Problem 4.7, 10: Let x be the width of the box and y be the height of the box. The volume is x^2y, so $x^2y = 32000$. The material used in the construction of the box can be assessed using its area. The area of the base is x^2 and the area of each side is xy, so the total area used is $A = x^2+4xy$. We must minimize A. Rewrite it in terms of x alone by using $y = 32000/x^2$ to eliminate y. We get $A = x^2 + 4x(32000/x^2) = x^2 + 128000/x$. Compute $dA/dx = 2x - 128000/x^2$. Solve $dA/dx = 0$, i.e., $2x = 128000/x^2$ to get $x = 64000^{1/3} = 40$. Thus $y = 32000/x^2 = 32000/1600 = 20$. Since $dA/dx = (2x^3 - 128000)/x^2$, by examining the numerator $2x^3 - 128000$ we see that dA/dx is positive for $x > 40$ and negative for $x < 40$, and thus the critical point is a local minimum. Since it's the only critical point, it's the global minimum. The box is $40 \times 40 \times 20$.

Problem 4.7, 18: Minimize the distance D from $(0, -3)$ to $(-y^2, y)$. We see that $D^2 = (y^2)^2 + (y + 3)^2$, and we might as well minimize $E = D^2$ instead. Simplify $E = y^4 + (y + 3)^2$. Compute $dE/dy = 4y^3 + 2(y + 3) = 4y^3 + 2y + 6$. Solve $dE/dy = 0$. By inspection we see that $dE/dy = 0$ when $y = -1$. Since $d^2E/dy^2 = 12y^2 + 2 > 0$, we see that dE/dy is an increasing function, so has only one root. Therefore $y = 1$ is the only critical number. From the geometry, it must give a minimum. The point is $(-1, -1)$.

Problem 4.7, 30: Let x be the width and y be the height of the poster. Then $180 = xy$ and we are to maximize $P = (x-2)(y-3)$. Eliminate y to get $P = (x-2)(180/x-3)$. Solve $dP/dx = 0$ to get $x = 2\sqrt{30}$, so $y = 180/(2\sqrt{30})$. It's the only critical point of a positive function, and at the endpoints of the interval we have $P = 0$, so it must be a maximum.

Problem 4.7, 40: Using high school geometry as explained in class we see that $AB = 4\cos\theta$ and $BC = 4\theta$. The total time is $T(\theta) = AB/2 + BC/4 = 2\cos\theta + \theta$. Solve $dT/d\theta = 0$ to get $\sin\theta = 1/2$, i.e., $\theta = \pi/6$. Comparing numerical values: $T(0) = 2$, $T(\pi/6) \approx 2.26$, $T(\pi/2) \approx 1.57$, we see that $\theta = \pi/2$ gives the smallest value, so she should walk all the way.

Problem 4.7, 48: As suggested in class, we avoid trigonometry by letting $x = RT$ be our parameter, so that $QR = QT - x$. We must minimize the length of the rope $L = \sqrt{x^2 + ST^2} + \sqrt{(QT-x)^2 + PQ^2}$. Compute $dL/dt = x/\sqrt{x^2 + ST^2} - (QT - x)/\sqrt{(QT-x)^2 + PQ^2}$. Setting $dL/dt = 0$ we see that $x/\sqrt{x^2 + ST^2} = (QT - x)/\sqrt{(QT-x)^2 + PQ^2}$, i.e., that $RT/RS = RQ/RP$. By high school geometry (side-angle-side) , the two triangles are similar, and thus $\theta_1 = \theta_2$. Alternatively, we see that $\cos\theta_1 = \cos\theta_2$, and thus $\theta_1 = \theta_2$, because both θ_1 and θ_2 lie in the interval $[0, \pi/2]$, on which cos is decreasing, and thus one-to-one.

Problem 4.7, 50: As explained in class, we want to find the minimum length the blue line can have. There are many ways to do this problem. Following the book we may use the angle θ as a parameter; we find that the length of the pipe is $L = 9/\sin(\theta) + 6/\cos(\theta)$. Differentiating, we get $dL/d\theta = -9\cos(\theta)/(\sin(\theta))^2 + 6\sin(\theta)/(\cos(\theta))^2$. Setting $dL/d\theta = 0$ we get $9\cos(\theta)/(\sin(\theta))^2 = 6\sin(\theta)/(\cos(\theta))^2$; multiplying out we get $6(\sin(\theta))^3 = 9(\cos(\theta))^3$, or $(\tan(\theta))^3 = 9/6$, so that $\theta = \tan^{-1}(\sqrt[3]{9/6}) \approx 0.852771$. Substituting that in the original formula for L we get $L \approx 21.0704$ feet. To see that it's an absolute minimum, observe that it is the only critical point, and that L approaches ∞ as θ approaches either 0 or $\pi/2$.

One can also this one without trigonometry. Let A be the point where the pipe touches the left wall, B the point where it touches the inside corner, and C the point where it touches the upper wall. Drop a perpendicular from A to the opposite wall to meet it at a point D. Drop a perpendicular from C to the opposite wall to meet it at a point E. Let $x = BE$ be our main parameter. Let $w = AB$, $z = BC$, and observe that $AD = 9$ and $CE = 6$. The two triangles ADB and BEC are similar, so $w/9 = z/x$, from which it follows that $w = 9z/x$; this seems simpler that using the theorem of Pythagoras to get a formula for w. We want to minimize the length $L = z + w = z + 9z/x = z(1 + 9/x)$. Notice that $z = \sqrt{x^2 + 6^2}$, and that can be used to write L as a function of x, but let's keep the symbol z around for a while. We want to compute dL/dx, but first let's compute $dz/dx = (d/dx)\sqrt{x^2 + 6^2} = (1/2)(x^2 + 6^2)^{-1/2} \cdot 2x = x/\sqrt{x^2 + 6^2} = x/z$. Now by the product rule $dL/dx = (d/dx)(z(1 + 9/x)) = dz/dx \cdot (1 + 9/x) + z \cdot (d/dx)(1 + 9/x) = (x/z)(1 + 9/x) + z \cdot (-9/x^2) = (x + 9)/z - (9z)/x^2 = ((x + 9)x^2 - 9(z^2))/(zx^2) = (x^3 + 9x^2 - 9(x^2 + 6^2))/(zx^2) = (x^3 - 9 \cdot 6^2)/(zx^2)$.

Setting $dL/dt = 0$ we get $x^3 = 9 \cdot 6^2$, so grabbing a calculator, we see that $x = \sqrt[3]{54 \cdot 6} \approx 6.86829$, and $z = \sqrt{x^2 + 6^2} \approx 9.11994$, and thus gives $L = z(1 + 9/x) \approx 21.0704$ feet.

Problem 4.7, 52: We want to maximize the area $A = (10+10\cos(\theta))(10\sin(\theta)) = 100(1+\cos(\theta))(\sin(\theta))$. Compute $dA/dt = 100((-\sin(\theta))(\sin(\theta)) + (1 + \cos(\theta))(\cos(\theta))) = 100(-(\sin(\theta))^2 + \cos(\theta) + (\cos(\theta))^2)$. Writing $(\sin(\theta))^2 = 1 - (\cos(\theta))^2$ we get $dA/dt = 100(-1 + \cos(\theta) + 2(\cos(\theta))^2)$. Setting $dA/dt = 0$ we get $-1+\cos(\theta)+2(\cos(\theta))^2 = 0$. Setting $x = \cos(\theta)$ we get $2x^2 + x - 1 = 0$. Using the quadratic formula we find $x = 1/2$ or $x = -1$. Since $0 \leq \theta \leq \pi/2$, we can rule out the solution where $\cos(\theta) = x = -1$. That leaves $\cos(\theta) = x = 1/2$, so $\theta = \pi/3$. Since the area is 0 when $\theta = 0$, it must be increasing when $0 < \theta < \pi/3$. Since the quadratic polynomial had two roots, dA/dt changes sign as x crosses $1/2$, and so A is decreasing when $\pi/3 < \theta < \pi/2$. Thus A has a maximum at $\theta = \pi/3$.

Problem 4.7, 54: We want to maximize $\theta = \tan^{-1}((h + d)/x) - \tan^{-1}(d/x)$ for $x \geq 0$. Compute $d\theta/dx = (h^2 d + hd^2 - hx^2)/((x^2 + (h + d)^2)(x^2 + d^2))$ and try to solve $d\theta/dx = 0$, getting $x = \sqrt{d(h + d)}$. Since θ approaches 0 as x approaches 0 or ∞, this single critical point must be an absolute maximum.

Problem 4.7, 56: (a) Let D be the marked point at the right end of the main branch, so that $a = AD$. Thus $\sin(\theta) = b/BC$ and $\cos(\theta) = (a - AB)/BC$. Solving for the lengths AB and BC, plugging them into the formula for R, and adding the two results, gives the desired formula for the total resistance.

(b) Compute $dR/d\theta = bC(1/\sin(\theta))(1/(r_1^4 \sin(\theta)) - (1/(r_2^4 \tan(\theta))))$. Setting $dR/d\theta = 0$ leads to $\cos(\theta) = r_2^4/r_1^4$.

(c) In this case $\cos(\theta) = 2^4/3^4$, so we use a calculator and find that $\theta \approx 1.37196$, which is about 79 degrees.