal{C}$ is a category with an initial object or a final object, then $|\mathcal{C}|$ is contractible.*

PROOF. Say C is an initial object of $\mathcal{C}$. Let $\mathcal{D}$ be a category with one object $*$ and one arrow, let $G : \mathcal{D} \to \mathcal{C}$ be the functor which sends $*$ to the initial object, and let $F : \mathcal{C} \to \mathcal{D}$ be the constant map. Observe that G is a left adjoint for F, and apply Proposition 2.6.3. □

2.7. Geometric realization of products, continued

To go further, we need two things: a way to tell when the geometric realization of a simplicial set is locally compact; and the category of compactly generated Hausdorff spaces, to deal with the case where neither factor of the product is locally compact.

First we show that the geometric realization of a simplicial set is always Hausdorff. For this we need a normal form for points of $|X|$ analogous to the one developed for points of geometric realizations of partially ordered sets in Definition 2.4.7.

DEFINITION 2.7.1. Let X be a simplicial set. A simplex $x \in X_m$ is a *degeneracy* of an element $y \in X_n$ if there is a surjective map $f : \underline{m} \twoheadrightarrow \underline{n}$ such that $x = yf$. The map $X_n \to X_m$ induced by a surjective map f is called a *degeneracy map*. The simplex $x \in X_m$ is called *degenerate* if it is a degeneracy of an element $y \in X_n$ with $n < m$. Otherwise, it is called *nondegenerate*. The simplex $x \in X_m$ is called *totally degenerate* if it is a degeneracy of an element $y \in X_0$.

DEFINITION 2.7.2. Let X be a simplicial set. A simplex $x \in X_m$ is a *face* of an element $y \in X_n$ if there is a injective map $f : \underline{m} \twoheadrightarrow \underline{n}$ such that $x = yf$. The map $X_n \to X_m$ induced by an injective map f is called a *face map*.

WARNING 2.7.3. It may happen that a simplex $x \in X_n$ can be nondegenerate but the representing map $x : \Delta^n \to X$ is not injective.

EXERCISE 2.7.4. Let X be a simplicial set. Show that a simplex $x \in X_n$ can be written as a degeneracy $x = yf$ of a unique nondegenerate simplex $y \in X_m$, for some $m \leq n$, using a unique surjective map f.

The fact stated in the exercise above is easy if X is the nerve of a partially ordered set, and we've already used it. It's also easy if X is the nerve of a category: a chain of arrows $C_0 \to C_1 \to \cdots \to C_n$ is degenerate if it has any identity arrows, so just eliminate them by passing to the quotient of $\underline{n}$ that identifies the source and target of each identity arrow.

DEFINITION 2.7.5. Let X be a simplicial set. For each $n \geq 0$, let $\operatorname{sk}_n X$ be the subsimplicial set of X generated by the k-simplices of X for $k \leq n$.

Evidently, $X \mapsto \operatorname{sk}_n X$ is a functor SSet $\to$ SSet.

The simplicial set $\operatorname{sk}_{n-1} \Delta^n$ sends the object $\underline{p}$ to the set of nonsurjective order preserving functions $\underline{p} \to \underline{n}$. The image of each such function is contained in a proper ordered subset T of $\underline{n}$, and perhaps in several. According to lemma A.33, we can write $\operatorname{sk}_{n-1} \Delta^n = \operatorname{colim}_{T \subsetneq \underline{n}} NT$, where T runs over the proper subsets of $\underline{n}$ ordered by inclusion, because the intersection of two such subsets is again such a subset. By compatibility of geometric realization with inductive limits, we see that $|\mathrm{sk}_{n-1} \Delta^n| \cong \mathrm{colim}_{T \subsetneq \underline{n}} |T|$, and according to lemma A.33, we see that the map $\mathrm{colim}_{T \subsetneq \underline{n}} |T| \to \bigcup_{T \subsetneq \underline{n}} |T|$ is a bijection, and thus by Lemma 2.5.1, is a homeomorphism (we give the union the subspace topology). The union is the boundary $\partial|\Delta^n|$ of $|\Delta^n|$. so we have proved that $|\mathrm{sk}_{n-1} \Delta^n| \cong \partial|\Delta^n|$. This justifies introducing the notation $\partial\Delta^n := \mathrm{sk}_{n-1} \Delta^n$.

LEMMA 2.7.6. *If X is a simplicial set, then the natural map* $\mathrm{colim}_n |\mathrm{sk}_n X| \to |X|$ *is a homeomorphism.*

PROOF. The map $\mathrm{colim}_n \mathrm{sk}_n X \to X$ is an isomorphism, because the colimit is just the union in this case. Since geometric realization commutes with colimits, the result follows. □

PROPOSITION 2.7.7. *Let X be a simplicial set, and suppose $n \geq 0$. Let X_n^{nd} be the set of nondegenerate n-simplices $x \in X_n$. Define a map $X_n^{nd} \times \Delta^n \to X$ by $(x, f) \mapsto xf$ and consider also the induced map $X_n^{nd} \times \partial\Delta^n \to \mathrm{sk}_{n-1} X$. The following square is a pushout square.*

$$\begin{array}{ccc} X_n^{nd} \times \partial\Delta^n & \longrightarrow & \mathrm{sk}_{n-1} X \\ \downarrow & & \downarrow \\ X_n^{nd} \times \Delta^n & \longrightarrow & \mathrm{sk}_n X \end{array}$$

PROOF. Let P denote the pushout and consider the map $P \to \mathrm{sk}_n X$; we'll show it's a bijection. Surjectivity is clear, because every simplex $x \in \mathrm{sk}_n X$ is a degeneracy of a simplex $y \in X_m$ with $m \leq n$, and we may even assume that y is nondegenerate. Now let's prove injectivity. Suppose $(x, f) \in (X_n^{nd} \times \mathrm{sk}_n \Delta^n)_m$. If f is not surjective, then factoring it through its image shows that $f \in \mathrm{sk}_{n-1} \Delta^n$, so we may replace such an element by its image $y = xf \in \mathrm{sk}_{n-1} X$ without changing the element of P it represents. Thus, considering representatives of a pair of elements in P with the same image in $\mathrm{sk}_n X$, there are three cases to consider: (a) two elements y, z of sk_{n-1}; (b) an element $(x, f) \in X_n^{nd} \times \mathrm{sk}_n$ with f surjective and an element y of sk_{n-1}; and (c) two elements (x, f) and (x', f') of $X_n^{nd} \times \mathrm{sk}_n X$ with f and f' surjective. In case (a) we see that $y = z$ because $\mathrm{sk}_{n-1} X$ is a subsimplicial set of $\mathrm{sk}_n X$. In case (b) xf is a degeneracy of the nondegenerate simplex x, and thus $xf \notin \mathrm{sk}_{n-1} X$, because, according to Exercise 2.7.4, there is no other way to write xf as a degeneracy of a nondegenerate simplex; hence $y \neq xf$, contrary to assumption. Finally, in case (c), from $xf = x'f'$ and Exercise 2.7.4 we conclude that $(x, f) = (x', f')$. □

PROPOSITION 2.7.8. *If X is a simplicial set, then $|X|$ is a CW-complex.*

PROOF. Applying geometric realization to the pushout square in Proposition 2.7.7, thinking of the set X_n^{nd} as a discrete topological space, and using the commutativity of geometric realization with colimits, we see that the following diagram of topological spaces is a pushout square.

$$\begin{tikzcd} X_n^{nd} \times \partial|\Delta^n| \arrow[r] \arrow[d, tail] & |\mathrm{sk}_{n-1} X| \arrow[d, tail] \\ X_n^{nd} \times |\Delta^n| \arrow[r] & |\mathrm{sk}_n X| \end{tikzcd}$$

Hence $|\mathrm{sk}_{n-1} X|$ is a closed subset of $|\mathrm{sk}_n X|$, and $|\mathrm{sk}_n X|$ is obtained from it by gluing cells to it along their boundaries. We see now that the colimit in Lemma 2.7.6 is the union of an increasing family of closed subspaces, and thus that $|X|$ is a CW-complex. □

PROPOSITION 2.7.9. *If $Y \hookrightarrow X$ is an injective map of simplicial sets, then $|X|$ can be obtained from $|Y|$ by attaching cells. In particular, $|Y|$ is a closed subspace of $|X|$.*

PROOF. We argue as in 2.7.9, but we replace $\mathrm{sk}_n X$ with the subsimplicial set of X generated by $Y \bigcup \mathrm{sk}_n X$. □

Let X be a simplicial set. Recall the notation $[x, a] \in |X|$ introduced in Remark 2.3.10, which describes the set $|X|$ by generators and relations. If $x \in X_n$, then $[x, a]$ is the image of the point $a \in |\Delta^n|$ under the map $|x| : |\Delta^n| \to |X|$.

DEFINITION 2.7.10. Let $\mathrm{int}\,|\Delta^n|$ denote the interior of $|\Delta^n|$. It consists of the points all of whose coordinates are nonzero.

DEFINITION 2.7.11. We say that a generator $[x, a]$ is in *normal form* if $x \in X_n$ is nondegenerate and $a \in \mathrm{int}\,|\Delta^n|$, for some n.

COROLLARY 2.7.12. *Any point of $|X|$ can be written in normal form in just one way. This sets up a bijection $|X| \cong \coprod_n X_n^{nd} \times \mathrm{int}\,|\Delta^n|$.*

PROOF. This follows from the pushout square in the proof of 2.7.8, together with Lemma 2.7.6. □

COROLLARY 2.7.13. *If X is a simplicial set, then $|X|$ is Hausdorff, normal, and paracompact.*

PROOF. First we check that each point of $|X|$ is closed. Writing it in normal form $[x, a]$, we see that it is a closed point of $|\mathrm{sk}_n X|$, which in turn is a closed subset of $|X|$, as we saw above, so the point is closed in $|\mathrm{sk}_n X|$.

Now start with a closed subset W of $|X|$ and a continuous function $\psi : W \to \mathbb{R}$. We may extend it to a continuous function on all of $|X|$ by using the Tietze extension theorem [**3**, 3.6.2] and induction on n to extend it to $W \cup |\mathrm{sk}_n X|$. At each stage, we use only the normality of the space $|\Delta^n|$.

To check that $|X|$ is Hausdorff, take two points v and w of $|X|$, consider the map $\{v, w\} \to [0, 1]$ defined by $v \mapsto 0$ and $w \mapsto 1$, use normality to extend it to a continuous map $|X| \to [0, 1]$, and take preimages under that of disjoint neighborhoods of 0 and 1 in $[0, 1]$.

We omit the proof that $|X|$ is paracompact. □

COROLLARY 2.7.14. *If X is a simplicial set, then $|X|$ is compactly generated.*

PROOF. The space $|X|$ is a colimit of a diagram of the compact spaces $|\Delta^n|$, by definition. Any colimit $Y = \operatorname{colim} Y_i$ of compact spaces Y_i is compactly generated, for if $W \subseteq Y$ is a subset meeting each compact subset C of Y in a closed subset of C, then it meets the image of each map $Y_i \to Y$ in a closed subset, and thus its preimage in Y_i is closed, too. Since Y has the final topology with respect to the system of maps $Y_i \to Y$, the set W is closed. □

COROLLARY 2.7.15. *If X is a simplicial set, then $|X|$ is a compactly generated Hausdorff space.*

DEFINITION 2.7.16. Let CGHaus be the category of compactly generated Hausdorff spaces.

A topological space Y which is not Hausdorff can be made Hausdorff in a universal way as follows. Let $y \sim y'$ be the smallest equivalence relation on Y whose graph is a closed subset of $Y \times Y$. It can be defined as the intersection of all equivalence relations whose graph is closed. Let $Y' := Y/{\sim}$ be the quotient space by that equivalence relation. The map $Y \to Y'$ is a universal map from Y to a Hausdorff space.

A Hausdorff space Y can be made compactly generated in a universal way as follows. Consider the partially ordered set of compact subsets $C \subseteq Y$, which is closed under intersection, and let $kY := \operatorname{colim}_{C \subseteq Y} C$. Since kY is a colimit of a diagram of compact spaces, it is compactly generated. By lemma A.33 the map $kY \to Y$ is a bijection, because the intersection of two compact subsets of a Hausdorff space is compact.

PROPOSITION 2.7.17. *Limits and colimits exist in the category* CGHaus*, and for each $B \in$ CGHaus the functor $X \mapsto X \times B$ commutes with colimits.*

PROOF. A limit of Hausdorff spaces is Hausdorff, and a colimit of Hausdorff spaces can be made Hausdorff using the procedure sketched above. A colimit of compactly generated spaces is compactly generated, and a limit of compactly generated Hausdorff spaces can be made compactly generated using the procedure sketched above. Thus limits and colimits in Top can be made into limits and colimits in CGHaus. For details we refer to [8, VII.8, Proposition 2]. According to [8, VII.8, Theorem 3] the category CGHaus is Cartesian closed, and thus the functor $X \mapsto X \times B$ has a right adjoint $Y \mapsto Y^B$, hence, by Lemma A.45, preserves colimits. □

COROLLARY 2.7.18. *Suppose X and Y are simplicial sets. Then the natural map $|X \times Y| \to |X| \times |Y|$ is a homeomorphism if the product is formed in the category* CGHaus.

We remark that, by lemma 2.5.9, if either $|X|$ or $|Y|$ is locally compact, then the product $|X| \times |Y|$, formed in Top, is a geometric realization, hence is in CGHaus, and serves as the product formed in CGHaus. In fact, using Corollary 2.5.5 one can show that the product of a locally compact Hausdorff space with a compactly generated Hausdorff space is a compactly generated Hausdorff space.

PROOF. We argue as in Lemma 2.5.9, but we form products in the category CGHaus. The colimits can be regarded as colimits either in Top or in CGHaus, because in Top, the spaces that result are known to be compactly generated Hausdorff spaces, by Corollary 2.7.15. The same remark applies to the product $|\Delta^n| \times |Y|$.

$$\begin{aligned} |X \times Y| &\cong |(\operatorname{colim} \Delta^n) \times Y| \\ &\cong |\operatorname{colim}(\Delta^n \times Y)| \\ &\cong \operatorname{colim} |\Delta^n \times Y| \\ &\xrightarrow{\cong} \operatorname{colim}(|\Delta^n| \times |Y|) \\ &\cong (\operatorname{colim} |\Delta^n|) \times |Y| \\ &\cong |X| \times |Y| \end{aligned}$$

□

LEMMA 2.7.19. *Geometric realization preserves equalizers of maps of simplicial sets.*

PROOF. First we remark that equalizers of maps between compactly generated Hausdorff spaces are compactly generated Hausdorff spaces, so it doesn't matter whether we work in Top or in CGHaus.

Suppose $r, s : Y \rightrightarrows Z$ are maps of simplicial sets, and let $X = \{y \in Y \mid ry = sy\}$ be their equalizer, so the sequence $X \to Y \rightrightarrows Z$ is exact. Then we will show $|X| \to |Y| \rightrightarrows |Z|$ is an equalizer of topological spaces.

We have shown in 2.7.9 that the map $|X| \to |Y|$ is a closed embedding, so it is enough to check that its image contains every point of $[y, b] \in |Y|$ whose two images in $|Z|$ are the same. For this purpose we may assume that $[y, b]$ is in normal form, with $y \in Y_n$ nondegenerate, and $b \in |\Delta^n|$ interior. Our assumption is that $r[y, b] = [ry, b]$ and $s[y, b] = [sy, b]$ are equal, so let's put those in normal form by writing $ry = zf$, where z is nondegenerate and f in Ord is surjective, and $sy = wg$, where w is nondegenerate and g is surjective. Now $[ry, b] = [zf, b] = [zx, fb]$ and $[sy, b] = [wg, b] = [w, gb]$. The points fb and gb are interior because f and g are surjective. Thus $[z, fb]$ and $[w, gb]$ are equal and in normal form, so $z = w$ and $fb = gb$. Since b is interior (i.e., surjective as a map from $[0, 1]^*$), we see that $f = g$. Thus $ry = zf = wg = sy$, and $y \in X$, so $[y, b] \in |X|$. □

PROPOSITION 2.7.20. *Geometric realization, regarded as a functor* SSet $\to$ CGHaus*, preserves finite limits of simplicial sets.*

PROOF. Every finite limit can be expressed as an equalizer of two maps between two finite products of objects in the diagram, so this follows immediately from 2.7.19 and 2.7.18. □

The proposition above will be especially useful when applied to pullbacks.

Let's give a couple of examples to show that one cannot hope that geometric realization will preserve infinite products.

EXAMPLE 2.7.21. Let S be a finite set. The map $|\prod_{\mathbb{N}}(S \times \Delta^0)| \to \prod_{\mathbb{N}} |(S \times \Delta^0)|$ is a bijection but is not a homeomorphism because the source is infinite and discrete, but the target is a product of finite spaces, and thus is compact.

EXAMPLE 2.7.22. The map $|\prod_{\mathbb{N}} \Delta^1| \to \prod_{\mathbb{N}} |\Delta^q|$ is not surjective, because all of the points in its image have just a finite number of different values among its components. The map is not even surjective on fundamental groups: the image is the set of bounded vectors in $\prod_{\mathbb{N}} \mathbb{Z}$.

2.8. Covering spaces

Let X be a topological space. A *covering space* of X is a space Y with a map $p : Y \to X$ such that X can be covered by open sets $U \subseteq X$ which *trivialize* p in the sense that the preimage $p^{-1}(U)$ is a disjoint union (coproduct) $p^{-1}(U) = \coprod_\alpha V_\alpha$ and p induces homeomorphisms $V_\alpha \to U$ for each α. A *map* $g : Y \to Y'$ of covering spaces is one that fits into the following commutative diagram.

$$\begin{tikzcd} {|Y|} \arrow[r, "g"] \arrow[dr, "{|p|}"'] & Y' \arrow[d, "p'"] \\ & {|X|} \end{tikzcd}$$

We let Cov_X denote the category of covering spaces of X. If X has a base point $x_0 \in X$, then the functor $F : \mathrm{Cov}_X \to \mathrm{Set}$ defined by $p \mapsto p^{-1}(x_0)$ will be called the *fiber functor* for Cov_X.

By a standard compactness argument that uses the simple connectedness of $|\Delta^n|$, a map $|\Delta^n| \to X$ lifts uniquely to Y, once the destination of one point is specified, as illustrated in the following diagram.

$$\begin{tikzcd} {|\Delta^0|} \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ {|\Delta^n|} \arrow[r] \arrow[ur, dashed, "\exists!"] & X \end{tikzcd}$$

In the case $n = 1$ we call this *path lifting.*

Let $\mathcal{G}_X$ be the fundamental groupoid of X. It's a category whose objects are the points of X and where an arrow $f : x \to x'$ is a homotopy class $[\gamma]$ of paths starting at x and ending at x', i.e., $\gamma(0) = x$ and $\gamma(1) = x'$. The fundamental group at a point x is $\pi_1(X, x) := \mathrm{Hom}_{\mathcal{G}}(x, x)$. There is a functor $p^{-1} : \mathcal{G}_X \to \mathrm{Set}$ defined on objects by $x \mapsto p^{-1}(x)$ and on arrows by path lifting. Conversely, if X is locally simply connected, then any such functor comes from a covering space that can be constructed by gluing together spaces over X of the form $p^{-1}(x) \times U \to U$, where U is a simply connected open subset of X and $x \in U$. Thus we have an equivalence $\mathrm{Cov}_X \to \mathrm{Set}^G$. Taking the forgetful functor $W : \mathrm{Set}^G \to \mathrm{Set}$ as the fiber functor for Set^G, we remark that the equivalence makes the following diagram commute.

$$\begin{tikzcd} \mathrm{Cov}_X \arrow[r] \arrow[dr] & \mathrm{Set}^G \arrow[d] \\ & \mathrm{Set} \end{tikzcd}$$

The group G can be recovered from the fiber functor $W : \mathrm{Set}^G \to \mathrm{Set}$, regarded as an object in the category of such functors, as follows. If S is a G-set, then G acts on $W(S)$, regarded as an object of Set. (It doesn't act on S regarded as an object of Set^G, because for $g \in G$, the function $S \to S$ defined by $s \mapsto gs$ is just a function, not a map of G-sets.) The action of G on $W(S)$ is natural in S, and thus there is a map $G \to \mathrm{Aut}\, W$. (Discuss whether $\mathrm{Aut}\, W$ is a set. ...) Regarding G as a left G-set using left multiplication, we see that the functor W is represented by G in the sense that there is a natural bijection $W(S) \cong \mathrm{Hom}(G, S)$, defined by $s \mapsto (g \mapsto gs)$, with inverse defined by $\phi \mapsto \phi(1)$. By Yoneda's lemma, we see that

$\operatorname{Aut} W \cong \operatorname{Aut}_{\mathrm{Set}^G}(G)^{\mathrm{op}} \cong (G^{\mathrm{op}})^{\mathrm{op}} \cong G$, and this isomorphism is equal to the map $G \to \operatorname{Aut} W$. (Give the details of this computation. ...)

In particular, the fundamental group of X can be recovered from the category of covering spaces and its fiber functor.

It also follows that if G and H are two groups, and $\psi : \mathrm{Set}^G \to \mathrm{Set}^H$ is an equivalence of categories that makes the following diagram commute up to natural isomorphism, then ψ provides an isomorphism $G \cong H$.

$$\begin{tikzcd}
\mathrm{Set}^G \arrow[r] \arrow[dr] & \mathrm{Set}^H \arrow[d] \\
& \mathrm{Set}
\end{tikzcd}$$

Assume now that X is path connected and locally simply connected, pick a basepoint $x_0 \in X$, and let $G = \pi_1(X, x_0)$. Giving a functor $p^{-1} : \mathcal{G} \to \mathrm{Set}$ is then equivalent to giving the left G-set $p^{-1}(x_0)$. Choosing $p^{-1}(x_0) = G$ gives the *universal covering space* $\tilde{X} \to X$ of X. Since G is also a right G-set by right multiplication, G acts on $\tilde{X}$ on the right. If S is a left G-set, the corresponding covering space can be obtained as $\tilde{X} \times_G S$.

Now suppose X is a simplicial set and $p' : Y' \to |X|$ is a covering space. We define a simplicial set Y by declaring that an n-simplex of Y is a pair (x, y), where $x \in X_n$ and $y : |\Delta^n| \to Y'$ is a map that makes the following triangle commute.

$$\begin{tikzcd}
& Y' \arrow[d, "p'"] \\
|\Delta^n| \arrow[ur, "y"] \arrow[r, "x"'] & |X|
\end{tikzcd}$$

We make Y into a functor by defining $(x, y)f := (xf, yf)$, for an arrow $\underline{m} \to \underline{n}$, as in the following diagram.

$$\begin{tikzcd}
& & Y' \arrow[d, "p'"] \\
|\Delta^m| \arrow[r, "f"] \arrow[urr, bend left, "yf"] \arrow[rr, bend right, "xf"'] & |\Delta^n| \arrow[ur, "y"] \arrow[r, "x"] & |X|
\end{tikzcd}$$

Let $p : Y \to X$ be the map defined by $p(x, y) = x$. The natural map $|Y| \to Y'$ is a homeomorphism that makes the following triangle commute, showing that any covering space of a geometric realization is a geometric realization.

$$\begin{tikzcd}
|Y| \arrow[r, "\cong"] \arrow[dr, "|p|"'] & Y' \arrow[d, "p'"] \\
& |X|
\end{tikzcd}$$

The map $p : Y \to X$ has the following lifting property: a partial lifting of a simplex of X to Y extends uniquely to a complete lifting, as in the following diagram.

$$\begin{tikzcd} \Delta^m \arrow[r] \arrow[d, "f"'] & Y \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "\exists!"] & X \end{tikzcd}$$

Evidently, it is enough to check the lifting property when $m = 0$. A map $p : Y \to X$ with the lifting property will be called a *simplicial covering space*. Let Cov_X denote the category of simplicial covering spaces of X. The geometric realization of a simplicial covering space is a covering space, and we have an equivalence of categories $\mathrm{Cov}_X \to \mathrm{Cov}_{|X|}$, and hence, when there is a base point, the fundamental group of $|X|$ can be recovered combinatorial from Cov_X and its fiber functor.

Let's recast the notion of simplicial covering space. Given a simplicial covering space $p : Y \to X$ and a simplex x of X, we define a set $T(x)$ by setting $T(x) := p^{-1}(x)$. Thus $y \in T(x)$ if the following diagram commutes.

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ \Delta^n \arrow[r, "x"] \arrow[ur, "y"] & X \end{tikzcd}$$

We make T into a functor $T : \mathrm{Elem}(X)^{\mathrm{op}} \to \mathrm{Set}$ by using $y \mapsto yf$ to define the function $T(x) \to T(xf)$ for an arrow $f : xf \to x$ of $\mathrm{Elem}(X)$, as in the following diagram.

$$\begin{tikzcd} & & Y \arrow[d, "p"] \\ \Delta^m \arrow[r, "f"] \arrow[urr, bend left, "yf"] \arrow[rr, bend right, "xf"'] & \Delta^n \arrow[r, "x"] \arrow[ur, "y"] & X \end{tikzcd}$$

The lifting property says that T sends every arrow to a bijection, i.e., T is morphism inverting.

$$\begin{tikzcd} & & Y \arrow[d, "p"] \\ \Delta^m \arrow[r, "f"] \arrow[urr, bend left, "yf"] \arrow[rr, bend right, "xf"'] & \Delta^n \arrow[r, "x"] \arrow[ur, dashed, "\exists! y"] & X \end{tikzcd}$$

Starting from the functor T, we can recover Y by defining $Y_n = \coprod_{x \in X_n} T(x)$, and if T is morphism inverting, then $Y \to X$ is a covering space.

We have shown that there is an equivalence of categories

$$\mathrm{Cov}_X \xrightarrow{\cong} \mathrm{Map}(\mathrm{Elem}(X)^{\mathrm{op}}, \mathrm{Iso\,Set})$$

defined by sending Y to T as constructed above.

There is an equivalence of categories

gpdmap (2.8.1) $$\mathrm{Map}(X, \mathrm{Iso\,Set}) \xrightarrow{\cong} \mathrm{Map}(\mathrm{Elem}(X)^{\mathrm{op}}, \mathrm{Iso\,Set}),$$

which takes a bit of care to describe, so we embark on that now. There is a map $\mathrm{Elem}(X)^{\mathrm{op}} \to X$ which sends a simplex x to its first vertex x_0. Composition with it defines the map (2.8.1).

To go the other way, suppose we have a map $T : \mathrm{Elem}(X)^{\mathrm{op}} \to \mathrm{Iso\,Set}$. From it we define $U : X \to \mathrm{Iso\,Set}$ as follows. For $x \in X_0$ we let $U(x) := T(x)$. For $x \xrightarrow{e} y$ in X_1, we consider the diagram $x \to e \leftarrow y$ in $\mathrm{Elem}(X)^{\mathrm{op}}$. Applying T to it yields

$$U(x) = T(x) \xleftarrow{\cong} T(e) \xrightarrow{\cong} T(y) = U(y),$$

from which we define the isomorphism $U(e) : U(x) \xrightarrow{\cong} U(y)$ by composition of one map with the inverse of the other. (That isomorphism is what we would expect to get by lifting the path corresponding to e to our covering space.)

Now we show that U sends a 2-simplex s of X to a 2-simplex of $\mathrm{Iso\,Set}$. First we label the faces of s.

$$\begin{tikzcd}
x \arrow[r, "e"] \arrow[dr, "g"'] & y \arrow[d, "f"] \\
& z
\end{tikzcd}$$

Our goal is to check that the following triangle commutes.

$$\begin{tikzcd}
U(x) \arrow[r, "U(e)"] \arrow[dr, "U(g)"'] & U(y) \arrow[d, "U(f)"] \\
& U(z)
\end{tikzcd}$$

Now we draw the corresponding diagram in $\mathrm{Elem}(X)$.

$$\begin{tikzcd}
x \arrow[rr] \arrow[drrr] \arrow[ddrr] & & e \arrow[dr] & & y \arrow[ll] \arrow[dl] \arrow[dd] \\
& & & \sigma & \\
& & g \arrow[ur] & & f \arrow[ul] \\
& & & & z \arrow[u] \arrow[uul] \arrow[ull]
\end{tikzcd}$$

Applying T yields the following diagram of isomorphisms, yielding the desired commutativity.

$$
\begin{tikzcd}
 & & & & & U(y) \arrow[dddddl, bend left, "U(f)"] \\
U(x) \arrow[r, equals] \arrow[urrrrr, bend left, "U(e)"] \arrow[ddddrrrr, bend right, "U(g)"'] & T(x) & T(e) \arrow[l] \arrow[rr] & & T(y) \arrow[ur, equals] & \\
 & & & T(\sigma) \arrow[ull] \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] \arrow[ddr] & & \\
 & & T(g) \arrow[uul] \arrow[ddrr] & & T(f) \arrow[uu] \arrow[d] & \\
 & & & & T(z) \arrow[d, equals] & \\
 & & & & U(z) &
\end{tikzcd}
$$

That U sends an n-simplex of X to an n-simplex of $\mathrm{Iso\,Set}$ for $n > 2$ follows from the statement for $n = 2$ just proved. In effect, for any category $\mathcal{C}$, a map $\mathrm{sk}_2\,\Delta^n \to \mathcal{C}$ extends uniquely to a map $\Delta^n \to \mathcal{C}$.

We have shown that there is an equivalence of categories

$$\mathrm{Cov}_X \xrightarrow{\cong} \mathrm{Map}(X, \mathrm{Iso\,Set})$$

.

Now define the *fundamental groupoid* $\mathcal{G}_X$ of X to be the groupoid generated by X. More precisely, its objects are the elements of X_0, its arrows are generated by X_1, and the relations are those coming from the triangles in X_2 (give more details here ...). We have an isomorphism $\mathrm{Map}(\mathcal{G}_X, \mathrm{Iso\,Set}) \cong \mathrm{Map}(X, \mathrm{Iso\,Set})$, induced by the natural map $X \to \mathcal{G}$. If $x \in X_0$, then we have shown that $\pi_1(|X|, x_0) \cong \mathrm{Aut}_{\mathcal{G}_X}(x_0)$.

It is also fairly clear that the evident map $\mathcal{G}_X \to \mathcal{G}_{|X|}$ is an equivalence.[2]

Now suppose M is a monoid, regarded as a category with one object $*$, and let M^+ be the *group completion* of M. It is the group defined by generators and relations, with generators the elements of M, and with relations the multiplication table of M. We see that $\pi_1|M| = M^+$. The *classifying space* of M is $BM = |M|$. If G is a group, then $G \cong \pi_1 BG$.

DEFINITION 2.8.1. If G is a monoid acting on a set S, let $\langle G, S\rangle$ be the category whose objects are the elements of S, where $\mathrm{Hom}(s, s') = \{g \in G \mid gs = s'\}$, and where composition of arrows is multiplication in G. We call it the *translation category* for G acting on S.

[2] More details needed here. Perhaps rewrite the section using representations of $\mathcal{G}_{|X|}$ instead of representationsn of the fundamental group.

Let G be a group. Observe that $\pi_0|\langle G,S\rangle|$ is the quotient set $G\backslash S = \{Gs \mid s \in S\}$. If $S = *$ is the one point set, then $\langle G,S\rangle \cong G$ and $|\langle G,S\rangle| \cong BG$. The map $|\langle G,S\rangle| \to BG$ induced by the map $S \to *$ is the covering space of BG whose fiber is S. Replacing S with G, regarded as a G-set by left multiplication, we define $EG := |\langle G,G\rangle|$; as basepoint we choose 1. The map $EG \to BG$ is the universal covering space of BG. The group G acts on the right side of G also, hence also on EG, and $EG/G \cong BG$.

LEMMA 2.8.2. *The space EG is contractible, and $\pi_n BG = 0$ for $n \neq 1$.*

PROOF. In the category $\langle G,G\rangle$ any object is an initial object, hence $EG = |\langle G,G\rangle|$ is contractible. Let $\alpha : |S^n| \to BG$ represent a homotopy class. Writing $|S^n| = |\Delta^n|/\partial|\Delta^n|$ we compose with the quotient map to get a map $\beta|\Delta^n| \to BG$. Pick any boundary point of $|\Delta^n|$ as the basepoint and lift the basepoint to the basepoint of EG. Now extend the lifting to get a map $\tilde{\beta} : |\Delta^n| \to EG$. Since it's a lifting, it sends $\partial|\Delta^n|$, which is connected, to the fiber of $EG \to BG$, which is discrete; thus it sends all of $\partial|\Delta^n|$ to the basepoint, and thereby factors to yield a map $\tilde{\alpha} : |S^n| \to EG$, which is a lifting of α. The map $\tilde{\alpha}$ may be contracted to the basepoint, because EG is contractible, hence so may α. $\square$

CHAPTER 3

Topological Techniques

3.1. Subdivision

In this section we present edgewise subdivision, which was presented in [14] and is due to Quillen.

DEFINITION 3.1.1. If T and U are partially ordered sets, define their *join* $T * U$ to be the disjoint union $T \sqcup U$ with a certain ordering. For $t \in T$ let t' denote the corresponding element of $T \sqcup U$, and for $u \in U$ let u'' denote the corresponding element of $T \sqcup U$. The ordering is defined as follows.

$$\begin{array}{rcl} t_1' \leq t_2' & \iff & t_1 \leq t_2 \\ u_1'' \leq u_2'' & \iff & u_1 \leq u_2 \\ t' \leq u'' & & \text{always} \\ u'' \leq t' & & \text{never} \end{array}$$

EXAMPLE 3.1.2. $\underline{m} * \underline{n} = \{0' < 1' < \cdots < m' < 0'' < 1'' < \ldots n''\}$

The join $\underline{m} * \underline{n}$ is totally ordered and is uniquely isomorphic to $\underline{m+n+1}$. This will allow us to regard $\underline{m} * \underline{n}$ as an object of Ord when necessary. Alternatively, we could replace Ord by a larger but equivalent small category of finite nonempty totally ordered sets, closed under join, as mentioned in 2.3.14.

EXERCISE 3.1.3. Observe that $|\underline{m} * \underline{n}|$ is homeomorphic to the join $|\underline{m}| * |\underline{n}|$ of two simplices. Show that for simplicial sets X and Y, the join $|X| * |Y|$ is homeomorphic to the geometric realization of the simplicial set $X * Y$ defined by $\underline{n} \mapsto \coprod_{f : \underline{n} \to \underline{1}} X(f^{-1}(0)) \times Y(f^{-1}(1))$, provided we define $X(\phi)$ and $Y(\phi)$ to be sets with one element.

DEFINITION 3.1.4. Define a functor $e : \text{Poset} \to \text{Poset}$ by $e(T) := T^{\text{op}} * T$.

Since $e(\underline{n})$ is totally ordered, we may also regard it as an object of Ord, and thus we may regard e as a functor from Ord to Ord. Thus if X is a simplicial set, the composite functor $X \circ e$ is again a simplicial set.

DEFINITION 3.1.5. If $\mathcal{I}$ is a category, let $\operatorname{Sub}\mathcal{I}$ denote the category whose objects are the arrows $k \leftarrow j$ of $\mathcal{I}$, and whose arrows from $k' \leftarrow j'$ to $k \leftarrow j$ are the commutative diagrams

$$\begin{array}{ccc} k' & \longleftarrow & j' \\ \downarrow & & \uparrow \\ k & \longleftarrow & j \end{array}$$

of $\mathcal{I}$. Composition of arrows of $\operatorname{Sub}\mathcal{I}$ is accomplished by composing the underlying arrows of $\mathcal{I}$. If T is a partially ordered set (regarded as a category), so is $\operatorname{Sub} T$. It can be regarded as the partially ordered set of *subintervals* of T, i.e., the partially ordered set of pairs (j,k) of elements of T with $j \leq k$, where $(j',k') \leq (j,k)$ is defined to mean $j \leq j'$ and $k' \leq k$.

Here is an illustration of $\mathrm{Sub}(\underline{2})$. Notice that its geometric realization is a triangle, subdivided into four smaller triangles.

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[dr] & (0,1) \arrow[d] & (1,1) \arrow[l] \arrow[dl] \arrow[d] \\
& (0,2) & (1,2) \arrow[l] \\
& & (2,2) \arrow[ul] \arrow[u]
\end{tikzcd}$$

LEMMA 3.1.6. *The functors e and* Sub *are adjoint functors on the category of partially ordered sets, i.e., there is a natural isomorphism* $\mathrm{Hom}_{\mathrm{Poset}}(e(T),U) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Poset}}(T,\mathrm{Sub}\,U)$.

PROOF. Define it by $f \mapsto (t \mapsto (f(t'),f(t'')))$. The following illustration shows the situation when $t_0 \leq t_1$, and hence $t_0' \geq t_1'$ and $t_0'' \leq t_1''$. Each vertical arrow is an element of $\mathrm{Sub}\,U$.

$$\begin{tikzcd}
f(t_0') \arrow[d] & f(t_1') \arrow[l] \arrow[d] \\
f(t_0'') \arrow[r] & f(t_1'')
\end{tikzcd}$$

□

COROLLARY 3.1.7. *If T is a partially ordered set, then there is a natural isomorphism $(NT)\circ e \cong N(\mathrm{Sub}\,T)$ of simplicial sets.*

PROOF. We compute $((NT)\circ e)(\underline{n}) = NT(e(\underline{n})) \cong \mathrm{Hom}(e(\underline{n}),T) \cong \mathrm{Hom}(\underline{n},\mathrm{Sub}\,T) = (N(\mathrm{Sub}\,T))(\underline{n})$. □

The corollary prompts the following definition.

DEFINITION 3.1.8. If X is a simplicial set, define $\mathrm{Sub}\,X := X \circ e$.

Using this definition the corollary can be rephrased as saying that $\mathrm{Sub}(NT) \cong N(\mathrm{Sub}\,T)$, so there is no danger of confusion if we identify T with NT and $\mathrm{Sub}\,T$ with $N(\mathrm{Sub}\,T)$.

In order to handle geometric realizations, it is convenient to extend the notions above to the category Poset^* introduced in remark 2.4.2, as follows. Suppose T and U are objects of Poset^*. Make $T * U$ into an object of Poset^* by declaring its essential subsets to be those containing a set of the form $F \sqcup G$ where F is an essential subset of T and G is an essential subset of U. Make T^{op} into an object of Poset^* by using the same essential subsets. Define $e(T) := T^{\mathrm{op}} * T$. Make $\mathrm{Sub}(T)$ into an object of Poset^* by declaring its essential subsets to be those containing a subset of the form $\mathrm{Sub}(F)$, where F is an essential subset of T.

Recall that if T is a partially ordered set, then T^* denotes T with its finite subsets declared to be negligible. We point out that $T^* * U^* \cong (T * U)^*$, and thus $e(T^*) = e(T)^*$.

The following exercise is a generalization of lemma 3.1.6.

EXERCISE 3.1.9. Show that e and Sub are adjoint functors on the category Poset*, i.e., there is a natural isomorphism $\mathrm{Hom}_{\mathrm{Poset}^*}(e(T), U) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Poset}^*}(T, \mathrm{Sub}\, U)$.

COROLLARY 3.1.10. *There is a natural homeomorphism* $|\mathrm{Sub}\, \Delta^p| \cong |\Delta^p|$.

PROOF. We use the exercise above, but just the easy part, where U has no nonempty negligible subsets. Let $I := [0,1]^*$, and recall from remark 2.4.4 that $|U| \cong \mathrm{Hom}(I, U)$. Let $e(I) \xrightarrow{\cong} I$ be the piecewise linear isomorphism defined by $r' \mapsto (1-r)/2$ and $r'' \mapsto (r+1)/2$. Using that and corollary 3.1.7, we compute $|\mathrm{Sub}\, \Delta^p| = |\Delta^p \circ e| = |(N\underline{p}) \circ e| \cong |N\, \mathrm{Sub}(\underline{p})| = |\mathrm{Sub}(\underline{p})| \cong \mathrm{Hom}(I, \mathrm{Sub}(\underline{p})) \cong \mathrm{Hom}(e(I), \underline{p}) \cong \mathrm{Hom}(I, \underline{p}) \cong |\Delta^p|$. Naturality in Δ^p amounts to naturality in $\underline{p}$ by Yoneda's lemma, and is easy to check because composition with an arrow $\underline{p} \to \underline{q}$ is compatible with each of the steps. (Say something about continuity ...) $\square$

EXERCISE 3.1.11. Show that the homeomorphism in corollary 3.1.10 is piecewise affine. ...

THEOREM 3.1.12. *Let X be a simplicial set. There is a natural homeomorphism* $|\mathrm{Sub}\, X| \cong |X|$.

PROOF. As in Lemma 2.3.4 we write $X \cong \mathrm{colim}_{x:\Delta^p \to X} \Delta^p$. Using corollary 3.1.10 we compute $|\mathrm{Sub}\, X| = |X \circ e| \cong |(\mathrm{colim}_x \Delta^p) \circ e| \cong |\mathrm{colim}_x (\Delta^p \circ e)| \cong \mathrm{colim}_x |\Delta^p \circ e| \cong \mathrm{colim}_x |\Delta^p| \cong |X|$. $\square$

Now we introduce a variant of edgewise subdivision called k-fold edgewise subdivision. It was used in [7] for studying the action of exterior powers on K-theory.

DEFINITION 3.1.13. For $k \geq 1$ define a functor $e_k : \mathrm{Ord} \to \mathrm{Ord}$ by setting $e_k(\underline{n}) := \underline{n} * \cdots * \underline{n}$, the join of k copies of $\underline{n}$. For a simplicial set X define $\mathrm{Sub}_k\, X := X \circ e_k$.

THEOREM 3.1.14. *There is a natural homeomorphism* $|\mathrm{Sub}_k\, X| \cong |X|$.

PROOF. The proof is similar to the proof of theorem 3.1.12, with one important difference: when T is a partially ordered set, $\mathrm{Sub}_k\, NT$ is not the nerve of a partially ordered set. An n-simplex of $\mathrm{Sub}_k\, NT$ is a chain $t_{01} \leq t_{11} \leq \ldots t_{n1} \leq t_{02} \leq t_{12} \leq \ldots t_{n2} \leq \cdots \leq t_{0k} \leq t_{1k} \leq \ldots t_{nk}$. We can make do by pointing out that $\mathrm{Sub}_k\, NT$ is a subsimplicial set of NT^k, and by defining a map $[0,1]^* \to \mathrm{Sub}_k\, NT$ to be a map $[0,1]^* \to T^k$ whose image is an n-simplex for some n. $\square$

3.2. Techniques for bisimplicial sets

THEOREM 3.2.1. *Let $X_{\cdot\cdot} \to Y_{\cdot\cdot}$ be a map of bisimplicial sets. Suppose the map is a homotopy equivalence in each row, i.e., for each $n \in \mathbb{N}$ the map $|X_{\cdot n}| \to |Y_{\cdot n}|$ is a homotopy equivalence.*

PROOF. ... $\square$

THEOREM 3.2.2. *Let $X_{\cdot\cdot} \to Y_{\cdot\cdot} \to Z_{\cdot\cdot}$ be a sequence of pointed bisimplicial sets so that the composite map $X \to Z$ is constant. Suppose the sequence is a fibration sequence in each row, i.e., for each $n \in \mathbb{N}$, the sequence $|X_{\cdot n}| \to |Y_{\cdot n}| \to |Z_{\cdot n}|$ is a fibration sequence. Suppose also that each space $|Z_{\cdot n}|$ is connected. Then the sequence $|X| \to |Y| \to |Z|$ is a fibration sequence.*

PROOF. ... $\square$

3.3. Techniques for simplicial sets

Let $f : X \to Y$ be a map of topological spaces. Review the definition of (Serre) (weak) fibration. Give the construction of the long exact sequence of homotopy groups. Show that any map is equivalent to a fibration. Define the homotopy fiber Φf of the map f. Show there is a long exact sequence of homotopy groups and sets:

$$\cdots \to \pi_n X \to \pi_n Y \to \pi_{n-1}\Phi f \to \pi_{n-1} X \to \cdots \to \pi_1 Y \to \pi_0 \Phi f \to \pi_0 X \to \pi_0 Y$$

and that, in addition, $\pi_1 Y$ acts on $\pi_0 \Phi f$ in such a way that the set $\pi_1 Y \backslash \pi_0 \Phi f$ of orbits maps injectively to $\pi_0 X$. ...

Define and discuss *homotopy cartesian square*, including those where the lower left corner contains a contractible space. ...

Define *fibration sequence* $F \to E \to B$, both those where the composite map $F \to B$ is a constant, and those where a null-homotopy of it is provided. ...

Now let $f : X \to Y$ be a map of simplicial sets and consider its geometric realization $|f| : |X| \to |Y|$. We ask for a way to obtain the homotopy fiber $\Phi|f|$ as the geometric realization of a simplicial set. There is a way to do this in general, but the answers it gives are too complicated for our purposes. The following definition records the most naive simplicial set that has some similarity to the homotopy fiber.

DEFINITION 3.3.1. Let $f : X \to Y$ be a map of simplicial sets. For each $m \geq 0$ and for each $y \in Y_m$ we define the simplicial set y/f as follows.

$$(y/f)(\underline{n}) := \lim \left(\begin{tikzcd} & & X(\underline{n}) \arrow[d, "f"] \\ & Y(\underline{m} * \underline{n}) \arrow[r, "Y(j)"] \arrow[d, "Y(i)"] & Y(\underline{n}) \\ \{y\} \arrow[r] & Y(\underline{m}) & \end{tikzcd} \right)$$

Here $i : \underline{m} \hookrightarrow \underline{m} * \underline{n}$ and $j : \underline{n} \hookrightarrow \underline{m} * \underline{n}$ are the natural inclusions.

Alternatively, $(y/f)_n = \{(\gamma, x) \in Y(\underline{m} * \underline{n}) \times X(\underline{n}) \mid \gamma i = y \text{ and } \gamma j = f(x)\}$, as illustrated in the following diagram.

$$\begin{tikzcd} & \underline{n} \arrow[r, "x"] \arrow[d, "j"] & X \arrow[d, "f"] \\ \underline{m} \arrow[r, "i"] \arrow[rr, bend right, "y"'] & \underline{m} * \underline{n} \arrow[r, "\gamma"] & Y \end{tikzcd}$$

DEFINITION 3.3.2. Let Y be a simplicial set. For each $m \geq 0$ and for each $y \in Y_m$ define $y/Y := y/1_Y$.

EXERCISE 3.3.3. Let Y be a simplicial set. For each $m \geq 0$ and for each $y \in Y_m$ show that the space $|y/Y|$ is contractible.

THEOREM 3.3.4 (Theorem B'). *Let $f : X \to Y$ be a map of simplicial sets, and suppose for every $m \geq 0$, for every $y \in Y_m$, and for every map $g : \underline{m}' \to \underline{m}$, that* *the map* $|y/f| \to |yg/f|$ *is a homotopy equivalence. Then for any* y *the following square is homotopy cartesian.*

$$\begin{tikzcd} |y/f| \arrow[r] \arrow[d] & |X| \arrow[d, "|f|"] \\ |y/Y| \arrow[r] & |Y| \end{tikzcd}$$

As a consequence, for any vertex $x_0 \in X_0$*, the following sequence of homotopy groups and sets is exact.*

$$\cdots \to \pi_n(|X|, x_0) \to \pi_n(|Y|, f(x_0)) \to \pi_{n-1}(|y_0/f|, e_0) \to \pi_{n-1}(|X|, x_0) \to \dots$$

Here e_0 *is the vertex* $(y_0 \xrightarrow{=} f(x_0), x_0)$ *of* y_0/f*.*

EXERCISE 3.3.5. Let $\mathcal{C}$ be a small category. Recall the definition of the category $\operatorname{Sub}\mathcal{C}$ from 3.1.5. Let $\operatorname{tar} : \operatorname{Sub}\mathcal{C} \to \mathcal{C}$ be the functor which sends an object $C' \to C$ of $\operatorname{Sub}\mathcal{C}$ to C. Show that $|\operatorname{tar}| : |\operatorname{Sub}\mathcal{C}| \to |\mathcal{C}|$ is a homotopy equivalence.

CHAPTER 4

Definitions of K-theory

4.1. Direct sum K-theory

Suppose $\mathcal{C}$ is a small additive category. The direct sum Grothendieck group $K_0^{\oplus}\mathcal{C}$ (see 1.1.1) is the group completion of the monoid M whose elements are the isomorphism classes of $\mathcal{C}$ and whose binary operation comes from direct sum. A more detailed study of the structure of $\mathcal{C}$ would involve the isomorphisms more directly. Thus it is natural to to build an analogue of the classifying space BM based on the "monoid" of objects of M under direct sum. It would be the geometric realization $|X|$ of a simplicial set X with one vertex 0 (to be used as the basepoint in $|X|$), one edge for each object of $\mathcal{C}$, and one triangle for each isomorphism $C \cong C' \oplus C''$ in $\mathcal{C}$. The three edges of the triangle would be labelled with the three objects, as in the following diagram.

$$\begin{tikzcd} 0 \arrow[r, "C'"] \arrow[dr, "C"'] & 0 \arrow[d, "C''"] \\ & 0 \end{tikzcd}$$

By the results of the previous section there would be an isomorphism $K_0^{\oplus}\mathcal{C} \to \pi_1|X|$.

A natural way to extend this construction to incorporate higher dimensional simplices involves mimicking the construction of BM. Recall that an n-simplex in the nerve of M is a functor $f : \underline{n} \to M$. The statement that f is a functor means that for $i \leq j \leq k$ we have the following equation.

fcomp
$$f(k{\leftarrow}i) = f(k{\leftarrow}j) \circ f(j{\leftarrow}i) \tag{4.1.1}$$

For $i \leq j$ in $\underline{n}$ we see that

$$f(j{\leftarrow}i) = f(j{\leftarrow}j-1) \circ f(j-1{\leftarrow}j-2) \circ \ldots f(i+1{\leftarrow}i),$$

so f is determined uniquely by the elements $f(j{\leftarrow}j-1)$ for $1 \leq j \leq n$.

What if we replace f by something called F that assigns to an arrow $j{\leftarrow}i$ of $\underline{n}$ an object $F(j{\leftarrow}i)$ of $\mathcal{C}$, and we try to replace (4.1.1) with the following isomorphism?

subintsum
$$F(k{\leftarrow}i) \cong F(k{\leftarrow}j) \oplus F(j{\leftarrow}i) \tag{4.1.2}$$

Such an isomorphism could be specified by specifying its two components.

$$\begin{array}{ccc} F(k{\leftarrow}i) & \leftarrow & F(k{\leftarrow}j) \\ F(k{\leftarrow}i) & \leftarrow & F(j{\leftarrow}i) \end{array}$$

Notice that the intervals $[j, k]$ and $[i, j]$ are subintervals of $[i, k]$, so a unified way to obtain both components would be to provide an arrow

$$F(\ell{\leftarrow}i) \quad \leftarrow \quad F(k{\leftarrow}j)$$

whenever $i \le j \le k \le \ell$. Since such arrows are to be used as inclusion maps in direct sum diagrams, when $h \le i \le j \le k \le \ell \le m$ we would want the following diagram to commute.

$$\begin{tikzcd} F(m\leftarrow h) & F(\ell\leftarrow i) \arrow[l] \\ & F(k\leftarrow j) \arrow[u] \arrow[ul] \end{tikzcd}$$

The discussion above leads immediately to the following precise definitions.

Assume $\mathcal{C}$ is an additive category with a chosen zero object 0, and recall the definition of $\mathrm{Sub}\,\mathcal{I}$ in 3.1.5.

DEFINITION 4.1.1. A functor $F : \mathrm{Sub}\,\mathcal{I} \to \mathcal{C}$ is called *additive* if $F(i\leftarrow i) = 0$ for all i, and for all $i \le j \le k$ the map $F(k\leftarrow i) \xleftarrow{\cong} (F(k\leftarrow j) \oplus F(j\leftarrow i))$ is an isomorphism.

DEFINITION 4.1.2. We define a simplicial set $S.^{\oplus}\mathcal{C}$ by letting it send $\underline{n} \in \mathrm{Ord}$ to the set of additive functors $C : \mathrm{Sub}\,\underline{n} \to \mathcal{C}$. It sends an arrow $g : \underline{m} \to \underline{n}$ in Ord to composition with the functor $\mathrm{Sub}\,g : \mathrm{Sub}\,\underline{m} \to \mathrm{Sub}\,\underline{n}$. In other words, $(Cg)(j\leftarrow i) := C(g(j)\leftarrow g(i))$. It is easy to check that if C is additive, then so is Cg.

The definition above is due to Waldhausen, [**15**, p. 182], where it is introduced as a simplicial exact category, and in [**16**, 1.3], in the more general context of categories with cofibrations. The corresponding simplicial set (of objects), considered here, is introduced as $s.\mathcal{C}$ in [**16**, 1.4].

DEFINITION 4.1.3. For an additive category $\mathcal{C}$ define the direct sum K-theory space $K^{\oplus}(\mathcal{C}) := \Omega|S.^{\oplus}\mathcal{C}|$.

The following definition doesn't conflict with our previous definition 1.1.1 of $K_0^{\oplus}\mathcal{C}$.

DEFINITION 4.1.4. For $n \ge 0$ define the n-th direct sum K-group $K_n^{\oplus}\mathcal{C} := \pi_n K^{\oplus}(\mathcal{C})$. Observe that $K_n^{\oplus}\mathcal{C} \cong \pi_{n+1}|S.^{\oplus}\mathcal{C}|$.

We saw in exercise 1.2.20 that $K_0^{\oplus}\mathcal{C}$ can also be defined by generators and relations. There is a generator $[C, D]$ for each pair of objects C, D in $\mathcal{C}$, and there is a relation $[C, D] = [C \oplus E, D \oplus E]$ for all objects C, D, and E in $\mathcal{C}$. Any set given by generators and relations can be represented as the set of connected components of a graph: put down one vertex for each generator and one edge for each relation. By using the category $\mathcal{C}$ we can build a simplicial set more interesting than that graph whose set of connected components is $K_0^{\oplus}\mathcal{C}$.

DEFINITION 4.1.5. Given an ordered set $\underline{n}$, define a functor $\Gamma : \mathrm{Ord} \to \mathrm{Poset}$ by letting $\Gamma(\underline{n})$ denote the partially ordered set $\{L, R\} \cup \underline{n}$ obtained by adding two new incomparable minimal elements L and R to $\underline{n}$, each of which is less than all the other elements.

Here is an illustration of $\Gamma(\underline{n})$ as a category.

$$\begin{tikzcd}
L \arrow[dr] & & & & \\
& 0 \arrow[r] & 1 \arrow[r] & \cdots \arrow[r] & n \\
R \arrow[ur] & & & &
\end{tikzcd}$$

DEFINITION 4.1.6. We define a simplicial set $G.^{\oplus}\mathcal{C}$ by letting it send $\underline{n}$ to the set of additive functors $\operatorname{Sub}\Gamma(\underline{n}) \to \mathcal{C}$.

By construction, we see that $\pi_0|G.^{\oplus}\mathcal{C}| \cong K_0^{\oplus}(\mathcal{C})$. Later we will see that $|G.^{\oplus}\mathcal{C}|$ is homotopy equivalent to the loop space of $|S.^{\oplus}\mathcal{C}|$, so that $\pi_n|G.^{\oplus}\mathcal{C}| \cong K_n^{\oplus}(\mathcal{C})$ for all $n \geq 0$.

The following definition is due to Quillen.

DEFINITION 4.1.7. We define a category $S^{-1}S(\mathcal{C})$ whose objects are pairs (C, D) of objects of $\mathcal{C}$, and whose arrows from (C, D) to (C', D') are isomorphism classes of triples (X, α, β), where $X \in \mathcal{C}$, $\alpha : X \oplus C \xrightarrow{\cong} C'$, and $\beta : X \oplus D \xrightarrow{\cong} D'$. Composition of arrows is done by ...

4.2. Exact sequence K-theory

Now let's switch gears and consider a small exact category $\mathcal{C}$. We would like to generalize the discussion above and construct a new simplicial set Y whose fundamental group is $K_0\mathcal{C}$. As before, Y could have one vertex 0 (to serve as the basepoint), one edge for each object of $\mathcal{C}$, and one triangle for each short exact sequence $0 \to C' \to C \to C'' \to 0$ in $\mathcal{C}$. The three edges of the triangle would be labelled with the three objects, as in the following diagram.

$$\begin{tikzcd}
0 \arrow[r, "C'"] \arrow[dr, "C"'] & 0 \arrow[d, "C''"] \\
& 0
\end{tikzcd}$$

By the results of the previous section there would be an isomorphism $K_0\mathcal{C} \to \pi_1|Y|$. To extend this construction to incorporate higher dimensional simplices we may proceed as before and try to invent something called F that assigns to an arrow $j \leftarrow i$ of $\underline{n}$ an object $F(j \leftarrow i)$ of $\mathcal{C}$, but this time we replace (4.1.2) with the following short exact sequences, one for each $i \leq j \leq k$.

subintesq (4.2.1)
$$0 \to F(j \leftarrow i) \to F(k \leftarrow i) \to F(k \leftarrow j) \to 0$$

In order to obtain the arrows in the exact sequence above we may make F a functor on the following category.

DEFINITION 4.2.1. If $\mathcal{I}$ is a category, let $\operatorname{Arr}\mathcal{I}$ denote the category whose objects are the arrows $k \leftarrow j$ of $\mathcal{I}$, and whose arrows from $k' \leftarrow j'$ to $k \leftarrow j$ are the commutative diagrams

$$\begin{tikzcd} k' & j' \arrow[l] \arrow[d] \\ k & j \arrow[l] \arrow[from=1-1, to=2-1] \end{tikzcd}$$

of $\mathcal{I}$. Composition of arrows of $\operatorname{Arr}\mathcal{I}$ is accomplished by composing the underlying arrows of $\mathcal{I}$. If $\mathcal{I}$ is a partially ordered set (regarded as a category), so is $\operatorname{Arr}\mathcal{I}$; it is the partially ordered set of pairs (j,k) of elements of $\mathcal{I}$ with $j \leq k$, where this time $(j',k') \leq (j,k)$ is defined to mean $j' \leq j$ and $k' \leq k$.

DEFINITION 4.2.2. A functor $F : \operatorname{Arr}\mathcal{I} \to \mathcal{C}$ is called *exact* if $F(i{\leftarrow}i) = 0$ for all i, and for all $i \leq j \leq k$ the sequence $0 \to F(j{\leftarrow}i) \to F(k{\leftarrow}i) \to F(k{\leftarrow}j) \to 0$ is a short exact sequence of $\mathcal{C}$.

WARNING 4.2.3. The category $\operatorname{Arr}\mathcal{I}$ is not an exact category.

DEFINITION 4.2.4. We define a simplicial set $S.\mathcal{C}$ by letting it send $\underline{n}$ to the set of exact functors $C : \operatorname{Arr}\underline{n} \to \mathcal{C}$. It sends an arrow $g : \underline{m} \to \underline{n}$ in Ord to composition with the functor $\operatorname{Arr} g : \operatorname{Arr}\underline{m} \to \operatorname{Arr}\underline{n}$. In other words, $(Cg)(j{\leftarrow}i) := C(g(j){\leftarrow}g(i))$. It is easy to check that if C is exact, then so is Cg.

DEFINITION 4.2.5. For an exact category $\mathcal{C}$ define the K-theory space $K(\mathcal{C}) := \Omega|S.\mathcal{C}|$.

The following definition doesn't conflict with our previous definition 1.2.3 of $K_0\mathcal{C}$.

DEFINITION 4.2.6. For $n \geq 0$ define the n-th K-group $K_n\mathcal{C} := \pi_{n+1}|S.\mathcal{C}|$.

DEFINITION 4.2.7. If R is a ring, define $K_nR := K_n\mathcal{P}_R$ (see 1.2.9 for the definition of $\mathcal{P}_R$).

DEFINITION 4.2.8. If $F : \mathcal{C} \to \mathcal{D}$ is an exact functor between small exact categories, then (by composition) it gives a map $S.\mathcal{C} \to S.\mathcal{D}$. Let $F_* : K_n\mathcal{C} \to K_n\mathcal{D}$ denote the induced map on K-groups.

EXERCISE 4.2.9. If F and G are naturally isomorphic exact functors $\mathcal{C} \rightrightarrows \mathcal{D}$ between small exact categories, show the induced maps $S.\mathcal{C} \rightrightarrows S.\mathcal{D}$ are homotopic.

It follows from the exercise that if $F : \mathcal{C} \to \mathcal{D}$ is an exact equivalence, then the induced map $S.\mathcal{C} \rightrightarrows S.\mathcal{D}$ is a homotopy equivalence, and hence the induced map $K_n\mathcal{C} \to K_n\mathcal{D}$ is an isomorphism.

Here is Quillen's original construction of algebraic K-theory.

DEFINITION 4.2.10. For a small exact category $\mathcal{C}$ we define a new category $Q\mathcal{C}$ whose objects are the objects of $\mathcal{C}$. The arrows of $Q\mathcal{C}$ from C to D are the isomorphisms of C with an admissible subquotient object of D. Arrows are composed by expressing a subquotient object of a subquotient object of D as a subquotient object of D. ...

It is possible to motivate the definition of $Q\mathcal{C}$ by using exercise 2.6.1 and asking how close $\operatorname{Sub} S.\mathcal{C}$ is to being isomorphic to the nerve of a category. ...

We may also introduce a simplicial set analogous to $G.^{\oplus}\mathcal{C}$ that uses exact sequences instead of just direct sums.

DEFINITION 4.2.11. We define a simplicial set $G.\mathcal{C}$ by letting it send $\underline{n}$ to the set of exact functors $\operatorname{Arr}\Gamma(\underline{n}) \to \mathcal{C}$.

By construction, we see that $\pi_0|G.\mathcal{C}| \cong K_0(\mathcal{C})$. Later we will see that $|G.\mathcal{C}|$ is homotopy equivalent to the loop space of $|S.\mathcal{C}|$, so that $\pi_n|G.\mathcal{C}| \cong K_n(\mathcal{C})$ for all $n \geq 0$.

EXERCISE 4.2.12. Let $\mathcal{L}_R$ denote the groupoid of finitely generated projective R-modules of rank 1. Show that the determinant functor $\mathcal{P}_R \to \mathcal{L}_R$ (define it somewhere else ...) can be used to construct a map $G.\mathcal{P}_R \to \mathcal{L}_R$. Use it to construct a homomorphism $\pi_1 G.\mathcal{P}_R \to R^\times$.

4.3. H-spaces

DEFINITION 4.3.1. An *H-space* is a topological space X with a base point x_0 and a binary operation $\mu : X \times X \to X$ such that the maps $x \mapsto \mu(x_0, x)$ and $x \mapsto \mu(x, x_0)$ are homotopic to the identity.

LEMMA 4.3.2. *Let X be an H-space with operation $\mu : X \times X \to X$. The fundamental group $\pi_1 X$ is abelian, and for all $n \geq 1$, the the map $\mu_* : \pi_n X \times \pi_n X \to \pi_n X$ induced by μ sends (α, β) to $\alpha + \beta$.*

PROOF. The zero element $0 \in \pi_n X$ is represented by the constant map to x_0, so the identity homotopies provide equations $\mu_*(\alpha, 0) = \alpha = \mu_*(0, \alpha)$. The map μ_* is a homomorphism of groups, so $\mu_*(\alpha, \beta) = \mu_*((\alpha, 0) + (0, \beta)) = \mu_*(\alpha, 0) + \mu_*(0, \beta) = \alpha + \beta$. We also see that $\mu_*(\alpha, \beta) = \mu_*((0, \beta) + (\alpha, 0)) = \mu_*(0, \beta) + \mu_*(\alpha, 0) = \beta + \alpha$, and thus $\beta + \alpha = \alpha + \beta$. $\square$

LEMMA 4.3.3. *Direct sum makes $|S.\mathcal{M}|$ into an H-space.*

PROOF. Define a simplicial map $S.\mathcal{M} \times S.\mathcal{M} \to S.\mathcal{M}$ by $(M, N) \mapsto M \oplus N$. The natural isomorphisms $M \oplus 0 \cong M \cong 0 \oplus M$ give rise to the needed homotopies, by 4.2.9. $\square$

The method in the lemma above applies also to $S^{-1}S(\mathcal{M})$, $|S.^\oplus\mathcal{M}|$, $|G.^\oplus\mathcal{M}|$, and $|G.\mathcal{M}|$, all of which are H-spaces.

4.4. Comparison of definitions

THEOREM 4.4.1. *If $\mathcal{C}$ is a small additive category, then there is a natural homotopy equivalence $|G.^\oplus\mathcal{C}| \xrightarrow{\sim} |S^{-1}S(\mathcal{C})|$.*

PROOF. ... $\square$

THEOREM 4.4.2. *If $\mathcal{C}$ is a small additive category, then there is a natural homotopy equivalence $|G.^\oplus\mathcal{C}| \xrightarrow{\sim} \Omega|S.^\oplus\mathcal{C}|$.*

PROOF. ... $\square$

THEOREM 4.4.3. *If $\mathcal{C}$ is a small exact category, then there is a natural homotopy equivalence $|S.\mathcal{C}| \xrightarrow{\sim} |Q\mathcal{C}|$.*

PROOF. ... $\square$

THEOREM 4.4.4. *If $\mathcal{C}$ is a small exact category, then there is a natural homotopy equivalence $|G.\mathcal{C}| \xrightarrow{\sim} \Omega|S.\mathcal{C}|$.*

PROOF. ... ☐

THEOREM 4.4.5. *If $\mathcal{C}$ is a small exact category, then there is a natural homotopy equivalence* $|G.\mathcal{C}| \xrightarrow{\sim} \Omega|S.\mathcal{C}|$.

PROOF. ... ☐

DEFINITION 4.4.6. Suppose $\mathcal{C}$ is an exact category. It is also an additive category, so both $S.^{\oplus}\mathcal{C}$ and $S.\mathcal{C}$ are defined. There is a natural map $S.^{\oplus}\mathcal{C} \to S.\mathcal{C}$ defined as follows. ... There is a natural map $G.^{\oplus}\mathcal{C} \to G.\mathcal{C}$ defined as follows. ...

THEOREM 4.4.7. *If $\mathcal{C}$ is a small exact category in which every short exact sequence splits, then the natural maps $|S.^{\oplus}\mathcal{C}| \to |S.\mathcal{C}|$ and $|G.^{\oplus}\mathcal{C}| \to |G.\mathcal{C}|$ are homotopy equivalences.*

PROOF. ... ☐

Conversely, we may use the following lemma to regard an additive category as an exact category (in which every sequence splits) whenever necessary.

LEMMA 4.4.8. *If $\mathcal{C}$ is a small additive category, then if the sequences isomorphic to one of the form $0 \to C' \to C' \oplus C'' \to C'' \to 0$ are taken as the short exact sequences, the result is an exact category (see Definition 1.2.1).*

PROOF. ... ☐

CHAPTER 5

Basic theorems of K-theory

5.1. The additivity theorem

In this section we offer a proof of Quillen's additivity theorem [13]. Quillen's original proof in [13] is based on the Q-construction, and uses both pullbacks and pushouts. We follow Waldhausen's proof [16, Theorem 1.4.2], based on Waldhausen's S-construction. The proof uses just pushouts or just pullbacks; this offers an important advantage, allowing categories more general than exact categories to be used in K-theory, leading to Waldhausen's development of algebraic K-theory of topological spaces (which is not the same as topological K-theory of topological spaces.

Our proof differs slightly from Waldhausen's in that we use Theorem B' instead of Theorem B. McCarthy gave a proof in [9] using just Theorem A', with some details omitted.

Throughout this section $\mathcal{M}$ is an arbitrary small exact category with a chosen zero object $0 \in \mathcal{M}$.

Definition 5.1.1. For any $n \geq 0$ the simplicial set $S_n\mathcal{M}$ is the set of objects of an exact category $\mathcal{S}_n\mathcal{M}$; the arrows are the natural transformations, and the short exact sequences $0 \to M' \to M \to M'' \to 0$ are those sequences such that for all $0 \leq i \leq j \leq n$ the sequence $0 \to M'_{j/i} \to M_{j/i} \to M''_{j/i} \to 0$ is an exact sequence of $\mathcal{M}$.

To see that $\mathcal{S}_n\mathcal{M}$ is an exact category, suppose $\mathcal{M}$ is a full additive subcategory, closed under extensions, of an abelian category $\mathcal{A}$. The category $\mathrm{Fun}(\mathrm{Arr}\,\underline{n}, \mathcal{A})$ is an abelian category, according to A.40, and we can check that $\mathcal{S}_n\mathcal{M}$ is a full additive subcategory of it closed under extensions. To see that, suppose $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of functors $\mathrm{Arr}\,\underline{n} \to \mathcal{A}$ with M' and M'' in $\mathcal{S}_n\mathcal{M}$. For each $i \leq j$ the exact sequence $0 \to M'_{j/i} \to M_{j/i} \to M''_{j/i} \to 0$ has $M'_{j/i}$ and $M''_{j/i}$ in $\mathcal{M}$, so $M_{j/i}$ is an object of $\mathcal{M}$, too. For each $i \leq j \leq k$ the diagram in Figure 1 has exact rows, exact first column and exact last column. By the 3-by-3 lemma, the middle column is also exact, showing that M is an exact functor $\mathrm{Arr}\,\underline{n} \to \mathcal{M}$, and thus that $M \in \mathcal{S}_n\mathcal{M}$.

Let $\mathcal{E} = \mathcal{E}(\mathcal{M})$ be the exact category isomorphic to $\mathcal{S}_2\mathcal{M}$ whose objects are the short exact sequences $E : 0 \to M \to N \to P \to 0$ of $\mathcal{M}$. Define exact functors $s, t, q : \mathcal{E} \to \mathcal{M}$ that send E to M, N, and P, respectively. In other words, we may display E in the form $0 \to sE \to tE \to qE \to 0$. The letters s, t, and q used here are the initial letters of the phrases "subobject", "total object", and "quotient object".

Theorem 5.1.2 (Additivity). *The map* $K(\mathcal{E}) \xrightarrow{(s,q)} K(\mathcal{M}) \times K(\mathcal{M})$ *is a homotopy equivalence.*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M'_{j/i} \arrow[r] \arrow[d] & M_{j/i} \arrow[r] \arrow[d] & M''_{j/i} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M'_{k/i} \arrow[r] \arrow[d] & M_{k/i} \arrow[r] \arrow[d] & M''_{k/i} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M'_{k/j} \arrow[r] \arrow[d] & M_{k/j} \arrow[r] \arrow[d] & M''_{k/j} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

FIGURE 1.

PROOF. Fix a zero object 0 of $\mathcal{M}$. Let $\ell : \mathcal{M} \to \mathcal{E}$ be the exact functor defined by $M \mapsto (0 \to M \xrightarrow{=} M \to 0 \to 0)$, and let $r : \mathcal{M} \to \mathcal{E}$ be the exact functor defined by $M \mapsto (0 \to 0 \to M \xrightarrow{=} M \to 0)$. The composite functor $q\ell$ is zero.

It is enough to show that the following sequence is a fibration sequence.

addfseq (5.1.1)
$$|S.\mathcal{M}| \xrightarrow{\ell} |S.\mathcal{E}| \xrightarrow{q} |S.\mathcal{M}|$$

For then we will have the following long exact sequence of abelian groups.

$$\cdots \to K_{n+1}\mathcal{M} \to K_n\mathcal{M} \xrightarrow{\ell} K_n\mathcal{E} \xrightarrow{q} K_n\mathcal{M} \to K_{n-1}\mathcal{M} \to \cdots$$

The splittings $s\ell = 1$ and $qr = 1$ will break the long exact sequence up into split short exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & K_n\mathcal{M} \arrow[r, "\ell"'] & K_n\mathcal{E} \arrow[l, bend right, "s"'] \arrow[r, "q"'] & K_n\mathcal{M} \arrow[l, bend right, "r"'] \arrow[r] & 0
\end{tikzcd}$$

Hence the map $K_n\mathcal{E} \xrightarrow{(s,q)} (K_n\mathcal{M})^2$ will be an isomorphism, and the map $|S.\mathcal{E}| \xrightarrow{(s,q)} |S.\mathcal{M}|^2$ will be a weak homotopy equivalence, hence a homotopy equivalence.

In order to prove that (5.1.1) is a fibration sequence, we will apply Theorem B' 3.3.4 to the map $|S.\mathcal{E}| \xrightarrow{S.q} |S.\mathcal{M}|$ and identify the homotopy fiber with $|S.\mathcal{M}|$ via the map ℓ.

For that purpose, consider $m \geq 0$ and $M \in S_m\mathcal{M}$. For $n \geq 0$, an element of $(M/S.q)_n$ is a pair (P, E) with $P \in S.\mathcal{M}(\underline{m} * \underline{n})$ and $E \in S_n\mathcal{E}$ satisfying $M = Pi$ and $Pj = qE$. Here $i : \underline{m} \hookrightarrow \underline{m} * \underline{n}$ and $j : \underline{n} \hookrightarrow \underline{m} * \underline{n}$ are the natural inclusions.

Define a map $\Phi_M : S.\mathcal{M} \to M/S.q$ on a simplex $N \in S_n\mathcal{M}$ by $N \mapsto (Mp, \ell N)$. Here $p : \underline{m} * \underline{n} \to \underline{m}$ is the unique map that splits i in the sense that $pi = 1$; it sends all the elements arising from $\underline{n}$ to the top element m of $\underline{m}$. Since pj factors through the set $\{m'\}$ with one element, Mpj is a degeneracy of the unique 0-simplex of $S.\mathcal{M}$, hence is the zero functor, 0. To check that $\Phi_M(N) \in (M/S.q)_n$ we compute $Mpj = 0 = q\ell N$.

Suppose now that $g : \underline{k} \to \underline{m}$ is an arrow of Ord. The map $g^* : M/S.q \to Mg/S.q$ it induces between naive homotopy fibers is defined by $(P, E) \mapsto (P(g * 1_{\underline{n}}), E)$. Warning: the following diagram does not necessarily commute.

$$\begin{tikzcd} S.\mathcal{M} \arrow[r, "\Phi_M"] \arrow[dr, "\Phi_{Mg}"'] & M/S.q \arrow[d, "g^*"] \\ & Mg/S.q \end{tikzcd}$$

The reason is that the splitting map p, used in the definition of Φ_M, is not natural in $\underline{m}$: it depends on the top element of $\underline{m}$, and perhaps g does not send the top element of $\underline{k}$ to the top element of $\underline{m}$.

So we also consider the map $\Psi_M : M/S.q \to S.\mathcal{M}$ defined by $(P, E) \mapsto sE$. It splits Φ_M in the sense that $\Psi_M \circ \Phi_M = 1$, because $s\ell = 1$. For g as above, it makes the following diagram commute.

$$\begin{tikzcd} M/S.q \arrow[d, "g^*"'] \arrow[dr, "\Psi_M"] & \\ Mg/S.q \arrow[r, "\Psi_{Mg}"'] & S.\mathcal{M} \end{tikzcd}$$

We will show that the maps Ψ_M and Φ_M are inverse homotopy equivalences, for any M. It will follow from the commutative diagram above that the transition map $M/S.q \to Mg/S.q$ is a homotopy equivalence, and hence the hypothesis of Theorem B' is satisfied.

Consider the composite map $\Phi_M \circ \Psi_M : M/S.q \to M/S.q$. It sends a simplex (P, E) to $(Mp, \ell sE) = (Pip, \ell sE)$. We will construct a simplicial homotopy $\Phi_M \circ \Psi_M \sim 1$ as a map $H : \Delta^1 \times (M/S.q) \to M/S.q$. For that purpose, suppose $n \geq 0$ and $(\tau, (P, E)) \in {\Delta^1}_n \times (M/S.q)_n$.

The first component of $H(\tau, (P, E))$ (call it P^τ) will have to interpolate between P and Pip, so we begin by making an order preserving map $h_\tau : \underline{m} * \underline{n} \to \underline{m} * \underline{n}$ that interpolates between 1 and ip; it is defined by $h_\tau(a') = a'$ for $a \in \underline{m}$, by $h_\tau(b'') = m'$ if $\tau(b) = 0$, and by $h_\tau(b'') = b''$ if $\tau(b) = 1$. Evidently, $h_1 = 1$ and $h_0 = ip$, so that h_τ does what we want, and $P^\tau := Ph_\tau$ interpolates between P and Pip, i.e., $P^1 = P$ and $P^0 = Pip$. Notice also that $P^\tau i = Ph_\tau i = Pi = M$, as desired.

The second component of $H(\tau, (P, E))$ (call it E^τ_P) will have to interpolate between E and ℓsE, and it should also satisfy $qE^\tau_P = P^\tau j$. For all $\alpha \in \underline{m} * \underline{n}$ we have the inequality $h_\tau(\alpha) \leq \alpha$. The resulting natural transformation $h_\tau \to 1$ induces a map

pbmap (5.1.2) $$P^\tau j = Ph_\tau j \to Pj = qE.$$

Let E^τ_P be the pullback of E along that map. Here we identify $E \in \mathcal{S}_n\mathcal{E}$ with the short exact sequence $0 \to sE \to tE \to qE \to 0$ in $\mathcal{S}_n\mathcal{M}$, and form the pullback in the exact category $\mathcal{S}_n\mathcal{M}$, which ensures that E^τ_P is an exact sequence in $\mathcal{S}_n\mathcal{M}$, and hence can be identified with an object of $\mathcal{S}_n\mathcal{E}$. Evidently $sE^\tau_P = sE$ and $qE^\tau_P = P^\tau j$.

Let's be more precise about how to form the pullback. We choose in advance all possible pullbacks in $\mathcal{M}$, and then we use those to form $(E^\tau_P)_{j/i}$, for each $i \leq j \in \underline{n}$. (Actually, we will be modifying that shortly, for it doesn't quite work.) That specifies the functor E^τ_P on objects; the values on arrows are determined by the universal property of pullbacks.

Now let's check what happens to E^τ_P at the endpoints of the 1-simplex. When $\tau = 1$ the map (5.1.2) is the identity, and thus $E^1_P \cong E$. When $\tau = 0$ then $P^\tau j = Pipj = 0$, i.e., the source of (5.1.2) is zero, and thus $E^0_P \cong \ell s E$. Isomorphisms aren't the same as identities, so we don't quite get the desired homotopy $\Phi_M \circ \Psi_M \sim 1$. There are two ways to proceed at this point. One way is to check that the natural isomorphisms $E^1_P \cong E$ and $E^0_P \cong \ell s E$ provide homotopies; then we could compose the three homotopies to get the homotopy we want. Another way is to arrange, by artifice, for the two isomorphisms to be identities. It seems, when choosing pullbacks, we should consider whether the source of the arrow (5.1.2) is zero and whether the arrow is an identity. But there is a conflict: both conditions might be fullfilled simultaneously. In that case, for the pullback, we would have to choose between E and $\ell s E$, and they may be different, despite $qE = 0$. We resolve this by peeking at the values of τ on $i \le j \in \underline{n}$: when $\tau(i) = \tau(j) = 1$, then we take $(E^\tau_P)_{j/i} := E_{j/i}$; and when $\tau(i) = \tau(j) = 0$ then we take $(E^\tau_P)_{j/i} := \ell s E_{j/i}$. Only when $\tau(i) = 0$ and $\tau(j) = 1$ do we use the previously mentioned choice of pullbacks in $\mathcal{M}$ to form $(E^\tau_P)_{j/i}$.

Now let's check that H is a simplicial map. We suppose that $r : \underline{c} \to \underline{n}$ is an arrow of Ord and proceed to check that the following square commutes.

$$\begin{array}{ccc}
\Delta^1{}_n \times (M/S.q)_n & \xrightarrow{H} & (M/S.q)_n \\
\Big\downarrow g^* & & \Big\downarrow g^* \\
\Delta^1{}_c \times (M/S.q)_c & \xrightarrow{H} & (M/S.q)_c
\end{array}$$

Using the previous notation, we must show $(H(\tau, (P, E)))r = H(\tau r, (P, E)r)$. Computing both sides, we obtain

$$\begin{aligned}
(H(\tau, (P, E)))r &= (P^\tau, E^\tau_P)r \\
&= (P^\tau(1 * r), E^\tau_P r) \\
&= (Ph_\tau(1 * r), E^\tau_P r)
\end{aligned}$$

and

$$\begin{aligned}
H(\tau r, (P, E)r) &= H(\tau r, (P(1 * r), Er)) \\
&= ((P(1 * r))^{\tau r}, (Er)^{\tau r}_{P(1*r)}) \\
&= (P(1 * r)h_{\tau r}, (Er)^{\tau r}_{P(1*r)})
\end{aligned}$$

so we must show $Ph_\tau(1*r) = P(1*r)h_{\tau r}$ and $E^\tau_P r = (Er)^{\tau r}_{P(1*r)}$. The first equation follows from the identity $h_\tau(1*r) = (1*r)h_{\tau r}$, which follows immediately from the definition of h_τ. For the second equation, let $j' : \underline{c} \hookrightarrow \underline{m} * \underline{c}$ be the natural inclusion. We remark that $E^\tau_P r$ is the pullback of Er along the map $P^\tau jr = Ph_\tau jr \to Pjr = qEr$, whereas $(Er)^{\tau r}_{P(1*r)}$ is the pullback of qEr along the map $P(1 * r)h_{tr}j' \to P(1 * r)j'$. It is enough to check that the two maps are equal, because of the way we are computing pullbacks (objectwise). It follows from the first equation that the sources of the two maps are equal. The maps themselves are equal because they arise from two natural transformations $h_\tau jr \to jr$ and $(1*r)h_{\tau r}j' \to (1*r)j'$ between functors $\underline{c} \to \underline{m} * \underline{n}$. Any two such natural transformations are equal because any arrows in the ordered set $\underline{m} * \underline{n}$ with the same source and target are equal. $\square$

5.2. Applications of the additivity theorem

DEFINITION 5.2.1. A sequence of exact functors $0 \to F \to G \to H \to 0 : \mathcal{M} \to \mathcal{N}$ is *exact* if for each $M \in \mathcal{M}$ the sequence $0 \to F(M) \to G(M) \to H(M) \to 0$ is a short exact sequence of $\mathcal{N}$.

PROPOSITION 5.2.2. *If $0 \to F \to G \to H \to 0 : \mathcal{M} \to \mathcal{N}$ is a sequence of exact functors, then G and $F \oplus H$ induce homotopic maps $|S.\mathcal{M}| \to |S.\mathcal{N}|$.*

PROOF. ... $\square$

COROLLARY 5.2.3. *For all n, we have $F_* + H_* = G_* : K_n\mathcal{M} \to K_n\mathcal{N}$.*

PROOF. Direct sum makes $K(\mathcal{N})$ into an H-space, by 4.3.3, so $(F \oplus H)_* = F_* + H_*$, according to 4.3.2. $\square$

DEFINITION 5.2.4. Given $\underline{m}$ and $\underline{n} \in$ Ord, let $\underline{m} \star \underline{n}$ denote ordered set that is the quotient of the join $\underline{m} * \underline{n}$ obtained by identifying the top element of $\underline{m}$ (actually, m') with the bottom element of $\underline{n}$ (actually, $0''$). It is isomorphic to $\underline{m+n}$. Warning: this construction is not natural in $\underline{m}$ or in $\underline{n}$, because maps of Ord need not preserve top or bottom elements.

An n-simplex of $S.\mathcal{M}$ may be thought of as a filtration of length n with extra information. The following lemma shows that such a filtration of length $m + n$ is canonically an extension of a filtration of length n by a filtration of length m.

LEMMA 5.2.5. *Let $\mathcal{M}$ be an exact category. Let $i : \underline{m} \hookrightarrow \underline{m} \star \underline{n}$ and $j : \underline{n} \hookrightarrow \underline{m} \star \underline{n}$ be the natural inclusions, and let $p : \underline{m} \twoheadleftarrow \underline{m} \star \underline{n}$ and $q : \underline{n} \twoheadleftarrow \underline{m} \star \underline{n}$ be the corresponding splittings. For each $M \in \mathcal{S}.\mathcal{M}(\underline{m} \star \underline{n})$, there is a short exact sequence $0 \to Mip \to M \to Mjq \to 0$ in $\mathcal{S}.\mathcal{M}$ that is natural in M.*

PROOF. The element $\gamma = m' = 0''$ comes both from $\underline{m}$ and $\underline{n}$. The maps ip and jq are the identity on one side of γ and send the elements on the other side to γ. For every $\alpha \in \underline{m} \star \underline{n}$ we have $ip(\alpha) \le \alpha \le jq(\alpha)$. The corresponding natural transformations $ip \to 1 \to jq$ provide the maps $Mip \to M \to Mjq$. Now for each $\alpha \le \beta \in \underline{m} \star \underline{n}$ we must check that $0 \to M_{ip(\beta)/ip(\alpha)} \to M_{\beta/\alpha} \to M_{jq(\beta)/jq(\alpha)} \to 0$ is an exact sequence of $\mathcal{M}$. If α comes from $\underline{m}$ and β comes from $\underline{n}$, then the sequence has the form $0 \to M_{\gamma/\alpha} \to M_{\beta/\alpha} \to M_{\beta/\gamma} \to 0$ and thus is exact. If both α and β come from $\underline{m}$, then the sequence reduces to $0 \to M_{\beta/\alpha} \to M_{\beta/\alpha} \to 0 \to 0$, which is exact. If both α and β come from $\underline{n}$, then the sequence reduces to $0 \to 0 \to M_{\beta/\alpha} \to M_{\beta/\alpha} \to 0$, which is exact. $\square$

COROLLARY 5.2.6. *The maps induced by i and j provide a homotopy equivalence $S.\mathcal{S}_{m+n}\mathcal{M} \xrightarrow{(i,j)} S.\mathcal{S}_m\mathcal{M} \times S.\mathcal{S}_n\mathcal{M}$.*

PROOF. ... $\square$

COROLLARY 5.2.7. *The maps induced by j and p provide a fibration sequence $|S.\mathcal{S}_m\mathcal{M}| \xrightarrow{p} |S.\mathcal{S}_{m+n}\mathcal{M}| \xrightarrow{j} |S.\mathcal{S}_n\mathcal{M}|$.*

PROOF. ... $\square$

COROLLARY 5.2.8. *The map $\underline{m} * \underline{0} \twoheadleftarrow \underline{m} * \underline{n}$ arising from the map $\underline{0} \twoheadleftarrow \underline{n}$ and the canonical inclusion $\underline{m} * \underline{n} \hookleftarrow \underline{n}$ provide a fibration sequence $|S.\mathcal{S}_{m+1}\mathcal{M}| \to |S.(\mathcal{S}.\mathcal{M}(\underline{m} * \underline{n}))| \to |S.\mathcal{S}_n\mathcal{M}|$ that is natural in $\underline{n}$.*

PROOF. Identify $\underline{m} * \underline{0}$ with $\underline{m+1}$ and $\underline{m+1} \star \underline{n}$ with $\underline{m} * \underline{n}$. Naturality in the variable $\underline{n}$ for a map $\underline{k} \to \underline{n}$ follows from the commutativity of the following diagram.

$$\begin{tikzcd}
 & \underline{m} * \underline{k} \arrow[dl] \arrow[d] & \underline{k} \arrow[l] \arrow[d] \\
\underline{m} * \underline{0} & \underline{m} * \underline{n} \arrow[l] & \underline{n} \arrow[l]
\end{tikzcd}$$

□

DEFINITION 5.2.9. Let $\mathcal{E}.\mathcal{M}$ be the simplicial exact category defined by $\underline{n} \mapsto \mathcal{E}_n\mathcal{M} := \mathcal{S}.\mathcal{M}(\underline{0} * \underline{n})$.

COROLLARY 5.2.10. *For each $n \geq 0$ there is a fibration sequence $|S.\mathcal{M}| \to |S.\mathcal{E}_n\mathcal{M}| \to |S.\mathcal{S}_n\mathcal{M}|$, natural in n.*

PROOF. Take $m = 0$ in the previous corollary. □

DEFINITION 5.2.11. Let $S.\mathcal{S}.\mathcal{M}$ be the bisimplicial set defined by $(\underline{m}, \underline{n}) \mapsto S_m\mathcal{S}_n\mathcal{M}$. Let $S.\mathcal{E}.\mathcal{M}$ be the bisimplicial set defined by $(\underline{m}, \underline{n}) \mapsto S_m\mathcal{E}_n\mathcal{M}$.

COROLLARY 5.2.12. *There is a fibration sequence $|S.\mathcal{M}| \to |S.\mathcal{E}.\mathcal{M}| \to |S.\mathcal{S}.\mathcal{M}|$.*

PROOF. Regard $S.\mathcal{M}$ as the bisimplicial set defined by $(\underline{m}, \underline{n}) \mapsto S_m\mathcal{M}$. Then we get a sequence of bisimplicial sets $S.\mathcal{M} \to S.\mathcal{E}.\mathcal{M} \to S.\mathcal{S}.\mathcal{M}$ which is a fibration sequence in each row. For each n the space $|S.\mathcal{S}_n\mathcal{M}|$ is connected. Now apply 3.2.2. □

CHAPTER 6

Low dimensional K-groups of rings

6.1. K-theory of rings

... define $K^{\oplus}(\mathcal{M}) = |G.^{\oplus}\mathcal{M}|$, $K(R) = |G.^{\oplus}\mathcal{P}_R|$ or $K(R) = |S^{-1}S(\mathcal{P}_R)|$ and $K_n(R) = \pi_n K(R)$.

6.2. $K_1(R)$

EXERCISE 6.2.1. Let $\mathcal{C}$ be a small additive category, and let I be the set of isomorphism classes $[C]$ of objects C of $\mathcal{C}$, and consider I as a commutative monoid with $[C] + [C'] = [C \oplus C']$. Let J be the translation category for I acting on I. Show J is filtering. For each $i \in I$ pick C_i with $i = [C_i]$. Show that $K_1^{\oplus}\mathcal{C} \cong \operatorname{colim}_J \operatorname{Aut}(C_i)^{\mathrm{ab}}$, where the transition maps in the inductive system arise by direct sum with identity automorphisms.

... compute $K_1^{\oplus}R = \mathrm{Gl}(R)^{\mathrm{ab}} = \mathrm{Gl}(R)/\mathrm{E}(R)$; show $\mathrm{E}(R)$ is perfect. For R a field, commutative local ring, or Euclidean domain, show $K_1(R) = R^{\times}$.

6.3. Homology of $\mathrm{Gl}(R)$

... generalities about homology: $H^n(|X|, \mathbb{Z}) = H^n(C\mathbb{Z}[X])$, $H_n(G, \mathbb{Z}) = H_n(BG, \mathbb{Z})$, bar resolution; Hurewicz theorem: $\pi_j(T) = 0$ for $j < n$ implies $\pi_n(T)^{\mathrm{ab}} = H^n(T, \mathbb{Z})$. Mention $H^n(B\,\mathrm{Gl}(R), \mathbb{Z}) = H^n(K(R), \mathbb{Z})$, the map $B\,\mathrm{Gl}(R) \to K(R)$ is acyclic, the plus-construction $B\,\mathrm{Gl}(R)^+$, and functoriality of it, gotten by adding one 2-cell and one 3-cell; $BE(R)^+$ is the universal covering space of $K(R)$, so $K_2(R) = H_2(E(R), \mathbb{Z})$.

6.4. $K_2(R)$

Matsumoto's presentation for $K_2(F)$ (see [11]).

APPENDIX A

Category Theory

DEFINITION A.1. A *category* $\mathcal{C}$ is a collection $\operatorname{Obj}\mathcal{C}$ of "objects" and a collection $\operatorname{Arr}\mathcal{C}$ of "arrows" or "maps". The notation $C \in \mathcal{C}$ will mean $C \in \operatorname{Obj}\mathcal{C}$, and the notation $f \operatorname{in} \mathcal{C}$ will mean $f \in \operatorname{Arr}\mathcal{C}$. Each arrow f has a "source" object $\operatorname{src} f$ and a target object $\operatorname{tar} f$. The notation $f : A \to B$ means $A = \operatorname{src} f$ and $B = \operatorname{tar} f$. For each object $C \in \operatorname{Obj}\mathcal{C}$ there is provided an "identity" arrow $1_{\mathrm{C}} \in \mathcal{C}$. For any arrows $f : A \to B$ and $g : B \to C$ there is provided a "composite" arrow $g \circ f : A \to C$. Composition of arrows is associative: $h \circ (g \circ f) = (h \circ g) \circ f$ and "identity" arrows are identities for composition: $f \circ 1_{\mathrm{A}} = f = 1_{\mathrm{B}} \circ f$. We use $\operatorname{Hom}_{\mathcal{C}}(A, B)$ or simply $\operatorname{Hom}(A, B)$ to denote the collection of arrows $f : A \to B$. We assume further that $\operatorname{Hom}_{\mathcal{C}}(A, B)$ is always a set.

DEFINITION A.2. An *isomorphism* is ...

DEFINITION A.3. A *groupoid* is a category all of whose arrows are isomorphisms.

DEFINITION A.4. If $\mathcal{C}$ is a category, we define $\operatorname{Iso}\mathcal{C}$ to be the subcategory of $\mathcal{C}$ whose arrows are the isomorphisms of $\mathcal{C}$. It is a groupoid.

DEFINITION A.5. A functor $H : \mathcal{C} \to \mathcal{D}$ that sends every arrow to an isomorphism is called *morphism inverting*. A morphism inverting functor is the same thing as a functor $H : \mathcal{C} \to \operatorname{Iso}\mathcal{D}$.

DEFINITION A.6. A *small* category is a category $\mathcal{C}$ where $\operatorname{Obj}\mathcal{C}$ is a set. Since we have already assumed that each collection $\operatorname{Hom}_{\mathcal{C}}(A, B)$ is a set, it follows that $\operatorname{Arr}\mathcal{C}$ is also a set.

DEFINITION A.7. A *linear* category is a category where each $\operatorname{Hom}_{\mathcal{C}}(A, B)$ is provided with an addition operation $f + g$ that makes it into an abelian group. In addition, composition is bilinear: $(f+g)\circ h = f\circ h + g\circ h$ and $f\circ(g+h) = f\circ g + f\circ h$.

DEFINITION A.8. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $G : \mathcal{C} \to \mathcal{D}$ assigns to each object $C \in \mathcal{C}$ an object $G(C) \in \mathcal{D}$ and assigns to each arrow $f : C \to C'$ of $\mathcal{C}$ an arrow $G(f) : G(C) \to G(C')$. It sends identity arrows to identity arrows, $G(1_{\mathrm{C}}) = 1_{\mathrm{G(C)}}$, and is compatible with composition, $G(g \circ f) = G(g) \circ G(f)$.

DEFINITION A.9. A *natural transformation* $p : F \to G$ is ...

DEFINITION A.10. ... composition of natural transformation with functors ...

DEFINITION A.11. A *natural isomorphism* $F \cong G$ is ...

DEFINITION A.12. A *faithful* functor $F : \mathcal{C} \to \mathcal{D}$ is a functor such that for any objects C and C' of $\mathcal{C}$, the function $F : \operatorname{Hom}_{\mathcal{C}}(C, C') \to \operatorname{Hom}_{\mathcal{D}}(FC, FC')$ is injective. If the function is always a bijection, then F is called a *fully* faithful functor or a *fully* faithful embedding.

Definition A.13. A *subobject* B of an object C is an equivalence class of monomorphisms $B \rightarrowtail C$, where two monomorphisms $B \rightarrowtail C$ and $B' \rightarrowtail C$ are equivalent if there is an isomorphism $B \xrightarrow{\cong} B'$ making the following diagram commute.

$$\begin{tikzcd} B \arrow[r] \arrow[d, "\cong"'] & C \\ B' \arrow[ur] & \end{tikzcd}$$

Definition A.14. In an additive category $\mathcal{C}$, the *direct sum* of a finite collection $\{C_1, \ldots, C_n\}$ of objects is an object C together with "inclusion" maps $\mathrm{in}_i : C_i \to C$ and "projection" maps $\mathrm{pr}_i : C \to C_i$ for $1 \leq i \leq n$. These maps satisfy the following identities: $\mathrm{pr}_i \circ \mathrm{in}_i = 1_{\mathrm{C_i}}$; $\mathrm{pr}_i \circ \mathrm{in}_j = 0$ if $i \neq j$; and $\sum_{i=1}^{n} \mathrm{in}_i \circ \mathrm{pr}_i = 1_{\mathrm{C}}$. We will write $C = C_1 \oplus \cdots \oplus C_n$.

Definition A.15. An *additive* category is an additive category $\mathcal{C}$ where any finite collection $\{C_1, \ldots, C_n\}$ of objects has a direct sum. The case where $n = 0$ and the collection is empty is included, providing an object called 0 whose identity arrow is zero.

Definition A.16. A *diagram* in a category $\mathcal{C}$ is a functor $F : \mathcal{I} \to \mathcal{C}$ where $\mathcal{I}$ is a small category, called the *index category*. The objects and arrows of $\mathcal{I}$ may be thought of as the "names" of the objects and arrows of the diagram, i.e., $i \in \mathcal{I}$ is the name attached to $F(i)$. The *empty* diagram is the diagram indexed by the empty category.

We often illustrate a diagram F by displaying just the objects $F_i := F(i)$ and the arrows $F(f)$ for $f : i \to j$. The shape of the index category can be inferred. Here are some examples of diagrams.

$$\begin{tikzcd} F_0 \arrow[r] \arrow[d] & F_1 \\ F_2 & \end{tikzcd} \qquad \begin{tikzcd} G_0 \arrow[r, shift left] \arrow[r, shift right] & G_1 \end{tikzcd}$$

$$\begin{tikzcd} H_0 \arrow[r] & H_1 & H_2 \end{tikzcd}$$

Definition A.17. The *limit* of a diagram $F : \mathcal{I} \to \mathcal{C}$ is an object C together with a map $C \to F(i)$ for each object $i \in \mathcal{I}$, such that, for every arrow $f : i \to j$ in $\mathcal{I}$, the triangle

$$\begin{tikzcd} C \arrow[r] \arrow[dr] & F(i) \arrow[d, "F(f)"] \\ & F(j) \end{tikzcd}$$

commutes. Moreover, C, with its arrows, is universal in the sense that if there is another object C' together with a map $C' \to F(i)$ for each object $i \in \mathcal{I}$, such that, for every arrow $f : i \to j$ in $\mathcal{I}$, the triangle

$$\begin{tikzcd} C' \arrow[r] \arrow[dr] & F(i) \arrow[d, "F(f)"] \\ & F(j) \end{tikzcd}$$

commutes, then there is a unique arrow $C' \to C$, such that, for every object $i \in \mathcal{I}$, the diagram

$$\begin{tikzcd} C' \arrow[d] \arrow[dr] & \\ C \arrow[r] & F(i) \end{tikzcd}$$

commutes.

We write $C = \lim F$ or $C = \lim_{i \in \mathcal{I}} F(i)$.

Another way to think about a limit is that it amounts to a universal natural transformation from a constant functor to the functor F.

The limit may not exist, but if it does exist it is unique up to a unique isomorphism.

DEFINITION A.18. A *terminal* or *final* object is the limit of the empty diagram. Alternatively, there is a unique arrow to it from any object.

DEFINITION A.19. A category is called *discrete* if every arrow in it is an identity arrow.

A discrete category is determined, up to isomorphism, by its set of objects. A diagram indexed by a discrete category amounts to the same things as an indexed collection of objects.

DEFINITION A.20. The *product* of an collection of objects $\{F_i \mid i \in \mathcal{I}\}$ indexed by a set $\mathcal{I}$ is the limit of the corresponding diagram indexed by the corresponding discrete category.

DEFINITION A.21. The *pullback* of a diagram

$$\begin{tikzcd} & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

is the limit of the diagram.

DEFINITION A.22. The *equalizer* of a pair of arrows $A \rightrightarrows B$ is the limit of the diagram.

DEFINITION A.23. In an additive category the *kernel* of a map $f : A \to B$ is the limit of the following diagram and is denoted by $\ker f$.

$$\begin{tikzcd} & 0 \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

LEMMA A.24 (Functoriality of limits). *Suppose $\mathcal{I}$ and $\mathcal{J}$ are small categories, $F : \mathcal{I} \to \mathcal{C}$ and $G : \mathcal{J} \to \mathcal{C}$ are functors, and the limits $\lim F$ and $\lim G$ exist. If $H : \mathcal{J} \to \mathcal{I}$ is a functor and $\eta : FH \to G$ is a natural transformation, then there is a unique map $h : \lim F \to \lim G$ such that, for every object $j \in \mathcal{J}$, the following* *diagram commutes.*

$$\begin{tikzcd}
\lim F \arrow[r] \arrow[d, "h"'] & F(H(j)) \arrow[d, "\eta_j"] \\
\lim G \arrow[r] & G(j)
\end{tikzcd}$$

PROOF. The proof proceeds by applying the universal property defining $\lim G$, using the following commutative diagram, where $f : j \to j'$ is an arbitrary arrow of $\mathcal{J}$.

$$\begin{tikzcd}
 & & F(H(j')) \arrow[dd, "\eta_{j'}"] \\
\lim F \arrow[urr] \arrow[r] \arrow[drr] \arrow[ddr] \arrow[dd, dashed, "\exists!"'] & F(H(j)) \arrow[ur, "F(H(f))"] \arrow[dd, "\eta_j"] & \\
 & & G(j') \\
\lim G \arrow[urr] \arrow[r] & G(j) \arrow[ur, "G(f)"'] &
\end{tikzcd}$$

□

Now we repeat all the definitions above, with the arrows going the other way, and the prefix “co-” attached to the terms. Alternatively, one could work in the opposite category.

DEFINITION A.25. The *colimit* of a diagram $F : \mathcal{I} \to \mathcal{C}$ is an object C together with a map $F(i) \to C$ for each object $i \in \mathcal{I}$, such that, for every arrow $f : i \to j$ in $\mathcal{I}$, the triangle

$$\begin{tikzcd}
F(i) \arrow[r] \arrow[d, "F(f)"'] & C \\
F(j) \arrow[ur] &
\end{tikzcd}$$

commutes. Moreover, C, with its arrows, is universal in the sense that if there is another object C' together with a map $F(i) \to C'$ for each object $i \in \mathcal{I}$, such that, for every arrow $f : i \to j$ in $\mathcal{I}$, the triangle

$$\begin{tikzcd}
F(i) \arrow[r] \arrow[d, "F(f)"'] & C' \\
F(j) \arrow[ur] &
\end{tikzcd}$$

commutes, then there is a unique arrow $C \to C'$, such that, for every object $i \in \mathcal{I}$, the diagram

$$\begin{tikzcd}
F(i) \arrow[r] \arrow[dr] & C \arrow[d] \\
 & C'
\end{tikzcd}$$

commutes.

We write $C = \operatorname{colim} F$ or $C = \operatorname{colim}_{i \in \mathcal{I}} F(i)$.

Another way to think about a colimit is that it amounts to a universal natural transformation from the functor F to a constant functor.

The colimit may not exist, but if it does exist it is unique up to a unique isomorphism.

DEFINITION A.26. An *initial* object is the colimit of the empty diagram. Alternatively, there is a unique arrow from it to any other object in the category.

DEFINITION A.27. The *coproduct* of an collection of objects $\{F_i \mid i \in \mathcal{I}\}$ indexed by a set $\mathcal{I}$ is the colimit of the corresponding diagram indexed by the corresponding discrete category.

DEFINITION A.28. The *pushout* of a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \\ C & \end{tikzcd}$$

is the colimit of the diagram.

DEFINITION A.29. The *coequalizer* of a pair of arrows $A \rightrightarrows B$ is the colimit of the diagram.

DEFINITION A.30. In an additive category the *cokernel* of a map $f : A \to B$ is the colimit of the following diagram and is denoted by $\operatorname{coker} f$.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \\ 0 & \end{tikzcd}$$

LEMMA A.31 (Functoriality of colimits). *Suppose $\mathcal{I}$ and $\mathcal{J}$ are small categories, $F : \mathcal{I} \to \mathcal{C}$ and $G : \mathcal{J} \to \mathcal{C}$ are functors, and the colimits* $\operatorname{colim} F$ *and* $\operatorname{colim} G$ *exist. If $H : \mathcal{I} \to \mathcal{J}$ is a functor and $\eta : F \to GH$ is a natural transformation, then there is a unique map $h : \operatorname{colim} F \to \operatorname{colim} G$ such that, for every object $i \in \mathcal{I}$, the following diagram commutes.*

$$\begin{tikzcd} F(i) \arrow[r] \arrow[d, "\eta_i"] & \operatorname{colim} F \arrow[d, "h"] \\ G(H(i)) \arrow[r] & \operatorname{colim} G \end{tikzcd}$$

PROOF. The proof proceeds by applying the universal property defining $\operatorname{colim} G$, using the following commutative diagram, where $f : i \to i'$ is an arbitrary arrow of

$\mathcal{I}$.

$$
\begin{tikzcd}
 & F(i') \arrow[r] \arrow[dd, "\eta_{i'}"] \arrow[ddr] & \mathrm{colim}\, F \arrow[dd, dashed, "\exists!"] \\
F(i) \arrow[ur, "F(f)"] \arrow[urr] \arrow[drr] \arrow[dd, "\eta_i"'] & & \\
 & G(H(i')) \arrow[r] & \mathrm{colim}\, G \\
G(H(i)) \arrow[ur, "G(H(f))"] \arrow[urr] & &
\end{tikzcd}
$$

□

DEFINITION A.32. A category $\mathcal{I}$ is *filtering* if ...

(Insert here some discussion of filtered colimits ...)

LEMMA A.33. *Let $\mathcal{F}$ be a collection of subsets of a set S which is closed under intersection, i.e., if $T \in \mathcal{F}$ and $T' \in \mathcal{F}$, then $T \cap T' \in \mathcal{F}$. Regard $\mathcal{F}$ as a diagram in which the arrows are the inclusions of one subset into another. Then the natural map $\eta : \mathrm{colim}_{T \in \mathcal{F}} T \to \bigcup_{T \in \mathcal{F}} T$ is an bijection.*

PROOF. We may regard the colimit $\mathrm{colim}_{T \in \mathcal{F}} T$ as being defined by generators (T, t) with $t \in T \in \mathcal{F}$ and by relations $(T, t) = (T', t)$ when $t \in T \subseteq T'$. The map η sends the generator (T, t) to t, and is surjective. To show injectivity, consider two generators, (T, t) and (T', t), with the same image under η. The relations $(T, t) = (T \cap T', t) = (T', t)$ show they give the same element in the colimit. □

DEFINITION A.34. In an additive category $\mathcal{C}$ where kernels and cokernels always exist, we define the *image* of a map $f : A \to B$ to be the kernel of the map $B \to \mathrm{coker}\, f$, and it is denoted by $\mathrm{im}\, f$.

DEFINITION A.35. In an additive category $\mathcal{C}$ where kernels and cokernels always exist, we define the *coimage* of a map $f : A \to B$ to be the cokernel of the map $\ker f \to A$. It is denoted by $\mathrm{coim}\, f$.

DEFINITION A.36. An *abelian* category is an additive category $\mathcal{C}$ where every map $f : A \to B$ has a kernel and a cokernel, and the natural map $\mathrm{coim}\, f \to \mathrm{im}\, f$ is always an isomorphism. An alternative for the latter condition is that every mononomorphism in $\mathcal{C}$ should be a kernel, and every epimorphism in $\mathcal{C}$ should be a cokernel.

DEFINITION A.37. An exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$ in an abelian category is one where $\ker g$ and $\mathrm{im}\, f$ are equivalent subobjects of B.

DEFINITION A.38. An *exact* functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories is an additive functor which sends exact sequences to exact sequences.

The main fact about abelian categories is that all the standard diagram chasing lemmas for diagrams of R-modules remain valid in any abelian category. These include the snake lemma, the 3-lemma, the 5-lemma, the 3-by-3 lemma, and all the diagram chasing arguments used in homological algebra. See the section "Diagram Lemmas" in [8, VIII.4] for a device to establish that statement; it involves "members" of an object of an abelian category as a replacement for elements of modules. Alternatively, a small abelian category can be embedded into a category of R-modules for a suitable ring R by a fully faithful exact functor; see [5, 12].

DEFINITION A.39. Let $\mathcal{A}$ be a category, and let $\mathcal{I}$ be a small category. The category $\mathrm{Fun}(\mathcal{I},\mathcal{A})$ has as its objects functors $M:\mathcal{I}\to\mathcal{A}$. The arrows $M'\to M$ are the natural transformations.

LEMMA A.40. *Let $\mathcal{A}$ be an abelian category, and let $\mathcal{I}$ be a small category. The category $\mathrm{Fun}(\mathcal{I},\mathcal{A})$ is an abelian category.*

PROOF. The kernel of an arrow $f:M'\to M$ of $\mathrm{Fun}(\mathcal{I},\mathcal{A})$ can be constructed by defining $(\ker f)(i) := \ker(f(i):M'(i)\to M(i))$. Similarly for cokernels, and an exact sequence $0\to M'\to M\to M''\to 0$ is a sequence of functors such that for all objects $i\in\mathcal{I}$, the sequence $0\to M'(i)\to M(i)\to M''(i)\to 0$ is an exact sequence of $\mathcal{A}$. $\square$

DEFINITION A.41. Supose $\mathcal{C}$ and $\mathcal{D}$ are categories and $F:\mathcal{C}\to\mathcal{D}$ and $G:\mathcal{D}\to\mathcal{C}$ are functors. The functors F and G are called *adjoint functors*, F is called the left adjoint functor of G, and G is called the right adjoint functor of F, if there is an isomorphism

adjeqn
$$\beta_{C,D}:\mathrm{Hom}_{\mathcal{C}}(C,G(D)) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{D}}(F(C),D) \tag{A.1}$$

which is natural in both variables C and D. The triple consisting of F, G, and β is called an *adjunction*.

REMARK A.42. Associated with an adjunction as above is a pair of natural transformations $\eta:1_{\mathcal{C}}\to GF$ (called the *unit* of the adjunction) and $\epsilon:FG\to 1_{\mathcal{D}}$ (called the *counit* of the adjunction). For example, to obtain η substitute $D = F(C)$ in (A.1) to get a natural isomorphism $\beta_{C,F(C)}:\mathrm{Hom}_{\mathcal{C}}(C,G(F(C)))\xrightarrow{\cong}\mathrm{Hom}_{\mathcal{D}}(F(C),F(C))$. Define $\epsilon_C = \beta^{-1}_{C,F(C)}(1_{\mathrm{F(C)}})$. Naturality of $\beta_{C,D}$ can be used to show that ϵ is a natural transformation.

For more details on adjunctions, see [8, IV.1].

DEFINITION A.43. We say that a functor $G:\mathcal{C}\to\mathcal{D}$ *preserves limits* if, whenever $F:\mathcal{I}\to\mathcal{C}$ is a diagram in $\mathcal{C}$ whose limit $\lim F$ exists, the natural maps $G(\lim F)\to G(F(i))$ express $G(\lim F)$ as the limit of the diagram GF.

DEFINITION A.44. We say that a functor $G:\mathcal{C}\to\mathcal{D}$ *preserves colimits* if, whenever $F:\mathcal{I}\to\mathcal{C}$ is a diagram in $\mathcal{C}$ whose limit $\mathrm{colim}\, F$ exists, the natural maps $G(F(i))\to G(\lim F)$ express $G(\mathrm{colim}\, F)$ as the colimit of the diagram GF.

LEMMA A.45. *Any functor that is a left adjoint preserves colimits. Any functor that is a right adjoint preserves limits.*

PROOF. We prove just the first statement. Suppose $F:\mathcal{C}\to\mathcal{D}$ is left adjoint to the functor $G:\mathcal{D}\to\mathcal{C}$, and suppose $C = \mathrm{colim}_i\, C_i$ in $\mathcal{C}$. Then we have the following sequence of natural isomorphisms.

$$\begin{aligned}
\operatorname{Hom}(F(C), D) &\cong \operatorname{Hom}(C, G(D)) \\
&= \operatorname{Hom}(\operatorname*{colim}_i C_i, G(D)) \\
&\cong \lim_i \operatorname{Hom}(C_i, G(D)) \\
&\cong \lim_i \operatorname{Hom}(F(C_i), D) \\
&\cong \operatorname{Hom}(\operatorname*{colim}_i F(C_i), D)
\end{aligned}$$

By (2.3.2) applied to $\mathcal{C}^{\mathrm{op}}$ we see that $F(C) \cong \operatorname{colim}_i F(C_i)$. $\square$

APPENDIX B

Notation

$\times$ as in $R^\times$, which denotes the multiplicative group of units in the ring R.
$\times_G$ as in $T \times_G S$, where G is a group acting on T on the right and acting on S on the left, denotes the set defined by generators $(t, s) \in T \times S$ and relations $(t, gs) = (tg, s)$ for all $g \in G$.
$\sim$ as in $x \sim y$, is used occasionally and briefly to denote an equivalence relation.
$\sim$ as in $f \sim g$, which means f and g are homotopic continuous maps.
$\sim$ as in $h : f \sim g$, which means h is a homotopy between f and g.
$+$ as in m^+, which denotes the pointed set $\{*, 1, 2, \ldots, m\}$.
$\simeq$ as in $f : X \xrightarrow{\simeq} Y$, which means the map f is a homotopy equivalence.
$\simeq$ as in $X \simeq Y$, means X and Y are homotopy equivalent topological spaces.
$\in$ as in $C \in \mathcal{C}$, where $\mathcal{C}$ is a category, means that C is an object of $\mathcal{C}$.
$\cong$ as in $X \cong Y$, which means X and Y are isomorphic objects of a category.
$\cong$ as in $f : X \xrightarrow{\cong} Y$ means the map f is an isomorphism.
$T \sqcup U$ is the disjoint union of two sets.
$*$ is the basepoint of a pointed set or space.
F_* is the map on K-groups induced by an exact functor, see 4.2.8.
$T * U$ is the join of two partially ordered sets, see 3.1.1.
$X * Y$ is the join of two topological spaces or of two simplicial sets, see 3.1.3.
T^* is the partially ordered set T with its finite subsets neglected; see 2.4.2.
$[0, 1]^*$ is the unit interval with finite subsets neglected; see 2.4.1 and 2.4.2.
$\underline{n}$ is the partially ordered set $\{0 < 1 < \cdots < n\}$, often regarded as a category. See 2.3.9.
$\#$ as in $\#A$, which denotes the cardinality of the set A.
1_X is the identity map $X \to X$.
ϕ is the empty set.
ΩX is the loop space of a pointed topological space X.
fx where $X : \mathcal{C} \to \mathrm{Set}$ is a functor, $C \in \mathcal{C}$, $x \in X(C)$, and $f : C \to C'$ is an arrow of $\mathcal{C}$, denotes $(X(f))(x)$.
xf where $X : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is a functor, $C \in \mathcal{C}$, $x \in X(C)$, and $f : C' \to C$ is an arrow of $\mathcal{C}$, denotes $(X(f))(x)$.
$\mathbb{C}$ is the field of complex numbers.
Cat is the category of small categories. The arrows are the functors.
Cov_X is the category of covering spaces of X.
D_k is the standard simplex presented in alternate coordinates. See 2.4.1.
Δ^n is the simplicial simplex of dimension n; it is the simplicial set represented by $\underline{n}$. See 2.3.9.
$|\Delta^n|$ is the standard simplex of dimension n. See 2.1.1.
FPSet is the category of finite pointed sets.
$\mathrm{Gl}_n\, R$ is the group of n by n invertible matrices with entries in the ring R.

I is the unit interval $[0,1]$.

in_j denotes the inclusion map corresponding to the j-th factor of a coproduct, or, in the case of pointed objects, a product.

$\mathrm{Iso}\,\mathcal{C}$ is the subcategory of the category $\mathcal{C}$ whose arrows are the isomorphisms of $\mathcal{C}$.

$K_0^{\oplus}\mathcal{C}$ is the direct sum Grothendieck group of an additive category, see 1.1.1.

$K_n^{\oplus}\mathcal{C}$ is the direct sum K-group of an additive category, see 4.1.4.

$K_0\mathcal{C}$ is the Grothendieck group of an exact category, see 1.2.3.

$\mathrm{Map}(\mathcal{C},\mathcal{D})$, where $\mathcal{C}$ and $\mathcal{D}$ are small categories, is the category of functors $\mathcal{C}\to\mathcal{D}$, in which the arrows are the natural transformations between such functors.

$\mathrm{Map}(X,Y)$, where X and Y are simplicial sets, is the simplicial set of maps from X to Y, in which an n-simplex is a map $|\Delta^n|\times X\to Y$. If either X or Y is a category, we replace it by its nerve to reduce to the previous case.

$\mathcal{M}_R$ is the exact category of finitely generated R-modules.

$\mathbb{N}=\{0,1,2,3,\dots\}$ is the set of natural numbers.

op as in $\mathcal{C}^{\mathrm{op}}$, where $\mathcal{C}$ is a category, which denotes the opposite category of $\mathcal{C}$, or as in T^{op}, where T is a partially ordered set, which denotes the opposite partially ordered set.

Ord is the category of finite nonempty (totally) ordered sets of the form $\underline{n}$ for some $n\in\mathbb{N}$. See 2.3.9.

pr_j denotes the projection onto the j-th factor of a product.

$\mathcal{P}_R$, where R is a ring, denotes the exact category of finitely generated projective left R-modules; see 1.2.9.

$\mathrm{Pre}\,\mathcal{O}$, where $\mathcal{O}$ is a small category, denotes the category of presheaves on $\mathcal{O}$.

$\mathcal{P}_X$, where X is a topological space, denotes the exact category of real vector bundles of finite rank over X; see 1.2.11.

Poset is the category of partially ordered sets, sets T with an ordering satisfying $a\le b\le c \implies a\le c$ and $a\le b\le a \implies a=b$. An arrow $f:T\to T'$ is a function satisfying $a\le b \implies f(a)\le f(b)$.

Poset^* is a category whose objects are partially ordered sets equipped with a family of essential subsets. See 2.4.2 for details.

$\mathbb{Q}$ is the field of rational numbers.

$\mathbb{R}$ is the field of real numbers.

$R^\times$ is the multiplicative group of units in the ring R.

$S.\mathcal{C}$ is the S-construction of Waldhausen, a simplicial set; see 4.2.4

$\mathcal{S}.\mathcal{C}$ is the S-construction of Waldhausen, a simplicial exact category; see 5.1.1

Set denotes the category of sets.

Simp is the category of standard simplices. See 2.3.8.

$\mathrm{Sub}\,\mathcal{I}$ is a certain category of arrows of $\mathcal{I}$, see 3.1.5.

Set^G, where G is a group, denotes the category of left G-sets.

Top is the category of topological spaces.

U_n is the Lie group of n by n unitary matrices.

$\mathbb{Z}$ is the ring or group of integers.

APPENDIX C

Bacground terminology

base point: A designated point of a set or space, usually called $*$.

compactly generated space: a space X homeomorphic to the colimit of its compact subspaces. Equivalently, a subset $W \subseteq X$ is closed if its intersection with any compact subspace $C \subseteq$ is closed in C.

concrete category: a category $\mathcal{C}$ equipped with a faithful functor $U : \mathcal{C} \to \mathrm{Set}$. The category of spaces, Spaces, is a concrete category. If $C \in \mathcal{C}$, then $U(C)$ is called the underlying set of C, and $c \in C$ will mean $c \in U(C)$. An arrow $f : C' \to C$ in $\mathcal{C}$ might as well be identified with the function Uf.

contractible space: a space Y for which the map $Y \to *$ is a homotopy equivalence. Here $*$ denotes the one point space.

deformation retraction: A *deformation retraction* of a space X onto a subspace $Y \subseteq X$ is a homotopy h from 1_X to $i \circ r$, where $i : Y \to X$ is the inclusion mapping, $r : X \to Y$ is a retraction, and $h(t, y) = y$ for all $y \in Y$. It follows that $X \simeq Y$, and i and r are inverse homotopy equivalences.

faithful functor: A functor $F : \mathcal{C} \to \mathcal{D}$ for which the corresponding functions $\mathrm{Hom}(C', C) \to \mathrm{Hom}(F(C'), F(C))$ are injective.

full subcategory: A subcategory $\mathcal{C}$ of a category $\mathcal{D}$ is called a full subcategory if $\mathrm{Hom}_{\mathcal{C}}(C', C) = \mathrm{Hom}_{\mathcal{D}}(C', C)$ for any $C, C' \in \mathcal{C}$.

fully faithful functor: A functor $F : \mathcal{C} \to \mathcal{D}$ is a *fully faithful functor* if the function $\mathrm{Hom}(C', C) \to \mathrm{Hom}(F(C'), F(C))$ is a bijection for any $C, C' \in \mathcal{C}$.

homotopic: Two continuous maps $f, g : X \to Y$ are *homotopic* if there is a homotopy between them. Being homotopic is an equivalence relation on $\mathrm{Hom}(X, Y)$,

homotopy: A homotopy between two continuous maps $f, g : X \to Y$ is a continuous map $h : I \times X \to Y$ such that $h(0, x) = f(x)$ and $h(1, x) = g(x)$ for all $x \in X$. If X and Y are pointed spaces and f and g are basepoint preserving maps, then we also require that $h(t, *) = *$ for all $t \in I$.

homotopy class: A homotopy class of continuous maps $X \to Y$ is an equivalence class in $\mathrm{Hom}(X, Y)$ for the equivalence relation "being homotopic". A homotopy class $X \to Y$ can be composed with a homotopy class $Y \to Z$ to yield a well-defined homotopy class $X \to Z$, and thus the homotopy classes can serve as the arrows in a new category, the "homotopy category".

homotopy equivalence: A continuous map $f : X \to Y$ is a homotopy equivalence if there is a map $g : Y \to X$ such that $f \circ g \sim 1_Y$ and $g \circ f \sim 1_X$. In other words, it's a map that becomes an isomorphism in the homotopy category. The maps f and g are called *inverse* homotopy equivalences.

join: The *join* $X * Y$ of two topological spaces X and Y is the quotient of $X \times [0, 1] \times Y$ that identifies all $(x, 0, y)$ with $(x, 0, y')$ and all $(x, 1, y)$ with $(x', 1, y)$. It contains a copy of X, a copy of Y, and and for every point of X and every point of Y it contains a line segment connecting them.

locally finite open covering: An open covering $\mathcal{U}$ of a topological space X such that each point $x \in X$ has an open neighborhood meeting only finitely many of the open sets in $\mathcal{U}$.

locally simply connected: A topological space X where any neighborhood of any point x contains a simply connected neighborhood of x.

opposite category: If $\mathcal{C}$ is a category, then $\mathcal{C}^{\mathrm{op}}$ denotes the opposite category of $\mathcal{C}$. The objects are the same, and the arrows are the same, except that they point in the opposite direction, i.e., $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(C', C) = \mathrm{Hom}_{\mathcal{C}}(C, C')$.

paracompact space: A Hausdorff topological space such that every open covering of it has a locally finite refinement.

partition of unity: A collection of continuous functions $f_i : X \to [0,1]$ on a topological space X ...

pointed set or space: A set or space with a base point. Maps $f : X \to Y$ satisfy $f(*) = *$.

retraction: A retraction of a space X onto a subspace $Y \subseteq X$ is a continuous map $r : X \to Y$ such that $r(y) = y$ for all $y \in Y$.

ring: The rings in this book are all assumed to contain an identity element 1 for the multiplication operation.

simply connected: A topological space X for which any loop $S^1 \to X$ is homotopic to a constant loop.

weak homotopy equivalence: A continuous map $f : X \to Y$ of topological spaces is a weak homotopy equivalence if for every point $x \in X$ and for every $n \geq 0$ the induced map $\pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism.

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Index