

The asymptotic behavior of least pseudo-Anosov dilatations

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For a surface S with n marked points and fixed genus $g \geq 2$, we prove that the logarithm of the minimal dilatation of a pseudo-Anosov homeomorphism of S is on the order of $\frac{\log n}{n}$. This is in contrast with the cases of genus zero or one where the order is $\frac{1}{n}$.

1 Introduction

Let $S = S_{g,n}$ be an orientable surface with genus g and n marked points. The **mapping class group** of S is defined to be the group of homotopy classes of orientation preserving homeomorphisms of S . We denote it by $\text{Mod}(S)$. Given a pseudo-Anosov element $f \in \text{Mod}(S)$, let $\lambda(f)$ denote the **dilatation** of f (see section 2.1). We define

$$\mathcal{L}(S_{g,n}) := \{\log \lambda(f) | f \in \text{Mod}(S_{g,n}) \text{ pseudo-Anosov}\}.$$

This is precisely the length spectrum of the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n marked points with respect to the Teichmüller metric; see [Iva88]. There is a shortest closed geodesic and we denote its length

$$l_{g,n} = \min\{\log \lambda(f) | f \in \text{Mod}(S_{g,n}) \text{ pseudo-Anosov}\}.$$

Our main theorem is the following:

Theorem 1.1 *For any fixed $g \geq 2$, there is a constant $c_g \geq 1$ depending on g such that*

$$\frac{\log n}{c_g n} < l_{g,n} < \frac{c_g \log n}{n},$$

for all $n \geq 3$.

To contrast with known results, we recall that in [Pen91] Penner proves that for $2g - 2 + n > 0$,

$$l_{g,n} \geq \frac{\log 2}{12g - 12 + 4n},$$

and for closed surfaces with genus $g \geq 2$,

$$\frac{\log 2}{12g - 12} \leq l_{g,0} \leq \frac{\log 11}{g}.$$

The bounds on $l_{g,0}$ have been improved by a number of authors; [Bau92], [McM00], [Min06],[HK06].

In [Pen91], Penner suggests that there may be an ‘‘analogous upper bound for $n \neq 0$ ’’. In [HK06], Hironaka and Kin use a concrete construction to prove that for genus $g = 0$,

$$l_{0,n} < \frac{\log(2 + \sqrt{3})}{\lfloor \frac{n-2}{2} \rfloor} \leq \frac{2 \log(2 + \sqrt{3})}{n - 3},$$

for all $n \geq 4$. The inequality is proven for even n in [HK06], but it follows for odd n by letting the fixed point of their example be a marked point. Combining this with Penner’s lower bound, one sees for $n \geq 4$,

$$\frac{\log 2}{4n - 12} \leq l_{0,n} < \frac{2 \log(2 + \sqrt{3})}{n - 3},$$

which shows that the upper bound is on the same order as Penner’s lower bound for $g = 0$. A similar situation holds for $g = 1$; see the Appendix 5.1.

Inspired by the construction of Hironaka and Kin, we tried to find examples of pseudo-Anosov $f_{g,n} \in \text{Mod}(S_{g,n})$ with $\log \lambda(f_{g,n}) = O\left(\frac{1}{|\chi(S_{g,n})|}\right)$, for $\chi(S_{g,n}) = 2 - 2g - n < 0$. However for any fixed $g \geq 2$, all attempts resulted in $f_{g,n} \in \text{Mod}(S_{g,n})$ pseudo-Anosov with $\log \lambda(f_{g,n}) = O_g\left(\frac{\log |\chi(S_{g,n})|}{|\chi(S_{g,n})|}\right)$ and not $O\left(\frac{1}{|\chi(S_{g,n})|}\right)$. This led us to prove Theorem 1.1.

The preceding discussion suggests that the asymptotic behavior of $l_{g,n}$ while varying both g and n can be quite complicated, in general. Hence, we will focus on understanding what happens along different (g, n) -rays. In addition to the results discussed above, there are other rays in which the asymptotic behavior of $l_{g,n}$ can be understood via examples (see Appendix 5.2) and Penner’s lower bound. The following table summarizes these behaviors for $\chi(S_{g,n}) < 0$.

(g, n) -rays	The asymptotic behavior of $l_{g,n}$
$g = 0$	$1/ \chi(S_{g,n}) $
$g = 1$ and n is even	$1/ \chi(S_{g,n}) $
$g = \text{constant} \geq 2$	$\log(\chi(S_{g,n})) / \chi(S_{g,n}) $
$n = 0, 1, 2, 3, \text{ or } 4$	$1/ \chi(S_{g,n}) $
$n = g, g + 1, \text{ or } g + 2$	$1/ \chi(S_{g,n}) $
$n = g - 1 \text{ or } 2(g - 1)$	$1/ \chi(S_{g,n}) $

Question What are asymptotic behaviors of $l_{g,n}$ along different (g,n) -rays in the (g,n) plane?

1.1 Outline of the paper

We will first recall some definitions and properties in section 2. In section 3 we prove the lower bound of Theorem 1.1. We construct examples in section 4 which give an upper bound for the genus 2 case, and we extend the example to arbitrary genus $g \geq 2$ to obtain the upper bound of Theorem 1.1. Finally, we construct a pseudo-Anosov element in $\text{Mod}(S_{1,2n})$ and obtain an upper bound on $l_{1,2n}$ in the Appendix.

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2 Preliminaries

2.1 Homeomorphisms of a surface

We say that a homeomorphism $f : S \rightarrow S$ is **pseudo-Anosov** if there are transverse singular foliations \mathcal{F}^s and \mathcal{F}^u together with transverse measures μ^s and μ^u such that for some $\lambda > 1$,

$$\begin{aligned} f(\mathcal{F}^s, \mu^s) &= (\mathcal{F}^s, \lambda\mu^s), \\ f(\mathcal{F}^u, \mu^u) &= (\mathcal{F}^u, \lambda^{-1}\mu^u). \end{aligned}$$

The number $\lambda = \lambda(f)$ is called the **dilatation** of f . We call f **reducible** if there is a finite disjoint union U of simple essential closed curves on S such that f leaves U invariant. If there exists $k > 0$ such that f^k is the identity, then f is **periodic**. A mapping class $[f]$ is pseudo-Anosov, reducible or periodic (respectively) if f is homotopic to a pseudo-Anosov, reducible or periodic homeomorphism (respectively). The following is proved in [FLP91].

Theorem 2.1 (Nielsen-Thurston) *A mapping class $[f] \in \text{Mod}(S)$ is either periodic, reducible, or pseudo-Anosov.*

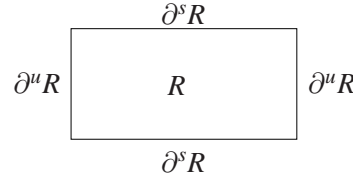
As a slight abuse of notation, we sometimes refer to a mapping class $[f]$ by one of its representatives f .

2.2 Markov partitions

Suppose $f : S \rightarrow S$ is pseudo-Anosov with stable and unstable measured singular foliations (\mathcal{F}^s, μ^s) and (\mathcal{F}^u, μ^u) . We define a rectangle R to be a map

$$\rho : I \times I \rightarrow S,$$

such that ρ is an embedding on the interior, $\rho(\text{point} \times I)$ is contained in a leaf of \mathcal{F}^u , and $\rho(I \times \text{point})$ is contained in a leaf of \mathcal{F}^s . We denote $\rho(\partial I \times I)$ by $\partial^u R$ and $\rho(I \times \partial I)$ by $\partial^s R$.



As a standard abuse of notation, we will write $R \subset S$ for the image of a rectangle map $\rho : I \times I \rightarrow S$.

Definition 2.2 A Markov partition for $f : S \rightarrow S$ is a decomposition of S into a finite union of rectangles $\{R_i\}_{i=1}^k$, such that:

- (1) $\text{Int}(R_i) \cap \text{Int}(R_j)$ is empty, when $i \neq j$
- (2) $f(\bigcup_{j=1}^k \partial^u R_j) \subset \bigcup_{j=1}^k \partial^u R_j$
- (3) $f^{-1}(\bigcup_{i=1}^k \partial^s R_i) \subset \bigcup_{i=1}^k \partial^s R_i$

Given a pseudo-Anosov homeomorphism $f : S \rightarrow S$, a Markov partition is constructed in [BH95] from a train track map for f . The advantage of this construction over [FLP91], for example, is that the number of rectangles is substantially smaller. From [BH95], one has the following:

Theorem 2.3 For any pseudo-Anosov homeomorphism $f : S \rightarrow S$ of a surface S with at least one marked point, there exists a Markov partition for f with at most $-3\chi(S)$ rectangles.

We say that a matrix is *positive* (respectively, *nonnegative*) if all the entries are positive (respectively, nonnegative).

We can define a *transition matrix* M associated to the Markov partition with rectangles $\{R_i\}_{i=1}^k$. The entry $m_{i,j}$ of M is the number of times that $f(R_j)$ wraps over R_i , so M

is a nonnegative integral $k \times k$ matrix. In Bestvina and Handel's construction, M is the same as the transition matrix of the train track map and they show it is an integral Perron-Frobenius matrix (i.e. it is irreducible with nonnegative integer entries); see [Gan59]. Furthermore, the Perron-Frobenius eigenvalue $\mu(M) = \lambda(f)$ is the dilatation of f . The width (respectively, height) of R_i is the i th entry of the corresponding Perron-Frobenius eigenvector of M (respectively, M^T), where the eigenvectors are both positive by the irreducibility of M .

The following proposition will be used in proving the lower bound.

Proposition 2.4 *Let M be a $k \times k$ integral Perron-Frobenius matrix. If there is a nonzero entry on the diagonal of M , then M^{2k} is a positive matrix and its Perron-Frobenius eigenvalue $\mu(M^{2k})$ is at least k .*

Proof We construct a directed graph Γ from M with k vertices $\{i\}_{i=1}^k$ such that the number of the directed edge from i to j in Γ equals $m_{i,j}$. We observe that for any $r > 0$ the (i,j) th entry $m_{i,j}^{(r)}$ of M^r is the number of directed edge paths from i to j of length r in Γ .

Since M is a Perron-Frobenius matrix, we know that Γ is path-connected by directed paths. Suppose M has a nonzero entry at the (l,l) th entry, then we will see at least one corresponding loop edge at the vertex l . For any i and j in Γ , path-connectivity ensures us that there are directed edge paths of length $\leq k$ from i to l and from l to j . This tells us that there is a directed edge path P of length $\leq 2k$ from i to j passing through l . Since we can wrap around the loop edge adjacent to l to increase the length of P , there is always a directed edge path of length $2k$ from i to j . In other words, $m_{i,j}^{(2k)}$ is at least 1 for all i and j , so M^{2k} is a positive matrix.

Let v be a corresponding Perron-Frobenius eigenvector, so that we have $M^{2k}v = \mu(M^{2k})v$. This implies that if $v = [v_1 \cdots v_k]^T$, for all i ,

$$\sum_{j=1}^k m_{i,j}^{(2k)} v_j = \mu(M^{2k}) v_i,$$

equivalently,

$$\mu(M^{2k}) = \sum_{j=1}^k m_{i,j}^{(2k)} \frac{v_j}{v_i}.$$

Choosing i such that $v_i \leq v_j$ for all j , we obtain

$$\mu(M^{2k}) \geq \sum_{j=1}^k m_{i,j}^{(2k)} \geq \sum_{j=1}^k 1 = k.$$

□

The following proposition will be used in proving the upper bound.

Proposition 2.5 *Let Γ be the induced directed graph of an integral Perron-Frobenius matrix M with Perron-Frobenius eigenvalue $\mu(M) = \mu$. Let $P_\Gamma(i, d)$ be the total number of paths of length d emanating from vertex i in Γ . Then, for all i ,*

$$\sqrt[d]{P_\Gamma(i, d)} \longrightarrow \mu(M) \text{ as } d \rightarrow \infty.$$

Proof Let M be an integral $k \times k$ Perron-Frobenius matrix with Perron-Frobenius eigenvalue μ and Perron-Frobenius eigenvector v . As above

$$\sum_{j=1}^k m_{i,j}^{(d)} v_j = \mu(M^d) v_i = \mu^d v_i.$$

Let $v_{max} = \max_i \{v_i\}$ and $v_{min} = \min_i \{v_i\}$. According to the Perron-Frobenius theory, the irreducibility of M implies that $v_i > 0$ for all i . For all i we have

$$\frac{v_{min} \left(\sum_j m_{i,j}^{(d)} \right)}{\mu^d} \leq \frac{\sum_j m_{i,j}^{(d)} v_j}{\mu^d} \leq \frac{v_{max} \left(\sum_j m_{i,j}^{(d)} \right)}{\mu^d},$$

hence

$$\frac{v_i}{v_{max}} \leq \frac{\sum_j m_{i,j}^{(d)}}{\mu^d} \leq \frac{v_i}{v_{min}}.$$

We are done, since $\sum_j m_{i,j}^{(d)} = P_\Gamma(i, d)$ and for all i ,

$$\sqrt[d]{\frac{v_i}{v_{max}}} \rightarrow 1 \text{ and } \sqrt[d]{\frac{v_i}{v_{min}}} \rightarrow 1, \text{ as } d \text{ tends to } \infty.$$

□

2.3 Lefschetz numbers

We will review some definitions and properties of Lefschetz numbers. A more complete discussion can be found in [GP74] and [BT82].

Let X be a compact oriented manifold, and $f : X \rightarrow X$ be a map. Define

$$\text{graph}(f) = \{(x, f(x)) | x \in X\} \subset X \times X$$

and let Δ be the diagonal of $X \times X$. The algebraic intersection number $I(\Delta, \text{graph}(f))$ is an invariant of the homotopy class of f , called the **(global) Lefschetz number** of f and it is denoted $L(f)$. As in [BT82], this can be alternatively described by

$$(1) \quad L(f) = \sum_{i \geq 0} (-1)^i \text{trace}(f_*^{(i)}),$$

where $f_*^{(i)}$ is the matrix induced by f acting on $H_i(X) = H_i(X; \mathbb{R})$. The Euler characteristic is the self-intersection number of the diagonal Δ in $X \times X$,

$$\chi(X) = I(\Delta, \Delta) = L(\text{id}).$$

As seen in [GP74], if f has isolated fixed points, we can compute the **local Lefschetz number** of f at a fixed point x in local coordinates as

$$L_x(f) = \deg \left(z \mapsto \frac{f(z) - z}{|f(z) - z|} \right),$$

where z is on the boundary of a small disk centered at x which contains no other fixed points. Moreover we can compute the Lefschetz number by summing the local Lefschetz numbers of fixed points,

$$L(f) = \sum_{f(x)=x} L_x(f).$$

This description of $L_x(f)$ is given for smooth f in [GP74], but it is equally valid for continuous f since such a map is approximated by smooth maps. We will be computing the Lefschetz number of a homeomorphism $f : S_{g,n} \rightarrow S_{g,n}$, *ignoring the marked points*.

Proposition 2.6 *If a homeomorphism $f : S_{g,n} \rightarrow S_{g,n}$ is homotopic (not necessarily fixing the marked points) to the identity or a multitwist, then*

$$L(f) = \chi(S_{g,0}) = 2 - 2g.$$

A *multitwist* is a composition of powers of Dehn twists on pairwise disjoint simple essential closed curves.

Proof If f is homotopic to the identity, the homotopy invariance of the Lefschetz number tells us $L(f) = L(\text{id}) = I(\Delta, \Delta)$ which is $\chi(S_{g,0})$.

Suppose f is homotopic to a multitwist. We will use (1) to compute $L(f)$. Note that $H_i(S_{g,0})$ is 0 for $i \geq 3$, $H_0(S_{g,0}) \cong H_2(S_{g,0}) \cong \mathbb{R}$ and $f_*^{(i)}$ is the identity when $i = 0$ or 2, so this implies $L(f) = 2 - \text{trace}(f_*^{(1)})$.

There exists a set $\{\gamma_i\}_{i=1}^k$ of disjoint simple essential closed curves with some integers $n_i \neq 0$ such that

$$f \simeq T_{\gamma_1}^{n_1} \circ \cdots \circ T_{\gamma_k}^{n_k},$$

where $T_{\gamma_i}^{n_i}$ is the n_i -th power of a Dehn twist along γ_i .

For any curve γ ,

$$T_{\gamma_i}^{n_i}([\gamma]) = [\gamma] + n_i \langle \gamma, \gamma_i \rangle [\gamma_i],$$

where $[\gamma]$ is the homology class of γ and $\langle \gamma, \gamma_i \rangle$ is the algebraic intersection number of $[\gamma]$ and $[\gamma_i]$. If any γ_i is a separating curve, then $[\gamma_i]$ is the trivial homology class and $T_{\gamma_i}^{n_i}$ acts trivially on $H_1(S_{g,0})$. We may therefore assume that each γ_i is nonseparating. After renaming the curves, we can assume that there is a subset $\{\gamma_1, \gamma_2, \dots, \gamma_s\}$ such that $\hat{\gamma} = \bigcup_{i=1}^s \gamma_i$ is nonseparating and $\hat{\gamma} \cup \gamma_j$ is separating for all $j > s$. Thus, for all $k \geq j > s$,

$$[\gamma_j] = \sum_{i=1}^s c_{ji} [\gamma_i],$$

for some constants $c_{ji} \in \mathbb{R}$. We can extend $\{[\gamma_i]\}_{i=1}^s$ to a basis of $H_1(S_{g,0})$,

$$\{\alpha_1, \alpha_2, \dots, \alpha_g, \beta_1, \beta_2, \dots, \beta_g\},$$

where $[\gamma_i] = \alpha_i$ for $i \leq s \leq g$ and $\langle \alpha_i, \beta_j \rangle = \delta_{ij}$, $\langle \alpha_i, \alpha_j \rangle = \langle \beta_i, \beta_j \rangle = 0$.

First suppose $s = k$, then $\langle \alpha_j, \gamma_i \rangle = \langle \alpha_j, \alpha_i \rangle = 0$ for all i and j . Therefore, for all j ,

$$\begin{aligned} f_*^{(1)}(\alpha_j) &= \alpha_j; \text{ and} \\ f_*^{(1)}(\beta_j) &= \beta_j + \sum_{i=1}^k n_i \langle \beta_j, \gamma_i \rangle [\gamma_i] = \beta_j + \sum_{i=1}^k n_i \langle \beta_j, \alpha_i \rangle \alpha_i = \beta_j - n_j \alpha_j. \end{aligned}$$

So we have

$$f_*^{(1)} = \left(\begin{array}{c|c} I_{g \times g} & * \\ \hline 0 & I_{g \times g} \end{array} \right)$$

and $L(f) = 2 - \text{trace}(f_*^{(1)}) = 2 - 2g$.

For $s < k$, we will have

$$\begin{aligned}
 f_*^{(1)}(\alpha_j) &= \alpha_j + \sum_{i=1}^k n_i \langle \alpha_j, \gamma_i \rangle [\gamma_i] \\
 &= \alpha_j + \sum_{i=1}^s n_i \langle \alpha_j, \alpha_i \rangle \alpha_i + \sum_{i=s+1}^k n_i \langle \alpha_j, \gamma_i \rangle [\gamma_i] \\
 &= \alpha_j + \sum_{i=s+1}^k n_i \sum_{t=1}^s c_{it} \langle \alpha_j, \gamma_t \rangle [\gamma_t] \\
 &= \alpha_j + \sum_{i=s+1}^k n_i \sum_{t=1}^s c_{it} \langle \alpha_j, \alpha_t \rangle \alpha_t \\
 &= \alpha_j
 \end{aligned}$$

and

$$\begin{aligned}
 f_*^{(1)}(\beta_j) &= \beta_j + \sum_{i=1}^k n_i \langle \beta_j, \gamma_i \rangle [\gamma_i] \\
 &= \beta_j + \sum_{i=1}^s n_i \langle \beta_j, \gamma_i \rangle [\gamma_i] + \sum_{i=s+1}^k n_i \sum_{t=1}^s c_{it} \langle \beta_j, \gamma_t \rangle [\gamma_t] \\
 &= \beta_j + \sum_{i=1}^s n_i \langle \beta_j, \alpha_i \rangle \alpha_i + \sum_{i=s+1}^k n_i \sum_{t=1}^s c_{it} \langle \beta_j, \alpha_t \rangle \alpha_t \\
 &= \begin{cases} \beta_j, & \text{if } j > s, \\ \beta_j - n_j \alpha_j - \sum_{i=s+1}^k n_i c_{ij} \alpha_j, & \text{if } j \leq s. \end{cases}
 \end{aligned}$$

Therefore, the diagonal of the matrix $f_*^{(1)}$ is still all 1's and

$$L(f) = 2 - \text{trace}(f_*^{(1)}) = 2 - 2g.$$

□

3 Bounding the dilatation from below

Lemma 3.1 *For any pseudo-Anosov element $f \in \text{Mod}(S_{g,n})$ equipped with a Markov partition, if $L(f) < 0$, then there is a rectangle R of the Markov partition, such that the interiors of $f(R)$ and R intersect.*

Proof Since f is a pseudo-Anosov homeomorphism, it has isolated fixed points. Suppose x is an isolated fixed point of f such that one of the following happens:

- (1) x is a nonsingular fixed point and the local transverse orientation of \mathcal{F}^s is reversed.
- (2) x is a singular fixed point and no separatrix of \mathcal{F}^s emanating from x is fixed.

A *separatrix* of \mathcal{F}^s is a maximal arc starting at a singularity and contained in a leaf of \mathcal{F}^s .

Claim $L_x(f) = +1$.

Let B be a small disk centered at x containing no other fixed point of f . First we show that (in local coordinates) for every $z \in \partial B$, $f(z) - z \neq \alpha z$ for all $\alpha > 0$.

It is easy to verify this in case 1 by choosing local coordinates (ξ_1, ξ_2) around x so that f is given by

$$f(\xi_1, \xi_2) = (-\lambda\xi_1, \frac{-1}{\lambda}\xi_2).$$

In case 2, we choose local coordinates around x such that the separatrices of \mathcal{F}^s emanating from x are sent to rays from 0 through the k th roots of unity in \mathbb{R}^2 . This means f rotates each of the sectors bounded by these rays through an angle $\frac{2\pi j}{k}$ for some $j = 1, \dots, k-1$, and so for all $z \in \partial B$ $f(z) - z \neq \alpha z$ for all $\alpha > 0$.

Define a smooth map $h_0 : \partial B \rightarrow S^1$ by $h_0(z) = \frac{f(z)-z}{|f(z)-z|}$, so $L_x(f) = \deg(h_0)$ by definition. Let $g : \partial B \rightarrow S^1$ be defined by $g(z) = \frac{z}{|z|}$ and $h_1 : S^1 \rightarrow S^1$ be defined by $h_1(\frac{z}{|z|}) = \frac{f(z)-z}{|f(z)-z|}$, so that $h_0 = h_1g$. Then

$$L_x(f) = \deg(h_0) = \deg(h_1g) = \deg(h_1)\deg(g) = \deg(h_1)$$

since $\deg(g) = 1$. Note that h_1 has no fixed point since for all $z \in \partial B$,

$$f(z) - z \neq \alpha z,$$

for all $\alpha > 0$. Therefore $L_x(f) = \deg(h_1) = (-1)^{(1+1)} = +1$.

The assumption of $L(f) < 0$ implies that there exists a fixed point x of f which is in neither of the cases above. In other words, it falls into one of the cases in Figure 1. As seen in Figure 1, there is a rectangle R of the Markov partition such that the interiors of $f(R)$ and R intersect. \square

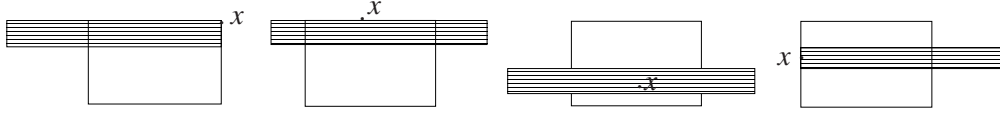


Figure 1: The intersection of $f(R)$ and R . R is the underlying rectangle and $f(R)$ is the shaded rectangle.

Let $\Gamma_S(3) \triangleleft \text{Mod}(S)$ denote the kernel of the action on $H_1(S; \mathbb{Z}/3\mathbb{Z})$, where $S = S_{g,0}$. In [Iva92], it is shown that $\Gamma_S(3)$ consists of pure mapping classes. Setting

$$\Theta(g) = [\text{Mod}(S) : \Gamma_S(3)],$$

we conclude the following.

Lemma 3.2 *Let $f \in \text{Mod}(S_{g,n})$ be a pseudo-Anosov element and $\widehat{f} \in \text{Mod}(S_{g,0})$ be the induced mapping class obtained by forgetting marked points. There exists a constant $1 \leq \alpha \leq \Theta(g)$ such that \widehat{f}^α satisfies exactly one of the following:*

- (1) \widehat{f}^α restricts to a pseudo-Anosov map on a connected subsurface.
- (2) $\widehat{f}^\alpha = \text{Id}$.
- (3) \widehat{f}^α is a multitwist map.

Remark For the first two cases of Lemma 3.2, one can find α bounded by a linear function of g , but in case 3, α may be exponential in g .

Theorem 3.3 *For $g \geq 2$, given any pseudo-Anosov $f \in \text{Mod}(S_{g,n})$, let α be as in Lemma 3.2. Then*

$$\log \lambda(f) \geq \min \left\{ \frac{\log 2}{\alpha(12g - 12)}, \frac{\log(6g + 3n - 6)}{2\alpha(6g + 3n - 6)} \right\}.$$

Proof We will deal with case 1 of Lemma 3.2 first.

If \widehat{f}^α restricts to a pseudo-Anosov homeomorphism on a connected subsurface \sum_{g_0, n_0} of $S_{g,0}$ of genus g_0 with n_0 boundary components (we have $2g_0 + n_0 \leq 2g$), then Penner's lower bound tells us

$$\log \lambda(\widehat{f}^\alpha) \geq \frac{\log 2}{12g_0 - 12 + 4n_0} \geq \frac{\log 2}{12g - 12}.$$

Hence $\log \lambda(f) \geq \log \lambda(\widehat{f}^\alpha) > \frac{\log 2}{\alpha(12g - 12)}$.

If \widehat{f}^α is homotopic to the identity or a multitwist map, from Proposition 2.6, we have $L(f^\alpha) = L(\widehat{f}^\alpha) = \chi(S_{g,0}) = 2 - 2g < 0$. Theorem 2.3 tells us that for any pseudo-Anosov f there is a Markov partition with k rectangles, where $k \leq -3\chi(S)$. Recall that the transition matrix M obtained from the rectangles is a $k \times k$ Perron-Frobenius matrix and the Perron-Frobenius eigenvalue $\mu(M)$ equals $\lambda(f)$.

By Lemma 3.1, there is a rectangle R such that the interiors of $f^\alpha(R)$ and R intersect. This implies that there is a nonzero entry on the diagonal of M^α . Applying Proposition 2.4, we obtain that $\mu((M^\alpha)^{2k}) = \mu(M^{2k\alpha})$ is at least k , so we have

$$(\lambda(f))^{2k\alpha} = \lambda(f^{2k\alpha}) = \mu(M^{2k\alpha}) \geq k.$$

One can easily check $\frac{\log x}{x}$ is monotone decreasing for $x \geq 3$. Since

$$3 \leq k \leq -3\chi(S) = 6g + 3n - 6,$$

hence

$$\log \lambda(f) \geq \frac{\log k}{2\alpha k} \geq \frac{\log(6g + 3n - 6)}{2\alpha(6g + 3n - 6)}.$$

□

Remark Penner's proof in [Pen91] does not use Lefschetz numbers which we used to conclude that $\mu(M^{2k\alpha})$ is at least k , so we obtain a sharper lower bound for $n \gg g$.

4 An example which provides an upper bound

4.1 For the genus two case

In this section, we will construct a pseudo-Anosov $f \in \text{Mod}(S_{2,n})$ for all $n \geq 31$ then we compute its dilatation which gives us an upper bound for $l_{2,n}$.

Let $S_{0,m+2}$ be a genus 0 surface with $m + 2$ marked points (i.e. a marked sphere), and recall an example of pseudo-Anosov $\phi \in \text{Mod}(S_{0,m+2})$ in [HK06]. We view $S_{0,m+2}$ as a sphere with $s + 1$ marked points X circling an unmarked point x and $t + 1$ marked points Y circling an unmarked point y , and a single extra marked point z . We can also draw this as a "turnover", as in Figure 2. Note that $|X \cap Y| = 1$, $|X| = s + 1$, $|Y| = t + 1$ and $m = s + t$.

We define homeomorphisms $\alpha_s, \beta_t : S_{0,m+2} \rightarrow S_{0,m+2}$ such that α_s rotates the marked points of X counterclockwise around x and β_t rotates the marked points of Y clockwise around y ; see Figure 3. Define $\phi_{s,t} := \beta_t \alpha_s$. In [HK06], it is shown that $\phi_{s,t}$ is pseudo-

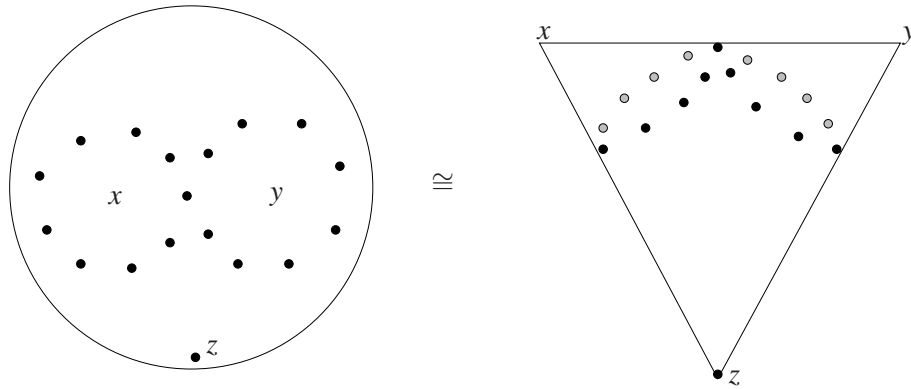


Figure 2: Two way of viewing a marked sphere. Black dots are marked points and the shaded dots on the right are marked points at the back.

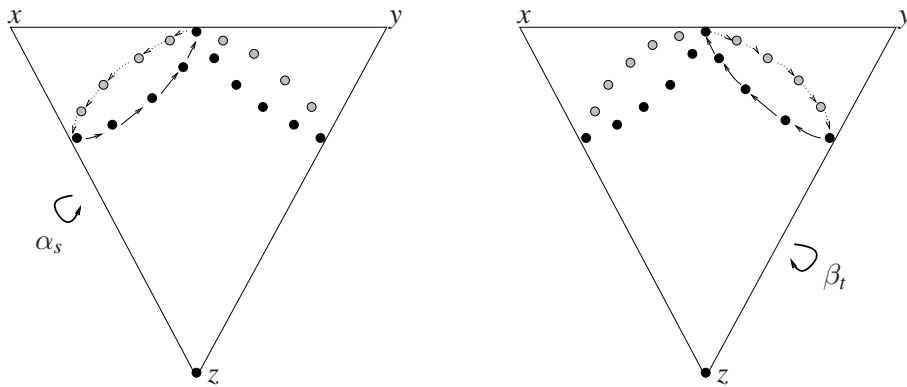


Figure 3: Homeomorphisms α_s and β_t .

Anosov by checking it satisfies the criterion of [BH95]. We also note that from this one can check that x , y and z are fixed points of a pseudo-Anosov representative of $\phi_{s,t}$. Moreover, for $s, t \geq 1$ the dilatation of $\phi_{s,t}$ equals the largest root of the polynomial

$$\begin{aligned} T_{s,t}(x) &= x^{t+1}(x^s(x-1) - 2) + x^{s+1}(x^{-s}(x^{-1} - 1) - 2) \\ &= (x-1)x^{(s+t+1)} - 2(x^{s+1} + x^{t+1}) - (x-1). \end{aligned}$$

The dilatation is minimized when $s = \lfloor \frac{m}{2} \rfloor$ and $t = \lceil \frac{m}{2} \rceil$. Let us define $\phi := \phi_{\lfloor \frac{m}{2} \rfloor, \lceil \frac{m}{2} \rceil}$ and its dilatation is the largest root of the polynomial

$$\begin{aligned} T_m(x) &:= T_{\lfloor \frac{m}{2} \rfloor, \lceil \frac{m}{2} \rceil}(x) \\ &= (x-1)x^{(m+1)} - 2\left(x^{\lfloor \frac{m}{2} \rfloor + 1} + x^{\lceil \frac{m}{2} \rceil + 1}\right) - (x-1). \end{aligned}$$

Proposition 4.1 *If $m \geq 5$, then the largest real root of $T_m(x)$ is bounded above by $m^{\frac{3}{m}}$.*

Proof For all m , we have $T_m(1) = -4$. It is sufficient to show that for all $x \geq m^{\frac{3}{m}}$, we have $T_m(x) > 0$. Dividing the inequality by $x^{(m+1)}$, it is equivalent to show

$$(x-1) + x^{-(m+1)} > 2\left(x^{\lfloor \frac{m}{2} \rfloor - m} + x^{\lceil \frac{m}{2} \rceil - m}\right) + x^{-m}.$$

For $m \geq 5$, one can verify the following inequalities hold for all $x \geq m^{\frac{3}{m}}$

- (1) $x-1 > \frac{3 \log m}{m} \geq \frac{9}{2m}$,
- (2) $x^{\lfloor \frac{m}{2} \rfloor - m} \leq x^{\lceil \frac{m}{2} \rceil - m} \leq \frac{1}{m}$,
- (3) $x^{-m} \leq \frac{1}{25m}$.

Therefore,

$$\begin{aligned} (x-1) + x^{-(m+1)} &> x-1 > \frac{9}{2m} > \frac{101}{25m} = 2\left(\frac{1}{m} + \frac{1}{m}\right) + \frac{1}{25m} \\ &\geq 2\left(x^{\lfloor \frac{m}{2} \rfloor - m} + x^{\lceil \frac{m}{2} \rceil - m}\right) + x^{-m}. \end{aligned}$$

□

Remark Proposition 4.1 fails if we try to replace the bound with $c^{\frac{1}{m}}$ where c is any constant.

Remark Hironaka and Kin [HK06] construct two infinite families of pseudo-Anosovs in $\text{Mod}(S_{0,m})$, with $\phi_{s,t}$ being one of them. Unlike $\phi_{s,t}$, the other family provides the sharp bound on $l_{0,m}$.

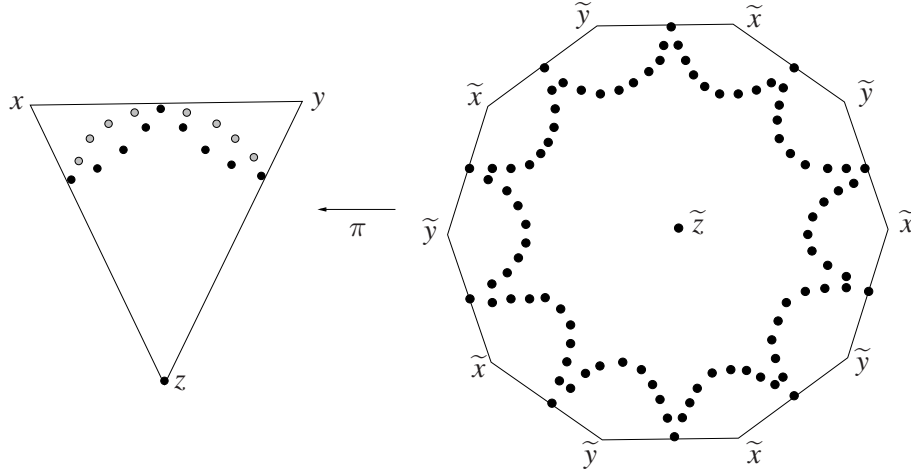


Figure 4: π is the covering map. To form $S_{2,n}$ from the decagon, identify the opposite sides. Then π is the quotient by the group generated by rotation of an angle $2\pi/5$.

Next, we take a cyclic branched cover $S_{2,n}$ of $S_{0,m+2}$ with branched points x , y , and z , where $n = 5(m + 1) + 1$ (See Figure 4.). Define $\tilde{X} = \{\text{marked points around } \tilde{x}\}$ and $\tilde{Y} = \{\text{marked points around } \tilde{y}\}$, so we have $|\tilde{X} \cap \tilde{Y}| = 5$, $|\tilde{X}| = 5(s + 1)$ and $|\tilde{Y}| = 5(t + 1)$.

We lift α_s, β_t to $S_{2,n}$ and call them $\tilde{\alpha}_s, \tilde{\beta}_t$, so that $\tilde{\alpha}_s$ rotates the marked points of \tilde{X} counterclockwise around \tilde{x} and $\tilde{\beta}_t$ rotates the marked points of \tilde{Y} clockwise around \tilde{y} ; see Figure 5. We define $\psi_{s,t} := \tilde{\beta}_t \tilde{\alpha}_s$. It follows that $\psi_{s,t}$ is a lift of $\phi_{s,t}$, and so is pseudo-Anosov with $\lambda(\psi_{s,t}) = \lambda(\phi_{s,t})$. An invariant train track for $\psi_{s,t}$ is obtained by lifting the one constructed in [HK06], and is shown in Figure 6 for $s = t = 3$.

Hence for $n = 5(m + 1) + 1 \geq 31$, we have constructed a pseudo-Anosov $\psi = \psi_{\lfloor \frac{m}{2} \rfloor, \lceil \frac{m}{2} \rceil} \in \text{Mod}(S_{2,n})$ with $\lambda(\psi) = \lambda(\phi) \leq m^{\frac{3}{m}}$ which implies

$$\log \lambda(\psi) \leq \frac{3 \log m}{m} = \frac{15 \log(n - 6) - 15 \log 5}{n - 6}.$$

We will now extend ψ so that n can be an arbitrary number ≥ 31 . We add an extra marked point p_1 on $S_{2,n}$ between points in \tilde{X} or \tilde{Y} except the places shown in Figure 7.

Without loss of generality we assume p_1 is added in \tilde{X} to obtain $S_{2,n+1}$ and we define $\psi_1 := \tilde{\beta}_t \tilde{\alpha}_s' \in \text{Mod}(S_{2,n+1})$ where $\tilde{\alpha}_s'$ is extended from $\tilde{\alpha}_s$ in the obvious way; see Figure 8. One can check that ψ_1 is pseudo-Anosov via the techniques of [BH95].

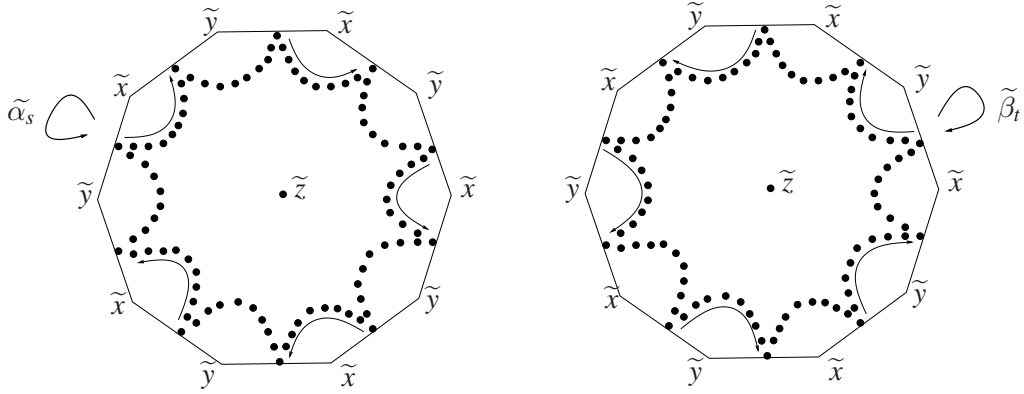


Figure 5: Homeomorphisms $\tilde{\alpha}_s$ and $\tilde{\beta}_t$.

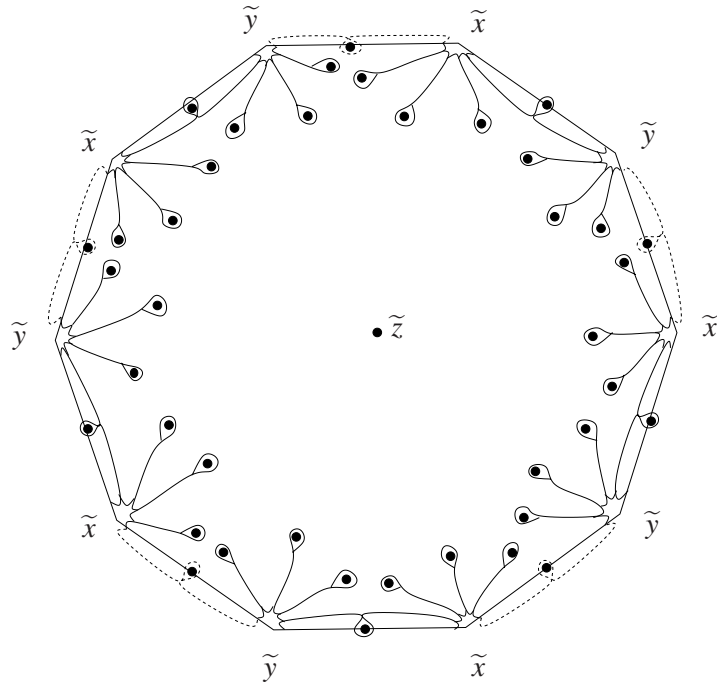


Figure 6: A train track for $\psi_{3,3}$

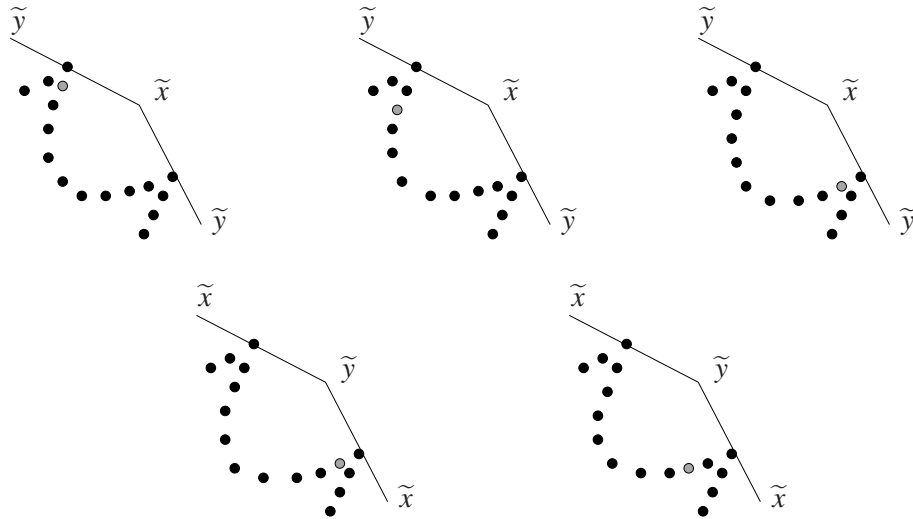


Figure 7: We are *not* allowed to add p_1 in the places indicated by a shaded point.

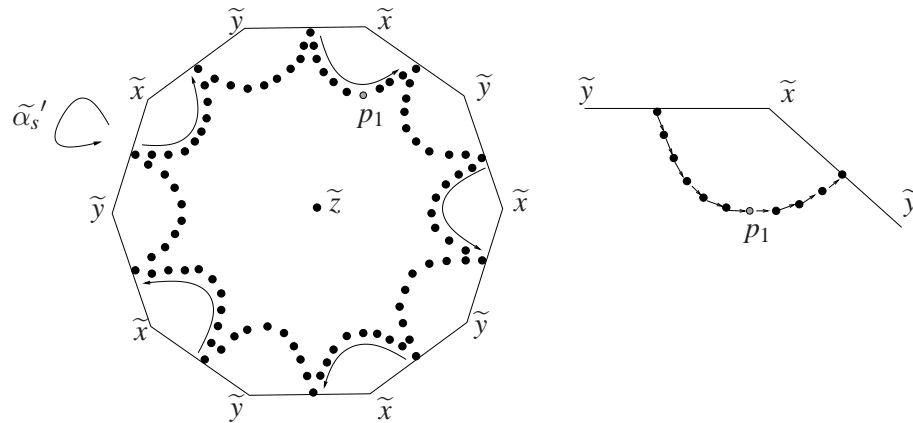


Figure 8: The homeomorphism $\tilde{\alpha}_s'$. The figure on the right is a local picture near the added point p_1 .

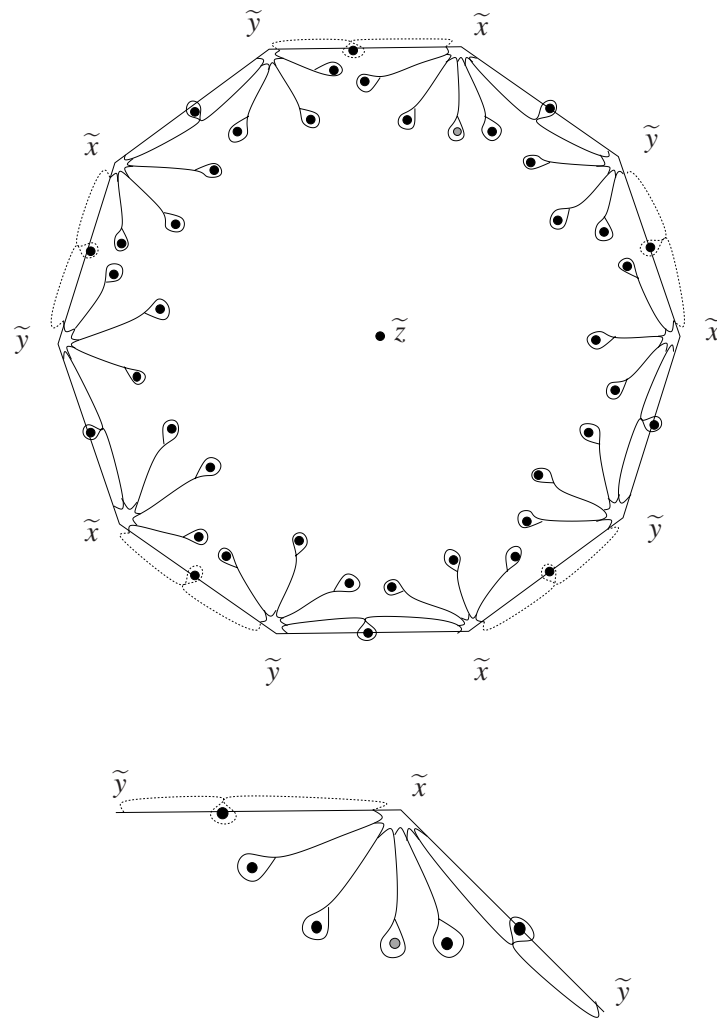


Figure 9: A train track for ψ_1 . The figure on the bottom is a local picture.

An invariant train track for ψ_1 is shown in Figure 9 and is obtained by modifying the invariant train track for ψ shown in Figure 6.

Next, we will show $\lambda(\psi_1) \leq \lambda(\psi)$. Let H (respectively, H_1) be the associated transition matrix of the train track map for ψ (respectively, ψ_1), and let Γ (respectively, Γ_1) be the induced directed graph as constructed in Section 2.2.

From the construction above (i.e. adding p_1), the directed graph Γ_1 is obtained by adding a vertex on the edge going out from some vertex i in Γ (that is, subdividing the edge going out from i) where i has exactly one edge coming in and exactly one edge going out. This implies $P_{\Gamma_1}(i, k+1) = P_{\Gamma}(i, k)$ and

$${}^{k+1}\sqrt{P_{\Gamma_1}(i, k+1)} \leq \sqrt[k]{P_{\Gamma_1}(i, k+1)} = \sqrt[k]{P_{\Gamma}(i, k)}$$

for all k . Since H and H_1 are Perron-Frobenius matrices with Perron-Frobenius eigenvalues corresponding to the dilatations of ψ and ψ_1 , and Proposition 2.5 tells us $\mu(H_1) \leq \mu(H)$, we have $\lambda(\psi_1) = \mu(H_1)$ is no greater than $\lambda(\psi) = \mu(H)$.

We can obtain ψ_2, ψ_3 and ψ_4 by repeating the construction above of adding more marked points without increasing dilatations (i.e. $\lambda(\psi_c) \leq \lambda(\psi)$ for $c = 1, 2, 3, 4$). Since $\frac{\log m}{m} \geq \frac{\log(m+1)}{m+1}$, we need not consider the cases with $c \geq 5$. Therefore, set $f : S_{2,n} \rightarrow S_{2,n}$ to be ψ_c , where $n = 5(m+1) + 1 + c$ with $c < 5$, and where $\psi_0 = \psi$. For $n \geq 31$, we have

$$\log \lambda(f) \leq \log \lambda(\psi) < \frac{3 \log m}{m} < \frac{3 \log \left(\frac{n-11}{5}\right)}{\left(\frac{n-11}{5}\right)},$$

where $m = \lfloor \frac{n-6}{5} \rfloor$.

Theorem 4.2 *There exists $\kappa_2 > 0$ such that $l_{2,n} < \frac{\kappa_2 \log n}{n}$, for all $n \geq 3$.*

Proof From the discussion above, for $n \geq 31$,

$$l_{2,n} < \frac{3 \log \left(\frac{n-11}{5}\right)}{\left(\frac{n-11}{5}\right)} < \frac{\kappa'_2 \log n}{n},$$

for some κ'_2 . For $3 \leq n \leq 30$, let $\kappa''_2 = \max\{l_{2,3}, l_{2,4}, \dots, l_{2,30}\}$ then

$$l_{2,n} \leq \kappa''_2 = \left(\kappa''_2 \frac{31}{\log 31}\right) \frac{\log 31}{31} < \left(\kappa''_2 \frac{31}{\log 31}\right) \frac{\log n}{n}.$$

Let $\kappa_2 := \max\{\kappa'_2, \kappa''_2 \frac{31}{\log 31}\}$. □

4.2 Higher genus cases

We can generalize our construction and extend to any genus $g > 2$. For any fixed $g > 2$, we define ψ to be a homeomorphism of $S_{g,n}$ in the same fashion with $n = (2g + 1)(m + 1) + 1$ by taking an appropriate branched cover over $S_{0,m+2}$, and we can again extend to arbitrary n by adding c extra marked points and constructing ψ_c . Define $f : S_{g,n} \rightarrow S_{g,n}$ to be ψ_c where $n = (2g + 1)(m + 1) + 1 + c$. If $n \geq 6(2g + 1) + 1$, then

$$\begin{aligned} \log \lambda(f) &< \frac{3 \log m}{m}, \text{ where } m = \left\lfloor \frac{n-1}{2g+1} \right\rfloor - 1 \\ &< \frac{3 \log \left(\frac{n-4g-3}{2g+1} \right)}{\left(\frac{n-4g-3}{2g+1} \right)}. \end{aligned}$$

Theorem 4.3 *For any fixed $g \geq 2$, there exists $\kappa_g > 0$ such that $l_{g,n} < \frac{\kappa_g \log n}{n}$, for all $n \geq 3$.*

Proof This is similar to the proof of Theorem 4.2, where κ_g is defined to be

$$\kappa_g := \max \left\{ \kappa'_g, \kappa''_g \frac{12g+7}{\log(12g+7)} \right\}.$$

□

Proof of Theorem 1.1 We only need to prove that the lower bounds on $\log \lambda(f)$ of Theorem 3.3 are bounded below by $\frac{\log n}{\omega_g n}$ for some ω_g depending only on g , then let $c_g = \max\{\kappa_g, \omega_g\}$. We use the monotone decreasing property of $\frac{\log n}{n}$ for $n \geq 3$. Let

$$\omega'_g(\alpha) := \frac{\alpha(12g-12) \log 3}{\log 2 \cdot 3} \geq \frac{\alpha(12g-12) \log n}{\log 2 \cdot n}$$

and so

$$\frac{\log 2}{\alpha(12g-12)} \geq \frac{\log n}{\omega'_g(\alpha)n}.$$

For $n \geq g - 1$,

$$\frac{\log(6g+3n-6)}{2\alpha(6g+3n-6)} \geq \frac{\log 9n}{2\alpha 9n} > \frac{1}{18\alpha} \frac{\log n}{n}.$$

For $3 \leq n < g - 1$,

$$\frac{\log(6g+3n-6)}{2\alpha(6g+3n-6)} > \frac{\log(9(g-1))}{2\alpha 9(g-1)} > \frac{\log g}{18\alpha g \log 3} \frac{3 \log n}{n}.$$

Let $\omega_g := \max\{\omega'_g(\alpha), 18\alpha, \frac{6\alpha g \log 3}{\log g}\}$, where $0 \leq \alpha \leq \Theta(g)$.

□

5 Appendix

5.1 Torus with marked points

We will construct an example to prove that $l_{1,2n}$ has an upper bound of the same order as Penner's lower bound in [Pen91], i.e. $l_{1,2n} = O(\frac{1}{n})$. The construction is analogous to the one given by Penner for $S_{g,0}$ in [Pen91].

Let $S_{1,2n}$ be a marked torus of $2n$ marked points. Let a and b be essential simple closed curves as in Figure 10. Let T_a^{-1} be the left Dehn twist along a and T_b be the

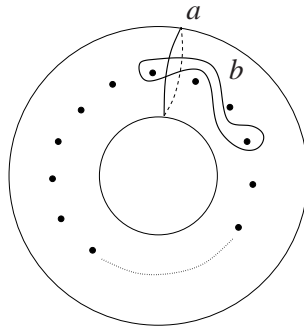


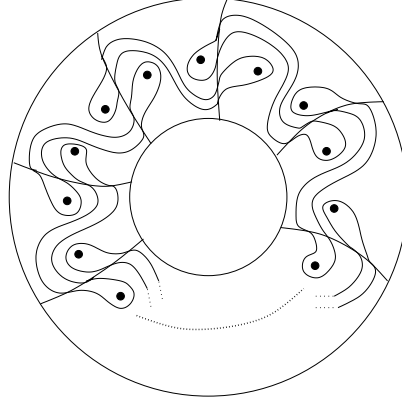
Figure 10: Essential simple closed curves a and b on a marked torus.

right Dehn twist along b , then we define

$$f := \rho \circ T_b \circ T_a^{-1} \in \text{Mod}(S_{1,2n})$$

where ρ rotates the torus clockwise by an angle of $2\pi/n$, so it sends each marked point to the one which is two to the right. As in [Pen88], f^n is shown to be pseudo-Anosov, and thus so is f . Figure 11 shows a bigon track for f^n .

We obtain the $2n \times 2n$ transition matrix M^n associated to the train track map of f^n where M^n is an integral Perron-Frobenius matrix and the Perron-Frobenius eigenvalues

Figure 11: A bigon track for f^n .

$\mu(M^n)$ is the dilatation $\lambda(f^n)$ of f^n . For $n \geq 5$, we have $M^n = N$, where

$$N = \begin{pmatrix} A_1 & B_1 & 0 & 0 & \cdots & 0 & 0 & D_1 \\ A_2 & B_2 & B_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & B_3 & B_2 & B_1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & B_3 & B_2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & B_3 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & B_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & B_2 & B_1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & B_3 & B_2 & D_2 \\ A_3 & C & 0 & 0 & \cdots & 0 & B_3 & D_3 \end{pmatrix},$$

and

$$\begin{aligned} A_1 &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \\ B_1 &= \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, B_2 = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}, B_3 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \\ D_1 &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, D_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, D_3 = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}. \end{aligned}$$

For $n \geq 5$, the greatest column sum of M^n is 9 and the greatest row sum of M^n is 11. One can verify that both the greatest column sum and the greatest row sum are ≤ 11

for $0 < n \leq 4$. Therefore, for $n \geq 1$,

$$\begin{aligned} 11 &\geq \mu(M^n) = \lambda(f^n) = (\lambda(f))^n \\ &\Rightarrow l_{1,2n} \leq \log \lambda(f) \leq \frac{\log 11}{n}. \end{aligned}$$

5.2 Higher genus with marked points

In all of the following examples we obtain a mapping class $\tilde{f} \in \text{Mod}(S_{g,n})$ from $f \in \text{Mod}(S_{g,0})$ by adding marked points on the closed surface $S_{g,0}$, where f is a composition of Dehn twists along some set \mathcal{T} of closed geodesics. We can add one marked point in each of the complementary disks of the curves in \mathcal{T} without creating essential reducing curves. By [Pen88, Theorem 3.1], the induced mapping class $\tilde{f} \in \text{Mod}(S_{g,n})$ is pseudo-Anosov with dilatation $\lambda(\tilde{f}) = \lambda(f)$.

Example 1 Penner [Pen91] constructed a pseudo-Anosov mapping class $f \in \text{Mod}(S_{g,0})$ with dilatation $\lambda(f) \leq \frac{\log 11}{g}$ for $g \geq 2$, where

$$f := \rho \circ T_c \circ T_a^{-1} \circ T_b.$$

and T_α is the Dehn twist along α . Here $\mathcal{T} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ with

$$\mathcal{A} = \sqcup_{i=1}^g a_i, \mathcal{B} = \sqcup_{i=1}^g b_i, \text{ and } \mathcal{C} = \sqcup_{i=1}^g c_i.$$

We can add g marked points as in the Figure 12 so that $\tilde{f} \in \text{Mod}(S_{g,g})$ is pseudo-Anosov. Therefore,

$$l_{g,g} \leq \log \lambda(\tilde{f}) \leq \frac{\log 11}{g}.$$

We can also add extra marked points at the fixed points of the rotation. For $g \geq 2$, we will have for $c = 0, 1$ and 2 ,

$$l_{g,g+c} \leq \log \lambda(\tilde{f}) \leq \frac{\log 11}{g},$$

where $\tilde{f} \in \text{Mod}(S_{g,g+c})$.

Example 2 For all $g \geq 3$, define $f : S_{g,0} \rightarrow S_{g,0}$ to be

$$f := \rho \circ T_{b_1} \circ T_{a_1}^{-1},$$

where,

$$\begin{aligned} \rho(a_1) &= a_{g+1}, \rho(b_1) = b_{g+1}, \text{ and} \\ \rho(a_i) &= a_{i-1}, \rho(b_i) = b_{i-1}, \text{ for } i = 2, \dots, g+1. \end{aligned}$$

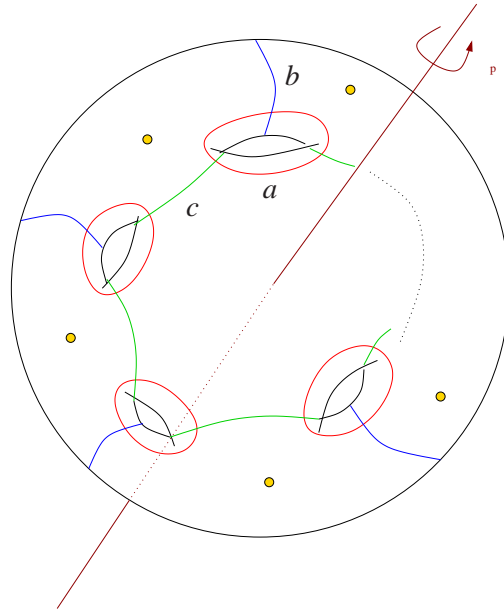


Figure 12: a pseudo-Anosov $\tilde{f} \in \text{Mod}(S_{g,g})$.

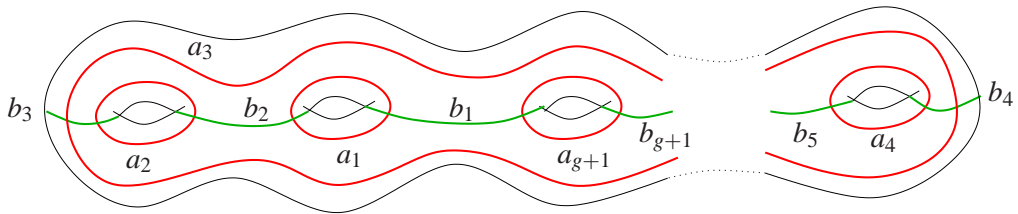


Figure 13: a pseudo-Anosov $f \in \text{Mod}(S_{g,0})$.

We construct the $(2g + 2) \times (2g + 2)$ transition matrix $M^{(g+1)}$ with respect to the spanning vectors associated with geodesics in \mathcal{T} . We will get $M^{(g+1)} = N$ for $g \geq 3$, where the matrices are the same as in Appendix 5.1. Therefore for $g \geq 3$ we have

$$\log \lambda(f) \leq \frac{\log 9}{g + 1}.$$

Here $\mathcal{T} = \mathcal{A} \cup \mathcal{B}$ with

$$\mathcal{A} = \sqcup_{i=1}^g a_i, \text{ and } \mathcal{B} = \sqcup_{i=1}^g b_i.$$

For $g \geq 3$ and $c = 0, 1, 2, 3, 4$, we have

$$l_{g,c} \leq \log \lambda(\tilde{f}) \leq \frac{\log 9}{g + 1},$$

where $\tilde{f} \in \text{Mod}(S_{g,c})$.

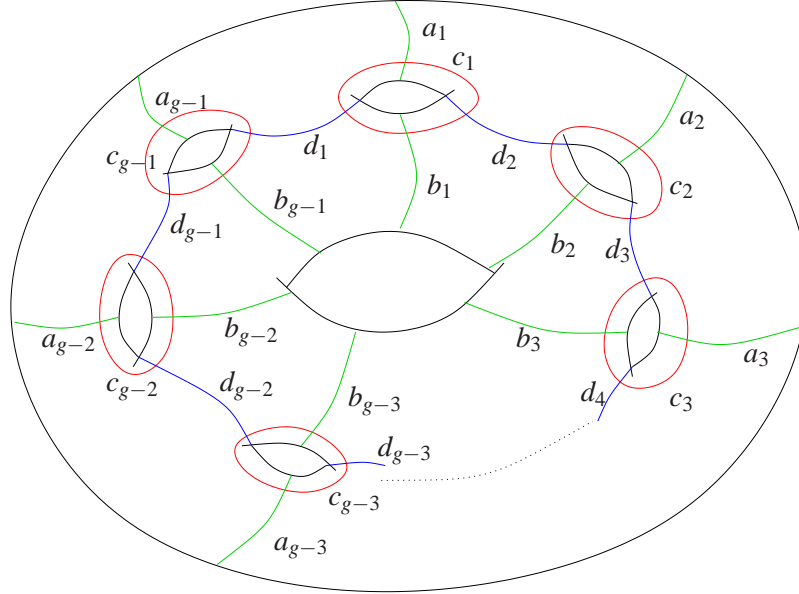
Example 3 For $g \geq 5$, define $f : S_{g,0} \rightarrow S_{g,0}$ by

$$f := \rho \circ T_{d_1} \circ T_{c_1}^{-1} \circ T_{b_1} \circ T_{a_1},$$

where

$$\begin{aligned} \rho(a_1) &= a_{g-1}, \rho(b_1) = b_{g-1}, \rho(c_1) = c_{g-1}, \rho(d_1) = d_{g-1}, \text{ and} \\ \rho(a_i) &= a_{i-1}, \rho(b_i) = b_{i-1}, \rho(c_i) = c_{i-1}, \rho(d_i) = d_{i-1}, \text{ for } i = 2, \dots, g - 1. \end{aligned}$$

Similarly, we have the $(4g - 4) \times (4g - 4)$ transition matrix $M^{(g-1)}$ with respect to the spanning vectors associated with the geodesics in \mathcal{T} . For $g \geq 5$ we have $M^{(g-1)} = N$

Figure 14: a pseudo-Anosov $f \in \text{Mod}(S_{g,0})$.

where

$$\begin{aligned}
 A_1 &= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 3 & 2 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 3 & 1 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 3 & 2 \\ 1 & 1 & 3 & 0 \end{pmatrix}, \\
 B_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, B_2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 3 & 3 \end{pmatrix}, B_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 3 & 1 \end{pmatrix}, \\
 C &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
 D_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, D_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, D_3 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 4 & 3 \end{pmatrix}.
 \end{aligned}$$

For $g \geq 5$, the greatest column sum of $M^{(g-1)}$ is 17 and the greatest row sum of $M^{(g-1)}$

is 21, hence

$$\log \lambda(f) \leq \frac{\log 17}{g-1}.$$

Here $\mathcal{T} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$ with

$$\mathcal{A} = \sqcup_{i=1}^g a_i, \mathcal{B} = \sqcup_{i=1}^g b_i, \mathcal{C} = \sqcup_{i=1}^g c_i, \text{ and } \mathcal{D} = \sqcup_{i=1}^g d_i.$$

For $c = 1$ and 2 , we can induce $\tilde{f} \in \text{Mod}(S_{g,c(g-1)})$ with

$$l_{g,c(g-1)} \leq \log \lambda(\tilde{f}) \leq \frac{\log 17}{g-1},$$

when $g \geq 5$.

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