MINIMAL PSEUDO-ANOSOV TRANSLATION LENGTHS ON THE TEICHMÜLLER SPACE

BY

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DISSERTATION

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ABSTRACT

This thesis is a study of the asymptotic behavior of minimal pseudo-Anosov translation lengths on the Teichmüller space.

For tori with n marked points, we find an upper bound for the minimal pseudo-Anosov translation length. The upper bound is on the order of $\frac{1}{|\chi(S)|}$. We find similar asymptotics for genus g surfaces with n marked points as g and n vary in certain prescribed ways.

However, for a surface S with fixed genus $g \geq 2$ and varied n, we prove that the asymptotic behavior (in n) is different. The main result of this thesis is that the least pseudo-Anosov translation length shrinks to zero on the order of $\frac{\log|\chi(S)|}{|\chi(S)|}$ as $|\chi(S)| \to \infty$. This is in contrast with the previously-known results for the cases of closed surfaces and marked spheres, in which the behavior is on the order of $\frac{1}{|\chi(S)|}$.

These results are published in [Tsa09].

To my family and my husband.

ACKNOWLEDGEMENTS

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CHAPTER 1

INTRODUCTION

In this thesis, we will only consider hyperbolic surfaces $S = S_{g,n}$ of genus g and n marked points with negative Euler characteristic, $\chi(S) = 2 - 2g - n < 0$.

In the 1920s and 1930s, Nielsen initiated a program to classify homeomorphisms of surfaces S (up to isotopy). A complete classification was obtained by Thurston in the 1970s, which says *a mapping class is finite order, reducible or pseudo-Anosov.* This initiated an active research program in the study of the elements of the mapping class group, especially its pseudo-Anosov elements. Each pseudo-Anosov mapping class is equipped with a quantity which is called its **pseudo-Anosov dilatation**, which has multiple interpretations as a measure of the complexity of the mapping class. For example, the logarithm of the pseudo-Anosov dilatation is equal to the translation length of a pseudo-Anosov mapping class acting on the Teichmüller space with respect to the Teichmüller metric. The set of all such logarithms is thus the same as the set of lengths of closed geodesics in the moduli space. In the area of dynamics, the logarithm of the pseudo-Anosov dilatation is equal to the minimal topological entropy of any representative. For more details, see Section 2.1 and 2.2.

For a surface S, the set of pseudo-Anosov dilatations of S has a minimal element (see [AY81], [Iva88]). We will refer to its logarithm as the **minimal (pseudo-Anosov) translation length** in the Teichmüller space, denoted $l_{g,n}$. The exact values are hard to compute and are only known in a few cases (see [CH08], [Hir09], [KLS02], [LT09]). On the other hand, we can study its asymptotic behavior while varying the genus and number of marked points. Due to the work of Penner and Hironaka-Kin ([Pen91], [HK06]), we know that for closed surfaces of genus ≥ 2 and for spheres with at least four marked points, the minimal translation length has asymptotic behavior $\frac{1}{|\chi(S)|}$ as $|\chi(S)| \to \infty$. However, in general, the situation is more complicated. For a given genus $g \geq 2$, and variable number of marked points, we prove a sharper lower bound for $l_{g,n}$ than was previously known and find examples which provide an upper bound of the same order. This is the main result of this thesis which records as

Main Theorem. *For any fixed $g \geq 2$, the asymptotic behavior of $l_{g,n}$ is $\frac{\log|\chi(S)|}{|\chi(S)|}$ as $n \to \infty$.*

(See Theorem 3.3.1 for a more precise statement.) In particular, this theorem says that for a fixed $g \geq 2$, the minimal translation length goes to zero slower than $\frac{1}{|\chi(S)|}$. In Section 3.2, we discuss other cases which have the asymptotic behavior $\frac{1}{|\chi(S)|}$. We summarize the known behaviors of $l_{g,n}$ in the following table.

(g, n)-**rays**	**The asymptotic behavior of** $l_{g,n}$
$g = 0$	$1/\lvert\chi(S_{g,n})\rvert$
$g = 1$	$1/\lvert\chi(S_{g,n})\rvert$
$g =$ constant ≥ 2	$\log\left(\lvert\chi(S_{g,n})\rvert\right)/\lvert\chi(S_{g,n})\rvert$
$n = 0, 1, 2, 3,$ or 4	$1/\lvert\chi(S_{g,n})\rvert$
$n = g, g+1,$ or $g+2$	$1/\lvert\chi(S_{g,n})\rvert$
$n = g-1$ or $2(g-1)$	$1/\lvert\chi(S_{g,n})\rvert$

Naturally, we ask the following question.

Question 1.0.1. What are the asymptotic behaviors of $l_{g,n}$ along different rays in the (g, n) plane?

More generally, instead of restricting along rays, we want to understand the following.

Question 1.0.2. What is the general asymptotic behavior of $l_{g,n}$ going out toward infinity in the (g, n) plane?

CHAPTER 2

PRELIMINARIES

In this chapter, we define the main objects and follow with some examples. We also introduce some useful tools which are used later in our proofs.

2.1 Mapping class group

For more detailed discussions of surface homeomorphisms and the mapping class group see [FLP91], [FM10], and [Thu88].

Let $f : S_{g,n} \to S_{g,n}$ be an orientation preserving homeomorphism of $S_{g,n}$.

Definition 2.1.1. f is **periodic** (or finite order) if there exists $k > 0$ such that f^k is the identity.

The identity map is obviously finite order. A less trivial example is a rotation map.

Example 2.1.2. $\rho : S_{3,0} \to S_{3,0}$ is a rotation map which rotates counterclockwise along the z-axis by angle $\frac{2\pi}{3}$ with $S_{3,0} \subset \mathbb{R}^3$ as shown in Figure 2.1, then ρ^3 is the identity.

A **simple closed curve** is an embedding $\alpha : S^1 \to S$. We identify a simple closed curve with its image $\alpha \subset S$. We say $\alpha \subset S$ is **essential** if it does not bound a disc with 0 or 1 marked point. We are primarily interested in simple closed curves up to iostopy, and whenever considering a collection of simple closed curves we will assume that they pairwise intersect transversely and minimally.

Definition 2.1.3. If there exists a disjoint union of essential simple closed curves, $\bigsqcup_{i=1}^{k} \alpha_i$ on $S_{g,n}$, such that $f(\bigsqcup_{i=1}^{k} \alpha_i) = \bigsqcup_{i=1}^{k} \alpha_i$, then f is **reducible**.

If f is not reducible, we say f is **irreducible**.

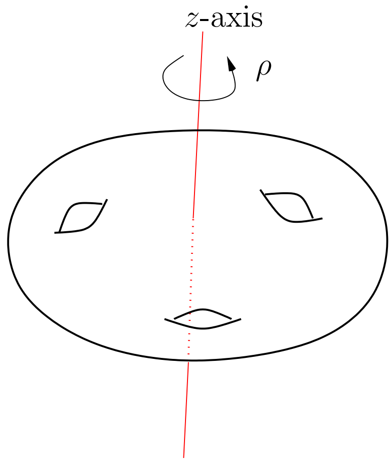

Figure 2.1: A rotation map of $\frac{2\pi}{3}$.

A basic example of a reducible homeomorphism is the (positive) **Dehn twist** T_α along an essential simple closed curve α. This is a homeomorphism $T_\alpha : S \to S$ for which there is a neighborhood A of α homeomorphic to an annulus $\{re^{i\theta} \in \mathbb{C} | 1 \leq r \leq 2\}$, so that T_α is the identity outside A and on A is given by

$$re^{i\theta} \to re^{i(\theta+2\pi r)}.$$

See Figure 2.2. The negative Dehn twist T_α^{-1} is the inverse which is given by

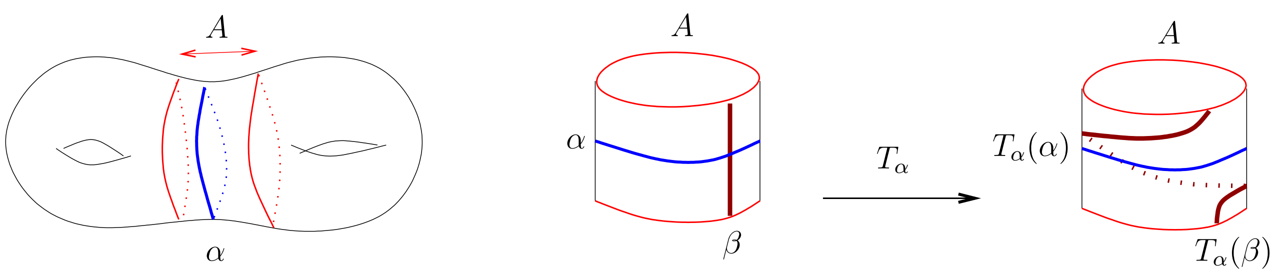

Figure 2.2: Dehn twist along α, β is an arc intersecting α once.

$re^{i\theta} \to re^{i(\theta-2\pi r)}$ on A. A **multitwist** is the composition of Dehn twists on a collection of pairwise disjoint essential simple closed curves.

We now define a third type of homeomorphism which will be the primary focus of our investigation. First we require a few preliminary definitions.

A **singular foliation** $\mathcal{F}$ is a decomposition of S into disjoint union of one-dimensional injectively immersed submanifolds, which are called *leaves*, and a finite set of points, called *singularities*, such that

- for every non-singular point x, there is a chart from a neighborhood of x to $\mathbb{R}^2$, which maps leaves to horizontal line segments,
- for every singularity y, there is a chart from a neighborhood of y to $\mathbb{R}^2$, which maps leaves to the level set of a k-prong saddle, k is a positive integer and $k \neq 2$; see Figure 2.3.

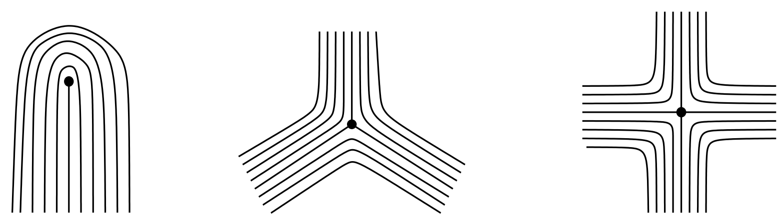

Figure 2.3: Neighborhoods of $1, 3, 4$-prong singularities.

A **transverse measure** μ on $\mathcal{F}$ is a function that assigns a positive real number to each arc transverse to $\mathcal{F}$ such that it is invariant under leaf preserving isotopy, and for each point, there is a smooth chart from a neighborhood to $\mathbb{R}^2$, so that the measure is induced by $|dy|$. A foliation equipped with transverse measure is a **measured foliation**.

Definition 2.1.4. A homeomorphism $f : S_{g,n} \to S_{g,n}$ is **pseudo-Anosov** if there are transverse singular foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ together with transverse measures μ^s and μ^u such that for some $\lambda > 1$,

$$f(\mathcal{F}^s, \mu^s) = (\mathcal{F}^s, \lambda^{-1}\mu^s),$$
$$f(\mathcal{F}^u, \mu^u) = (\mathcal{F}^u, \lambda\mu^u).$$

The number $\lambda = \lambda(f)$ is called the **dilatation** of f.

A collection of pseudo-Anosov homeomorphisms is produced in [Pen88, Theorem 3.1]. This construction is used in Section 3.2, so we will restate it here.

Given two collections of pairwise disjoint essential simple closed curves $\mathcal{C} = c_1 \sqcup \cdots \sqcup c_n$, $\mathcal{D} = d_1 \sqcup \cdots \sqcup d_m$, we say that $\mathcal{C} \cup \mathcal{D}$ **fills** S if the components of $S \setminus (\mathcal{C} \cup \mathcal{D})$ are discs or discs with exactly one marked point (recall our convention that $\mathcal{C}$ intersects $\mathcal{D}$ minimally).

Theorem 2.1.5 ([Pen88]). *Suppose $\mathcal{C} = c_1 \sqcup \cdots \sqcup c_n$ and $\mathcal{D} = d_1 \sqcup \cdots \sqcup d_m$ are two sets of pairwise disjoint essential simple closed curves in S so that $\mathcal{C} \cup \mathcal{D}$ fills S. If f is any product of T_{c_i} and $T_{d_j}^{-1}$, where each c_i and each d_j occur at least once, then f is isotopic to a pseudo-Anosov.*

Two other well known constructions of pseudo-Anosov homeomorphisms are due to Thurston. The first one is compositions of Dehn twists which can be found in [Thu88, Theorem 7]. The second is a minor variation obtained by composing "half (Dehn) twists", which are examples of *point pushing* maps. We will demonstrate this second one in the following example. From this construction, we can easily see how the mapping class group relates to the braid group.

Example 2.1.6. Let D_1 be a disc containing the marked points x_1, x_2, and f_1 is the counterclockwise half twist in D_1 which maps x_1 to x_2 and x_2 to x_1 and is the identity outside D_1; see Figure 2.4. Similarly, let D_2 be a disc containing x_2, x_3, and f_2 is the

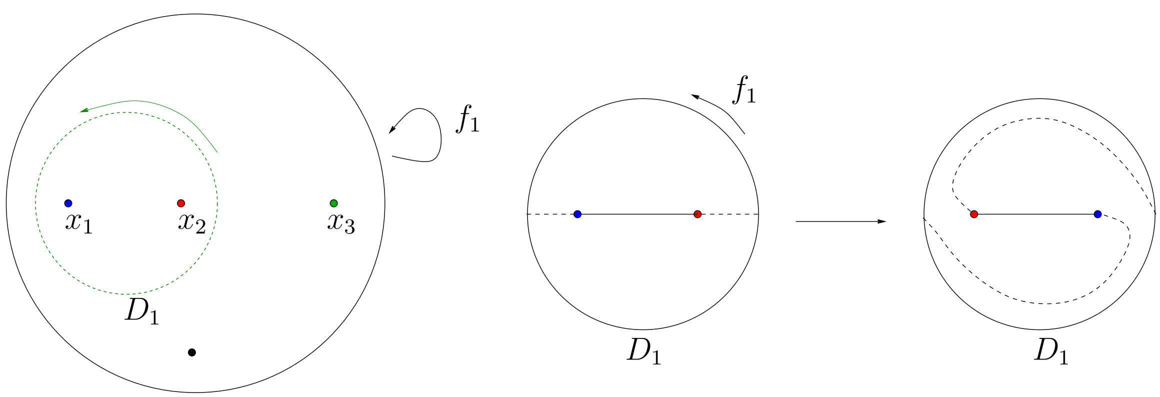

Figure 2.4: $f_1 : S \to S$.

clockwise half twist in D_2. Now, let $f : S_{0,4} \to S_{0,4}$ be $f_2 \circ f_1$ as shown in Figure 2.5. f is isotopic to a pseudo-Anosov (see Section 2.4 for further discussion of this example).

To picture this example, we have chosen an essential closed curve α and drawn the image of α under the first two iterations of f in Figure 2.6. We can see that the length of α

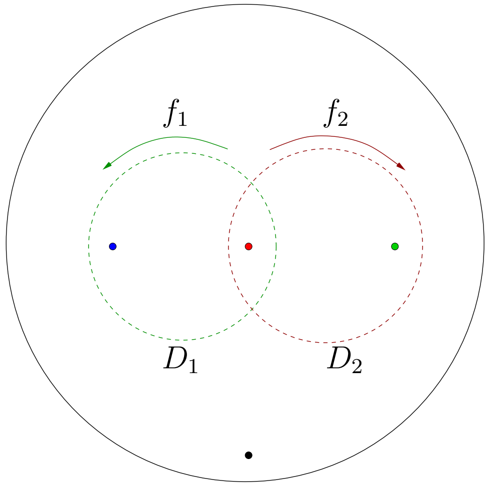

Figure 2.5: $f := f_2 \circ f_1 : S \to S$.

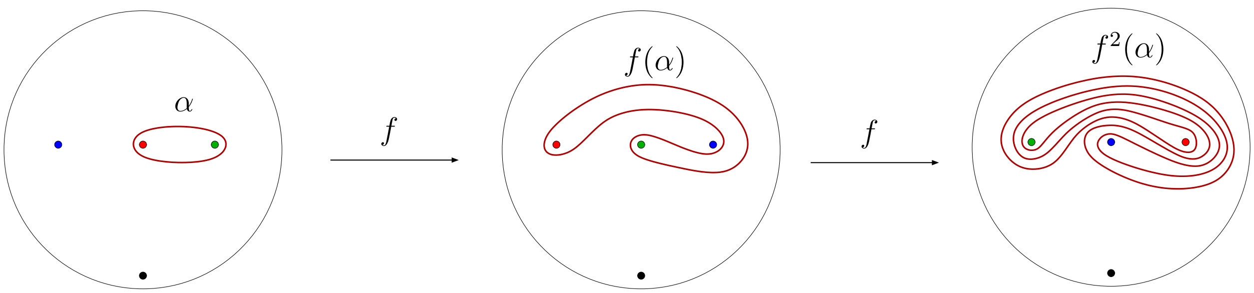

Figure 2.6: The image of α under the iteration of f.

increases whenever we apply f. In fact, we have the following, see [FLP91].

Theorem 2.1.7 (Thurston)**.** *Given any complete Riemannian metric on S (with marked points removed), if $F : S \to S$ is isotopic to a pseudo-Anosov homeomorphism $f : S \to S$,*

$$\lim_{k \to \infty} \sqrt[k]{\ell([F^k(\alpha)])} = \lambda(f),$$

for any essential closed curve α, where $\ell([\alpha]) :=$ length of a minimizing geodesic in the isotopy class of α.

The set of isotopy classes of orientation preserving homeomorphisms $f : S \to S$ forms the **mapping class group** $\mathrm{Mod}(S)$, where the group operation is induced by composition of functions. We say that a mapping class $[f]$ is pseudo-Anosov, reducible or periodic (respectively) if f is isotopic to a pseudo-Anosov, reducible or periodic homeomorphism (respectively). As a slight abuse of notation, we sometimes refer to a mapping class $[f]$ by one of its representatives f.

As a corollary of the Theorem 2.1.7, if $[f]$ is pseudo-Anosov, $\lambda(f)$ is an invariant of $[f]$, which means each pseudo-Anosov mapping class comes equipped with exactly one dilatation. There is an amazing result about the mapping class group. Its proof can be found in [FLP91].

Theorem 2.1.8. *(Nielsen-Thurston Classification) Each mapping class is periodic, reducible or pseudo-Anosov. Moreover, the pseudo-Anosov case is exclusive from the other two cases.*

There are finitely many conjugacy classes of periodic mapping classes. For any reducible homeomorphism, we can cut the surface open along the reducing curves, reduce the complexity of the surface, and arrive at an irreducible homeomorphism of the resulting surface. Hence, in studying the mapping class group, it is essential to understand its pseudo-Anosov elements.

Let us construct a different family of reducible homeomorphisms which when restricted to a proper subsurface is pseudo-Anosov, in which we reduce the complexity of the surface. The complexity of S is

$$\xi(S_{g,n}) = 3g + n - 3.$$

Example 2.1.9. Let $f : S \to S$ be the composition of positive Dehn twists along α_i's and negative Dehn twists along β_i's, that is

$$f := T_{\alpha_1} T_{\beta_1}^{-1} T_{\alpha_2} T_{\beta_2}^{-1};$$

see Figure 2.7. Clearly, γ is invariant under f, hence, f is reducible. If we cut S open along γ, the complexity of Σ is $6 + 1 - 3 = 4 < 6 = \xi(S)$, and f restricted to Σ induces a pseudo-Anosov homeomorphism $\Sigma \to \Sigma$ by Theorem 2.1.5.

We will need to following facts for the proofs in Section 3.3.1.

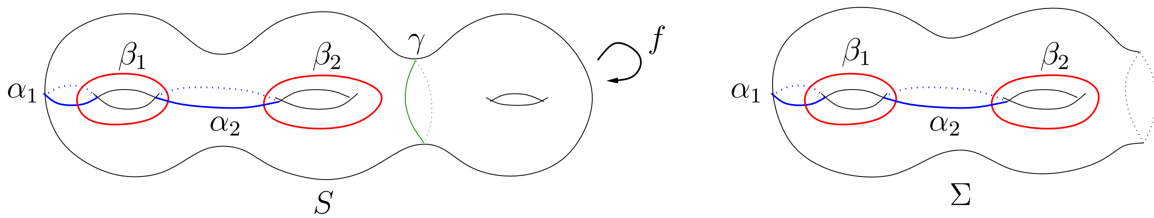

Figure 2.7: f restricted to Σ is pseudo-Anosov.

Definition 2.1.10. A homeomorphism $f : S \to S$ is **pure** if there exists a closed one-dimensional submanifold C of S satisfying the following:

The components of C are essential and pairwise nonisotopic; f is the identity on C (and so it does not rearrange the components of $S \setminus C$), and it induces on each component of $S \setminus C$ a homeomorphism isotopic to either a pseudo-Anosov or the identity homeomorphism.

An element $f \in \text{Mod}(S)$ is pure if f is isotopic to a pure homeomorphism. Such an element has infinite order if it is nontrivial.

Let $\Gamma_S(m) \lhd \text{Mod}(S)$ denote the kernel of the natural homorphsim

$$\text{Mod}(S) \to \text{Aut}(H_1(S : \mathbb{Z}/m\mathbb{Z}))$$

defined by the action of homeomorphism in the homology. Clearly, the subgroup $\Gamma_S(m)$ has finite index in $\text{Mod}(S)$.

Theorem 2.1.11 ([Iva92])**.** *Fix $m \geq 3$. For $f \in \text{Mod}(S)$, if $f_* : H_1(S : \mathbb{Z}/m\mathbb{Z}) \to H_1(S : \mathbb{Z}/m\mathbb{Z})$ is the identity, then f is pure.*

In other words, for $m \geq 3$, the $\Gamma_S(m)$ consists of pure elements.

2.2 Teichmüller space

We will review some basic definitions and properties of Teichmüller space, more details can be found in [Abi80] and [Gar87].

The **Teichmüller space** $\mathcal{T}(S)$ is defined to be the set of equivalence classes $[(X, f)]$, where $f : S \to X$ is orientation-preserving homeomorphism to a Riemann surface X, and (X, f) and (Y, g) are equivalent if and only if there exists a conformal map $h : X \to Y$ homotopic to $g \circ f^{-1}$. The Teichmüller space $\mathcal{T}(S_{g,n})$ is a metric space (see below) homeomorphic to an open ball $\mathbb{R}^{6g+2n-6}$. The mapping class group acts on $\mathcal{T}(S)$ by

$$[g] \in \mathrm{Mod}(S), [g] \cdot [(X, f)] = [(X, f \circ g^{-1})].$$

This action is by isometries with respect to the Teichmüller metric $d_{\mathcal{T}}$ defined below and properly discontinuous. The quotient space $\mathcal{M}(S)$ is precisely the **moduli space** of Riemann surface homeomorphic to S.

Let $X_1 = [(f_1, X_1)]$ and $X_2 = [(f_2, X_2)]$ be two points in $\mathcal{T}(S)$. The **Teichmüller map** from X_1 to X_2 is a homeomorphism $g : X_1 \to X_2$ homotopic to $f_2 \circ f_1^{-1}$ with the following property. There exist $K \geq 1$ and holomorphic quadratic differentials q_i on X_i such that g takes the zeros of q_1 to the zeros of q_2, and in any natural coordinates for q_1 with the base point p and for q_2 with the base point $g(p)$, we have

$$g(x + iy) = \sqrt{K}x + \frac{1}{\sqrt{K}}iy.$$

Such maps are examples of quasi-conformal maps. Teichmüller's Theorem asserts that there exists a unique Teichmüller map for any two points in $\mathcal{T}(S)$. We define the **Teichmüller metric**

$$d_{\mathcal{T}}(X_1, X_2) := \frac{1}{2}\log(K(g))$$

where $g : X_1 \to X_2$ is the unique Teichmüller map.

Given a pseudo-Anosov $f \in \mathrm{Mod}(S)$, there exists $X \in \mathcal{T}(S)$ and a Teichmuller map $g : X \to f \cdot X$ such that in the natural coordinates of the associated quadratic differentials, $g(x + iy) = \lambda(f)x + \frac{1}{\lambda(f)}iy$. Therefore,

$$d_{\mathcal{T}}(X, f \cdot X)) = \frac{1}{2}\log\left((\lambda(f))^2\right) = \log \lambda(f).$$

Similar to the classification of isometries of hyperbolic space, for any $f \in \text{Mod}(S)$, we define the **translation length** of f to be

$$\tau(f) := \inf_{X \in \mathcal{T}(S)} d_{\mathcal{T}}(X, f \cdot X),$$

where X is an point of $\mathcal{T}(S)$. Any isometry f falls into one of the following cases:

(1) $\tau(f)$ is achieved and equals to 0.

(2) $\tau(f)$ is not achieved.

(3) $\tau(f)$ is achieved and is positive.

From above, we observe that $\log \lambda(f) \geq \tau(f)$. In fact, for any pseudo-Anosov $f \in \text{Mod}(S)$,

$$\log \lambda(f) = \tau(f).$$

The main idea of Bers' proof in [Ber78] of the Nielsen-Thurston Classification Theorem stated in Theorem 2.1.8 is transforming the classification of mapping classes into the above categorization. He shows that for any $f \in \text{Mod}(S)$,

(1) If $\tau(f)$ is achieved and equals to 0, then f is periodic.

(2) If $\tau(f)$ is not achieved, then f is reducible.

(3) If $\tau(f)$ is achieved and is positive, then f is pseudo-Anosov.

2.3 Markov partition and Perron-Frobenious theory

Suppose $f : S \to S$ is pseudo-Anosov with stable and unstable measured singular foliations $(\mathcal{F}^s, \mu^s)$ and $(\mathcal{F}^u, \mu^u)$. We define a rectangle R to be a map

$$\rho : I \times I \to S,$$

such that ρ is an embedding on the interior, $\rho(point \times I)$ is contained in a leaf of $\mathcal{F}^s$, and $\rho(I \times point)$ is contained in a leaf of $\mathcal{F}^u$.

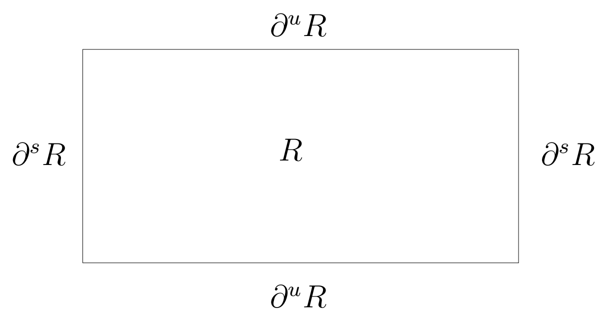

Figure 2.8: A rectangle of a Markov partition.

As a standard abuse of notation, we will write $R \subset S$ for the image of a rectangle map $\rho : I \times I \to S$. We denote $\rho(\partial I \times I)$ by $\partial^s R$ and $\rho(I \times \partial I)$ by $\partial^u R$. Also, we define $intR$ to be $\rho(int(I \times I))$.

Definition 2.3.1. A Markov partition for $f : S \to S$ is a decomposition of S into a finite union of rectangles $\{R_i\}_{i=1}^k$, such that:

1. $int(R_i) \cap int(R_j)$ is empty, when $i \neq j$
2. $f(\bigcup_{j=1}^{k} \partial^s R_j) \subset \bigcup_{j=1}^{k} \partial^s R_j$
3. $f^{-1}(\bigcup_{i=1}^{k} \partial^u R_i) \subset \bigcup_{i=1}^{k} \partial^u R_i$

For any pseudo-Anosov homeomorphism $f : S \to S$, there exist a Markov partition of f, see [CB88], [BH95] or [FLP91]. In particular, the construction in [BH95] has the advantage of controlling the number of rectangles. In particular, one has the following:

Theorem 2.3.2. *For any pseudo-Anosov homeomorphism $f : S \to S$ of a surface S with at least one marked point, there exists a Markov partition for f with at most $-3\chi(S)$ rectangles.*

We can define a **transition matrix** M associated to the Markov partition of f with rectangles $\{R_i\}_{i=1}^k$ by assigning the entry $m_{i,j}$ of M to be the number of times that $f(R_j)$ wraps over R_i. More precisely,

$$m_{i,j} = |f(int(R_j)) \cap int(R_i)|,$$

where $|A|$ is the number of the components of A. Since $f(\bigcup_{j=1}^{k} \partial^s R_j) \subset \bigcup_{j=1}^{k} \partial^s R_j$,

$$(M(f))^s = M(f^s)$$

for all integers s.

A matrix M is **positive** (respectively, nonnegative) if every entry of M is positive (respectively, nonnegative). If for any (i, j), there exist a constant s_{ij}, such that $M^{s_{ij}}$ has a positive (i, j)-th entry, then M is **irreducible**. M is **Perron-Frobenius** if it is irreducible and nonnegative. More details of the Perron-Frobenius theory can be found in [Gan59].

We will need the following two facts.

Theorem 2.3.3 ([Gan59])**.** *If a $k \times k$ matrix M is Perron-Frobenius, then there exists a unique maximal eigenvalue $\mu(M)$ with a eigenvector of positive entry and μ is bounded above both by the maximal row sum and the maximal column sum.*

Theorem 2.3.4 ([BH95])**.** *The transition matrix M of a Markov partition of a pseudo-Anosov homeomorphism f is an integral Perron-Frobenius matrix (which is a Perron-Frobenius matrix with integer entries). Moreover, $\mu(M(f)) = \lambda(f)$ the dilatation of f, and the height (respectively, width) of R_i is the ith entry of the corresponding Perron-Frobenius eigenvector of M (respectively, M^T).*

Associated to an integral Perron-Frobenius matrix M, we construct a directed graph Γ with k vertices $\{i\}_{i=1}^{k}$ such that the number of directed edges from i to j in Γ equals $m_{i,j}$. We observe that for any $r > 0$ the (i, j)th entry $m_{i,j}^{(r)}$ of M^r is the number of directed edge paths from i to j of length r in Γ.

The following property of an integral Perron-Frobenius matrix is used for proving lower bounds in Section 3.3.1.

Proposition 2.3.5. *Let M be a $k \times k$ integral Perron-Frobenius matrix. If there is a nonzero entry on the diagonal of M, then M^{2k} is a positive matrix and its Perron-Frobenius eigenvalue $\mu(M^{2k})$ is at least k.*

Proof. Since M is a Perron-Frobenius matrix, we know that the directed graph Γ described above is path-connected by directed paths. Suppose M has a nonzero entry at the (l,l)th entry, then we will see at least one corresponding loop edge at the vertex l. For any i and j in Γ, path-connectivity ensures us that there are directed edge paths of length $\leq k$ from i to l and from l to j. This tells us that there is a directed edge path P of length $\leq 2k$ from i to j passing through l. Since we can wrap around the loop edge adjacent to l to increase the length of P, there is always a directed edge path of length $2k$ from i to j. In other words, $m_{i,j}^{(2k)}$ is at least 1 for all i and j, so M^{2k} is a positive matrix.

Let v be a corresponding Perron-Frobenius eigenvector, so that we have $M^{2k}v = \mu(M^{2k})v$. This implies that if $v = [v_1 \cdots v_k]^T$, for all i,

$$\sum_{j=1}^{k} m_{i,j}^{(2k)} v_j = \mu(M^{2k}) v_i,$$

equivalently,

$$\mu(M^{2k}) = \sum_{j=1}^{k} m_{i,j}^{(2k)} \frac{v_j}{v_i}.$$

Choosing i such that $v_i \leq v_j$ for all j, we obtain

$$\mu(M^{2k}) \geq \sum_{j=1}^{k} m_{i,j}^{(2k)} \geq \sum_{j=1}^{k} 1 = k.$$

□

The following proposition will be used in proving the upper bound in Section 3.3.2.

Proposition 2.3.6. *Let Γ be the induced directed graph of an integral Perron-Frobenius matrix M with Perron-Frobenius eigenvalue $\mu(M) = \mu$. Let $P_\Gamma(i,d)$ be the total number of edge paths of length d emanating from vertex i in Γ. Then, for all i,*

$$\sqrt[d]{P_\Gamma(i,d)} \longrightarrow \mu(M) \text{ as } d \to \infty.$$

Proof. Let M be an integral $k \times k$ Perron-Frobenius matrix with Perron-Frobenius eigenvalue μ and Perron-Frobenius eigenvector v. As above

$$\sum_{j=1}^{k} m_{i,j}^{(d)} v_j = \mu(M^d) v_i = \mu^d v_i.$$

Let $v_{max} = \max_i\{v_i\}$ and $v_{min} = \min_i\{v_i\}$. According to the Perron-Frobenius theory, the irreducibility of M impies that $v_i > 0$ for all i. For all i we have

$$\frac{v_{min}\left(\sum_j m_{i,j}^{(d)}\right)}{\mu^d} \leq \frac{\sum_j m_{i,j}^{(d)} v_j}{\mu^d} \leq \frac{v_{max}\left(\sum_j m_{i,j}^{(d)}\right)}{\mu^d},$$

hence

$$\frac{v_i}{v_{max}} \leq \frac{\sum_j m_{i,j}^{(d)}}{\mu^d} \leq \frac{v_i}{v_{min}}.$$

We are done, since $\sum_j m_{i,j}^{(d)} = P_\Gamma(i,d)$ and for all i,

$$\sqrt[d]{\frac{v_i}{v_{max}}} \to 1 \text{ and } \sqrt[d]{\frac{v_i}{v_{min}}} \to 1, \text{ as } d \text{ tends to } \infty.$$

□

2.4 Train tracks

For a detailed discussion of train tracks, see [PH92]. We summarize the necessary definitions here.

A **train track** τ on the surface is an embedded 1-dimensional CW complex with some additional structure, part of which we now describe. The edges are called **branches** and the vertices are called **switches**. The branches are smoothly embedded on the interiors, and there is a common point of tangency to all branches meeting at a switch. The branches incident at a switch are divided into two sets, the "incoming" and "outgoing" branches.

A **train route** is a regular smooth path in τ. In particular, it traverses a switch only by passing from an incoming branch to an outgoing branch or vice versa. A train track σ is

carried by a train track τ, denoted by $\sigma \prec \tau$, if there is a map $f : S \to S$ homotopic to the identity on S such that every train route in σ is taken by f to a train route in τ. In particular, σ can be embedded in an ϵ neighborhood of τ.

An assignment of non-negative numbers, called **weights**, to the branches so that at every switch, the sum of the incoming weights equals the sum of the outgoing weights is called a **transverse measure** on the train track. A closed train route induces a counting measure on τ by counting the number of times the train route traverses the branches. A train track τ is called **recurrent** if there is a transverse measure which is positive on every branch of τ. A train track is called **large** if all the complementary regions are polygons or once-punctured polygons.

Definition 2.4.1. Given a pseudo-Anosov mapping class $f \in \text{Mod}(S)$, we say a train track τ is an (invariant) train track of f if τ is large and recurrent, and $f(\tau)$ is carried by τ.

Example 2.4.2. From the Example 2.1.6, τ is a train track of f as in Figure 2.9. We can

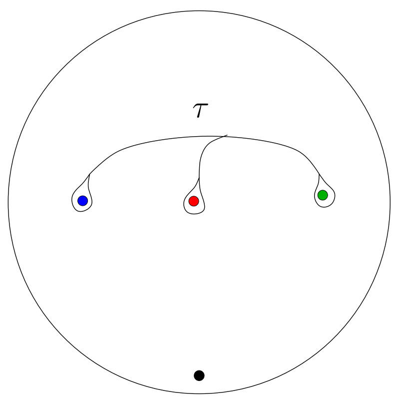

Figure 2.9: A train track τ of f.

see in Figure 2.10, $f(\tau)$ is carried by τ.

In [BH95], Bestvina and Handel construct a train track τ for f such that switches send to switches. We can define a transition matrix M associated to the branches b_i's of τ where the (i, j)-th entry $m_{i,j}$ of M is the number of times that $f(b_j)$ wraps over b_i. More precisely, pick a point $x_i \in b_i$, then

$$m_{i,j} = |f^{-1}(x_i) \cap b_j|.$$

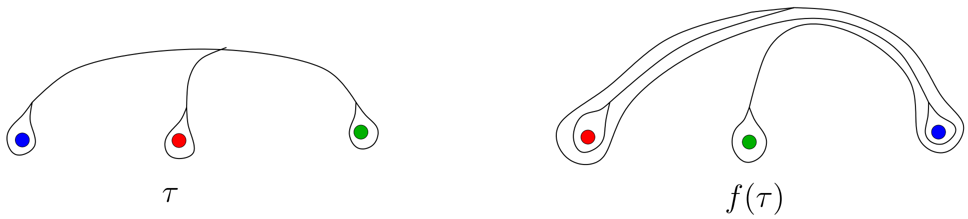

Figure 2.10: $f(\tau)$ is carried by τ.

M restricted to a certain set of "real" branches is an integral Perron-Frobenius matrix, denoted M_H. Moreover, M_H is the matrix of a Markov partition of f as described in Theorem 2.3.4. Hence, the maximal eigenvalue $\mu(M_H)$ is the dilatation of f.

2.5 Lefschetz numbers

We will review some definitions and properties of Lefschetz numbers. A more complete discussion can be found in [GP74] and [BT82].

Let X be a compact oriented smooth manifold, and $f : X \to X$ be a continuous map. Define

$$graph(f) = \{(x, f(x)) | x \in X\} \subset X \times X$$

and let Δ be the diagonal of $X \times X$. The algebraic intersection number $I(\Delta, graph(f))$ is an invariant of the homotopy class of f, called the **(global) Lefschetz number** of f and it is denoted $L(f)$. As in [BT82], this can be alternatively described by

$$L(f) = \sum_{i \geq 0} (-1)^i trace(f_*^{(i)}), \tag{2.5.1}$$

where $f_*^{(i)}$ is the matrix induced by f acting on $H_i(X) = H_i(X; \mathbb{R})$. The Euler characteristic is the self-intersection number of the diagonal Δ in $X \times X$,

$$\chi(X) = I(\Delta, \Delta) = L(id).$$

As seen in [GP74], if f is smooth and has isolated fixed points, we can compute the **local Lefschetz number** of f at a fixed point x in local coordinates as

$$L_x(f) = \deg\left(z \mapsto \frac{f(z)-z}{|f(z)-z|}\right),$$

where z ranges over the boundary of a small disc centered at x which contains no other fixed points. Moreover, we can compute the Lefschetz number by summing the local Lefschetz numbers of fixed points,

$$L(f) = \sum_{f(x)=x} L_x(f).$$

More generally, if $D \subset X$ is a disc containing no fixed points on the boundary, then

$$\sum_{f(x)=x,\ x\in D} L_x(f) = \deg\left(z \mapsto \frac{f(z)-z}{|f(z)-z|}\right),$$

where z ranges over the boundary of D.

The description of $L_x(f)$ is given for smooth f with isolated fixed points in [GP74]. However, it is equally valid for continuous f since such a map is approximated by smooth maps, as the following consequence of the Stone-Weirstrass Theorem explains.

Proposition 2.5.1. *A continuous map $f : X \to X$ of a compact smooth manifold X to itself can be approximated by a sequence of smooth maps $f_n : X \to X$ in the sup-metric. More precisely,*

$$\sup\{d_X(f(x), f_n(x)) | x \in X\} \to 0, \text{ as } n \to \infty.$$

Proof. First, we embed X in $\mathbb{R}^k$ and use the distance function from $\mathbb{R}^k$. Let V be a tubular neighborhood of X, which is diffeomorphic to $X \times (-1, 1)$, where X is identified with $X \times \{0\}$.

The function $f : X \to X$ can be viewed as a map $f : X \to \mathbb{R}^k$, which we can write as $f = (f^1, \cdots, f^k)$, where $f^j : X \to \mathbb{R}$. By the Stone-Weirstrass Theorem, there are sequences of smooth functions $\{F^j_n : X \to \mathbb{R}\}_{n=1}^{\infty}$ which converge to f^j in the sup-metric $\forall j$, as n tends to infinity. Putting these together we get a sequence of smooth maps

$$\left\{F_n = (F_n^1, \cdots, F_n^k) : X \to \mathbb{R}^k\right\}_{n=1}^{\infty}.$$

For n large enough, $F_n(X)$ lies in V, and now let $p : V = X \times (-1,1) \to X$ be the projection onto the first factor. Set $f_n = p \circ F_n$, so f_n is smooth, since it is the composition of smooth maps p and F_n. Moreover, f_n converges to f as n tends to infinity. □

Given a homeomorphism $f : X \to X$ of a compact manifold X to itself with isolated fixed points x_i, by Proposition 2.5.1, for all $\epsilon > 0$ there exist n_ϵ, such that if $f_n(x) = x$, then x is inside of the ϵ-neighborhoods of x_i's, for all $n \geq n_\epsilon$. We may also assume that f_n is homotopic to f for all $n \geq n_\epsilon$. Let D_i be a disc centered at x_i inside ϵ-neighborhood of x_i. Choose $n \geq n_\epsilon$ large enough, such that

$$\deg\left(z \mapsto \frac{f(z)-z}{|f(z)-z|}\right) = \deg\left(z \mapsto \frac{f_n(z)-z}{|f_n(z)-z|}\right),$$

where $z \in \partial D_i$, for all i. Fix such n, we apply the Transversality Theorem in [GP74] to the smooth map f_n to produce a sequence of smooth maps $g_k^{(n)}$ with isolated fixed points converging to f_n in the sup metric. Choose k large enough, such that the fixed points of $g_k^{(n)}$ lie only inside D_i, and

$$\deg\left(z \mapsto \frac{f_n(z)-z}{|f_n(z)-z|}\right) = \deg\left(z \mapsto \frac{g_k^{(n)}(z)-z}{|g_k^{(n)}(z)-z|}\right),$$

on ∂D_i, for all i. Again, by taking k large enough, $g_k^{(n)} \simeq f_n \simeq f$.

Hence, fixing such n and k, we can compute the Lefschetz numbers

$$L(f) = L\left(g_k^{(n)}\right) = \sum_{g_k^{(n)}(x)=x} L_x\left(g_k^{(n)}\right) = \sum_{f(x)=x} L_x(f).$$

In what follows, we will ignore marked points when computing Lefschetz numbers of homeomorphisms $f : S_{g,n} \to S_{g,n}$. A straightforward computation will give us the following proposition which is used in Section 3.3.1.

Proposition 2.5.2. *If a homeomorphism* $f : S_{g,n} \to S_{g,n}$ *is homotopic (not necessarily fixing the marked points) to the identity or a multitwist, then*

$$L(f) = \chi(S_{g,0}) = 2 - 2g.$$

Proof. If f is homotopic to the identity, the homotopy invariance of the Lefschetz number tells us $L(f) = L(id) = I(\Delta, \Delta)$ which is $\chi(S_{g,0})$.

Suppose f is homotopic to a multitwist. We will use the formula (2.5.1) to compute $L(f)$. Note that $H_i(S_{g,0})$ is 0 for $i \geq 3$, $H_0(S_{g,0}) \cong H_2(S_{g,0}) \cong \mathbb{R}$ and $f_*^{(i)}$ is the identity when $i = 0$ or 2, so this implies $L(f) = 2 - trace(f_*^{(1)})$.

There exists a set $\{\gamma_i\}_{i=1}^k$ of disjoint simple essential closed curves with some integers $n_i \neq 0$ such that

$$f \simeq T_{\gamma_1}^{n_1} \circ \cdots \circ T_{\gamma_k}^{n_k},$$

where $T_{\gamma_i}^{n_i}$ is the n_i-th power of a Dehn twist along γ_i.

For any curve γ,

$$T_{\gamma_i *}^{n_i}([\gamma]) = [\gamma] + n_i \langle \gamma, \gamma_i \rangle [\gamma_i],$$

where $[\gamma]$ is the homology class of γ and $\langle \gamma, \gamma_i \rangle$ is the algebraic intersection number of $[\gamma]$ and $[\gamma_i]$. If any γ_i is a separating curve, then $[\gamma_i]$ is the trivial homology class and $T_{\gamma_i *}^{n_i}$ acts trivially on $H_1(S_{g,0})$. We may therefore assume that each γ_i is nonseparating. After renaming the curves, we can assume that there is a subset $\{\gamma_1, \gamma_2, \cdots, \gamma_s\}$ such that $\widehat{\gamma} = \bigcup_{i=1}^{s} \gamma_i$ is nonseparating and $\widehat{\gamma} \cup \gamma_j$ is separating for all $j > s$. Thus, for all $k \geq j > s$,

$$[\gamma_j] = \sum_{i=1}^{s} c_{ji} [\gamma_i],$$

for some constants $c_{ji} \in \mathbb{R}$. We can extend $\{[\gamma_i]\}_{i=1}^{s}$ to a basis of $H_1(S_{g,0})$,

$$\{\alpha_1, \alpha_2, \cdots, \alpha_g, \beta_1, \beta_2, \cdots, \beta_g\},$$

where $[\gamma_i] = \alpha_i$ for $i \leq s \leq g$ and $\langle \alpha_i, \beta_j \rangle = \delta_{ij}$, $\langle \alpha_i, \alpha_j \rangle = \langle \beta_i, \beta_j \rangle = 0$.

First suppose $s = k$, then $\langle \alpha_j, \gamma_i \rangle = \langle \alpha_j, \alpha_i \rangle = 0$ for all i and j. Therefore, for all j,

$$f_*^{(1)}(\alpha_j) = \alpha_j; \text{ and}$$

$$f_*^{(1)}(\beta_j) = \beta_j + \sum_{i=1}^{k} n_i \langle \beta_j, \gamma_i \rangle [\gamma_i] = \beta_j + \sum_{i=1}^{k} n_i \langle \beta_j, \alpha_i \rangle \alpha_i = \beta_j - n_j \alpha_j.$$

So we have

$$f_*^{(1)} = \left(\begin{array}{c|c} I_{g\times g} & * \\ \hline 0 & I_{g\times g} \end{array} \right)$$

and $L(f) = 2 - trace(f_*^{(1)}) = 2 - 2g$.

For $s < k$, we will have

$$\begin{aligned} f_*^{(1)}(\alpha_j) &= \alpha_j + \sum_{i=1}^{k} n_i \langle \alpha_j, \gamma_i \rangle [\gamma_i] \\ &= \alpha_j + \sum_{i=1}^{s} n_i \langle \alpha_j, \alpha_i \rangle \alpha_i + \sum_{i=s+1}^{k} n_i \langle \alpha_j, \gamma_i \rangle [\gamma_i] \\ &= \alpha_j + \sum_{i=s+1}^{k} n_i \sum_{t=1}^{s} c_{it} \langle \alpha_j, \gamma_t \rangle [\gamma_t] \\ &= \alpha_j + \sum_{i=s+1}^{k} n_i \sum_{t=1}^{s} c_{it} \langle \alpha_j, \alpha_t \rangle \alpha_t \\ &= \alpha_j \end{aligned}$$

and

$$
\begin{aligned}
f_*^{(1)}(\beta_j) &= \beta_j + \sum_{i=1}^{k} n_i \langle \beta_j, \gamma_i \rangle [\gamma_i] \\
&= \beta_j + \sum_{i=1}^{s} n_i \langle \beta_j, \gamma_i \rangle [\gamma_i] + \sum_{i=s+1}^{k} n_i \sum_{t=1}^{s} c_{it} \langle \beta_j, \gamma_t \rangle [\gamma_t] \\
&= \beta_j + \sum_{i=1}^{s} n_i \langle \beta_j, \alpha_i \rangle \alpha_i + \sum_{i=s+1}^{k} n_i \sum_{t=1}^{s} c_{it} \langle \beta_j, \alpha_t \rangle \alpha_t \\
&= \begin{cases} \beta_j, \ \text{if } j > s, \\ \beta_j - n_j \alpha_j - \sum_{i=s+1}^{k} n_i c_{ij} \alpha_j, \ \text{if } j \leq s. \end{cases}
\end{aligned}
$$

Therefore, the diagonal of the matrix $f_*^{(1)}$ is still all 1's and

$$
L(f) = 2 - trace(f_*^{(1)}) = 2 - 2g.
$$

□

CHAPTER 3

MINIMAL PSEUDO-ANOSOV TRANSLATION LENGTHS ON THE TEICHMÜLLER SPACE

3.1 Known Results

From the discussion in Section 2.2, it follows that the set

$$\mathcal{L}(S_{g,n}) := \{\log \lambda(f) | f \in \mathrm{Mod}(S_{g,n}) \text{ pseudo-Anosov}\}.$$

is precisely the set of lengths of closed geodesics in the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n marked points with respect to the Teichmuller metric. Penner defines $\mathcal{L}(S_{g,n})$ to be the **length spectrum** of $S_{g,n}$. It is known to be a discrete set; see [AY81], [Iva88]. There is a minimal element

$$l_{g,n} = \min\{\log \lambda(f) | f \in \mathrm{Mod}(S_{g,n}) \text{ pseudo-Anosov}\}.$$

We call $l_{g,n}$ **the minimal (pseudo-Anosov) translation length** on $\mathcal{T}(S_{g,n})$. In [Pen91], Penner proves a general lower bound for $l_{g,n}$.

Theorem 3.1.1 (Penner)**.** *For* $2g - 2 + n > 0$*,*

$$l_{g,n} \geq \frac{\log 2}{12g - 12 + 4n}.$$

Furthermore, for closed surfaces he compliments it with an upper bound.

Theorem 3.1.2 (Penner)**.** *For closed surfaces with genus* $g \geq 2$*,*

$$\frac{\log 2}{12g - 12} \leq l_{g,0} \leq \frac{\log 11}{g}.$$

The bounds on $l_{g,0}$ have been improved by a number of authors ([Bau92], [McM00], [Min06], [Hir09], [HK06]). McMullen [McM00] points out that the lower bound can be sharped to

$$l_{g,0} \geq \frac{\log 2}{6g-6},$$

and the best known asymptotic upper bound is

$$\limsup_{g\to\infty} g l_{g,0} \leq \log\left(\frac{3+\sqrt{5}}{2}\right)$$

shown by Hironaka [Hir09].

In [Pen91], Penner suggests that there may be an "analogous upper bound for $n \neq 0$". In [HK06], Hironaka and Kin use a concrete construction to prove that for genus $g = 0$,

$$l_{0,n} < \frac{\log(2+\sqrt{3})}{\lfloor \frac{n-2}{2} \rfloor} \leq \frac{2\log(2+\sqrt{3})}{n-3},$$

for all $n \geq 4$. The inequality is proven for even n in [HK06], but it follows for odd n by letting the fixed point of their example be a marked point. Combining this with Penner's lower bound in Theorem 3.1.1, one sees

Theorem 3.1.3. *For $n \geq 4$,*

$$\frac{\log 2}{4n-12} \leq l_{0,n} < \frac{2\log(2+\sqrt{3})}{n-3},$$

It shows that the upper bound is on the same order as Penner's lower bound for $g = 0$. In the next section, we see a similar situation holds for $g = 1$ and for surfaces with $g \geq 2$ and $n = cg + d$ for certain small $c, d \geq 0$.

3.2 Asymptotic behavior I: $\frac{1}{|\chi(S)|}$

Here we will describe examples with $l_{g,n} \leq O(\frac{1}{|\chi(S)|})$. Combining this with Penner's lower bound in Theorem 3.1.1, $l_{g,n}$ has the asymptotic behavior $\frac{1}{|\chi(S)|}$ as $|\chi(S)| \to \infty$ in these cases.

3.2.1 Tori with marked points

The following example provides $l_{1,2n}$ an upper bound of the same order as Penner's lower bound, which means that $l_{1,2n}$ has the asymptotic behavior $\frac{1}{n}$ as $n \to \infty$. The construction is inspired by Penner's construction [Pen91].

Example 3.2.1. Let $S_{1,2n}$ be a torus with $2n$ marked points. Let a and b be essential simple closed curves as in Figure 3.1. Let T_a^{-1} be the negative Dehn twist along a and T_b

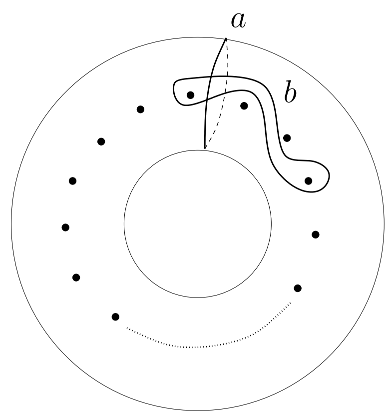

Figure 3.1: Essential simple closed curves a and b on a marked torus.

be the Dehn twist along b, then we define

$$f := \rho \circ T_b \circ T_a^{-1} \in \mathrm{Mod}(S_{1,2n})$$

where ρ rotates the torus clockwise by an angle of $2\pi/n$, so it sends each marked point to the one which is two to the right. By Theorem 2.1.5, f^n is shown to be pseudo-Anosov, and thus so is f. Figure 3.2 shows a train track for f. Let $\mathcal{T} := \mathcal{A} \cup \mathcal{B}$,

$$\mathcal{A} = \bigsqcup_{i=1}^{n} a_i \text{ and } \mathcal{B} = \bigsqcup_{i=1}^{n} b_i,$$

where $a_i = \rho^{n-i+1}(a)$ and $b_i = \rho^{n-i+1}(b)$. We obtain the $2n \times 2n$ transition matrix M associated to the train track map of f with respect to the spanning vectors associated with

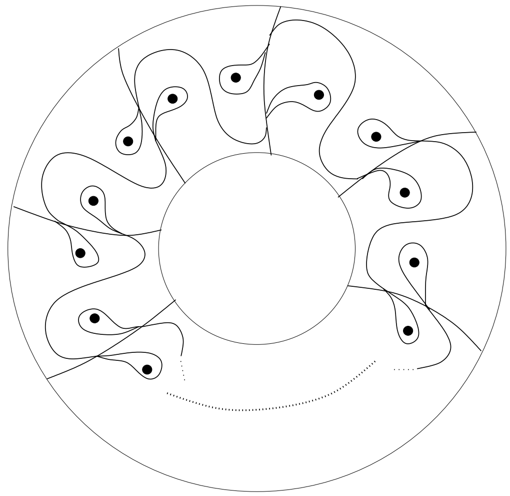

Figure 3.2: A train track for f.

geodesics in $\mathcal{T}$. Note that $M^n = M(f^n)$ and M^n is an integral Perron-Frobenius matrix where its Perron-Frobenius eigenvalues $\mu(M^n) = \lambda(f^n)$. For $n \geq 5$, we have $M^n = N$, where

$$N := \begin{pmatrix} A_1 & B_1 & 0 & 0 & \cdots & 0 & 0 & D_1 \\ A_2 & B_2 & B_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & B_3 & B_2 & B_1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & B_3 & B_2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & B_3 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & B_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & B_2 & B_1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & B_3 & B_2 & D_2 \\ A_3 & C & 0 & 0 & \cdots & 0 & B_3 & D_3 \end{pmatrix}, \tag{3.2.1}$$

and

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},$$
$$B_1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, B_2 = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}, B_3 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \tag{3.2.2}$$
$$D_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, D_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, D_3 = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}.$$

For $n \geq 5$, the greatest column sum of M^n is 9 and the greatest row sum of M^n is 11. Hence, for $n \geq 5$

$$(\lambda(f))^n = \lambda(f^n) = \mu(M^n) \leq 9$$
$$\Rightarrow l_{1,2n} \leq \log \lambda(f) \leq \frac{\log 9}{n}.$$

For $1 \leq n \leq 4$, one can verify that both the greatest column sum and the greatest row sum are ≤ 11, so we have $l_{1,2n} \leq \dfrac{\log 11}{n}$ where $1 \leq n \leq 4$.

Now, combining this upper bound with Penner's lower bound in Theorem 3.1.1, we obtain

Theorem 3.2.2. *For $n \geq 5$,*

$$\frac{\log 2}{8n} \leq l_{1,2n} \leq \frac{\log 9}{n}.$$

Remark. In fact, there is an upper bound of the same order when the number of marked points is odd. Consider $W = S^3 - N(\lambda_1 \cup \lambda_2)$ the complement of the Whitehead link shown in Figure 3.3; see also [Thu86]. W is hyperbolic and $H^1(W; \mathbb{R}) = \langle \lambda_1, \lambda_2 \rangle$. Given $p, q \geq 0$, the Thurston norm was computed in [Thu86], and is given by

$$x_T(p\lambda_1 + q\lambda_2) = p + q.$$

Moreover, for every $p, q > 0$, $p\lambda_1 + q\lambda_2$ is represented by a fiber which we denote $S_{(p,q)}$ with

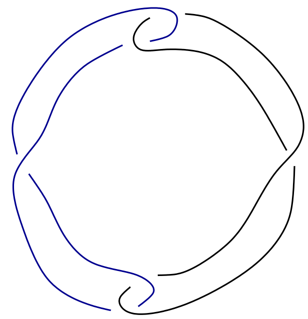

Figure 3.3: The Whitehead link.

monodromy $\phi_{(p,q)} : S_{(p,q)} \to S_{(p,q)}$. Since W is hyperbolic, $\phi_{(p,q)}$ is pseudo-Anosov.

One can check that $S_{(p,q)}$ has $p+q$ boundary components, so since $\chi(S_{(p,q)}) = -p-q$, it follows that if p and q are relatively prime, then $S_{(p,q)}$ is a torus with $p+q$ marked points. The sequence $\phi_{(n,n+1)} : S_{(n,n+1)} \to S_{(n,n+1)}$ consists of pseudo-Anosov homeomorphisms of tori with $2n+1$ marked points, and as in McMullen's paper [McM00],

$$\chi_T\left(n\lambda_1 + (n+1)\lambda_2\right) \cdot \log\left(\lambda(\phi_{n,n+1})\right) \to \chi_T\left(\lambda_1 + \lambda_2\right) \cdot \log\left(\lambda(\phi_{1,1})\right) = 2\log(2+\sqrt{3}).$$

The last equation is an explicit computation. Hence

$$\limsup_{n\to\infty}(2n+1)l_{1,2n+1} \leq 2\log(2+\sqrt{3}).$$

3.2.2 Surface with $g \geq 2$ and $n = cg + d$ for certain small $c, d \geq 0$

In all of the following examples we obtain a mapping class $\widetilde{f} \in \mathrm{Mod}(S_{g,n})$ from $f \in \mathrm{Mod}(S_{g,0})$ by adding an f-invariant set of marked points on the closed surface $S_{g,0}$, where a power of f is a composition of Dehn twists along some set $\mathcal{T}$ of closed geodesics. We can add one marked point in each of the complementary disks of the curves in $\mathcal{T}$ without creating essential reducing curves. By Theorem 2.1.5, the induced mapping class $\widetilde{f} \in \mathrm{Mod}(S_{g,n})$ is pseudo-Anosov with dilatation $\lambda(\widetilde{f}) = \lambda(f)$.

Example 3.2.3. Penner [Pen91] constructed a pseudo-Anosov mapping class

$f \in \mathrm{Mod}(S_{g,0})$ with dilatation $\lambda(f) \leq \frac{\log 11}{g}$ for $g \geq 2$, where

$$f := \rho \circ T_c \circ T_a^{-1} \circ T_b,$$

and ρ, a, b, c are as shown in Figure 3.4. Note that f^g is a composition of Dehn twists. Let

$$\mathcal{A} = \bigsqcup_{i=1}^{g} a_i, \ \mathcal{B} = \bigsqcup_{i=1}^{g} b_i, \text{ and } \mathcal{C} = \bigsqcup_{i=1}^{g} c_i$$

with the notational convention as in Example 3.2.1. Since $a_i \in \mathcal{A}$ and $c_j \in \mathcal{C}$ are disjoint for all i, j and $(\mathcal{A} \cup \mathcal{C}) \cup \mathcal{B}$ fills S, f^g is pseudo-Anosov by Theorem 2.1.5. Hence, f is pseudo-Anosov and using Perron-Frobenius theory Penner shows $\lambda(f) \leq \frac{\log 11}{g}$ for $g \geq 2$.

Next, we add g marked points as in Figure 3.4, so that $\widetilde{f} \in \mathrm{Mod}(S_{g,g})$ is still pseudo-Anosov by Theorem 2.1.5. Therefore,

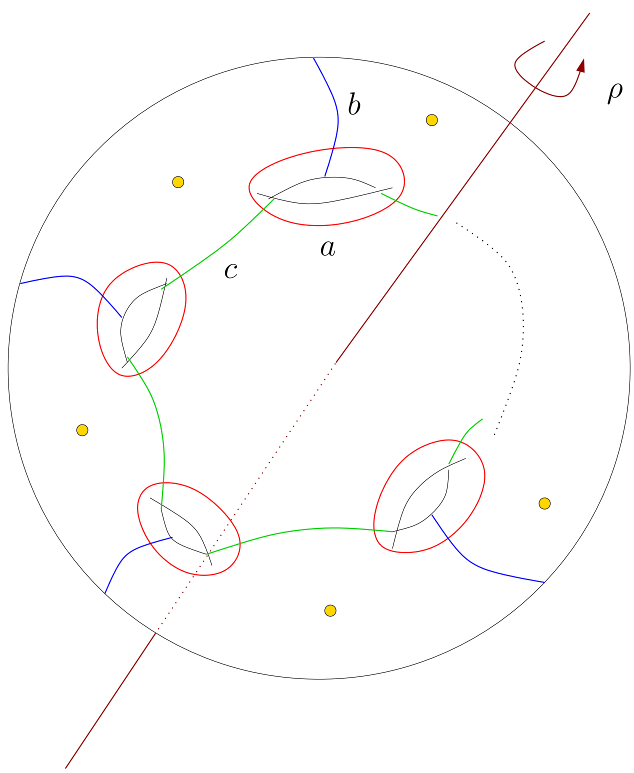

Figure 3.4: A pseudo-Anosov $\widetilde{f} \in \mathrm{Mod}(S_{g,g})$.

$$l_{g,g} \leq \log \lambda(\widetilde{f}) \leq \frac{\log 11}{g}.$$

We can also add extra marked points at the fixed points which are the intersection of S and the axis of the rotation so that the induced mapping class is still pseudo-Anosov. For $g \geq 2$, we will have for $c = 0, 1, 2$,

$$l_{g,g+c} \leq \log \lambda(\widetilde{f}) \leq \frac{\log 11}{g},$$

where $\widetilde{f} \in \mathrm{Mod}(S_{g,g+c})$.

Thus, together with Penner's lower bound in Theorem 3.1.1, we have verified

Theorem 3.2.4. *For $g \geq 2$, and $c = 0, 1, 2$,*

$$\frac{\log 2}{16g - 12 + c} \leq l_{g,g+c} \leq \frac{\log 11}{g}.$$

Example 3.2.5. For all $g \geq 3$, let $\mathcal{T} := \mathcal{A} \cup \mathcal{B}$ where

$$\mathcal{A} = \bigsqcup_{i=1}^{g+1} a_i \text{ and } \mathcal{B} = \bigsqcup_{i=1}^{g+1} b_i$$

are as in Figure 3.5. Define $f : S_{g,0} \to S_{g,0}$ to be

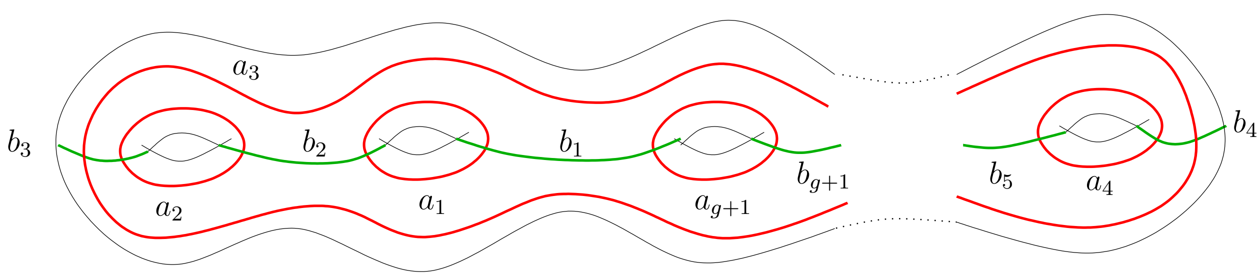

Figure 3.5: A pseudo-Anosov $f \in \mathrm{Mod}(S_{g,0})$.

$$f := \rho \circ T_{b_1} \circ T_{a_1}^{-1},$$

where ρ is a finite order element satisfying

$$\rho(a_1) = a_{g+1}, \rho(b_1) = b_{g+1}, \text{ and}$$
$$\rho(a_i) = a_{i-1}, \rho(b_i) = b_{i-1}, \text{ for } i = 2, \cdots, g+1.$$

By Theorem 2.1.5, f is pseudo-Anosov.

We construct the $(2g+2) \times (2g+2)$ transition matrix M with respect to the spanning vectors associated with geodesics in $\mathcal{T}$. We will get $M^{(g+1)} = N$ for $g \geq 3$, where N is shown in (3.2.1) whose matrix blocks are the same as (3.2.2). As in Example 3.2.1, for $g \geq 3$ we have

$$\log \lambda(f) \leq \frac{\log 9}{g+1}.$$

Similar to Example 3.2.3, we can add up to 4 marked points which are the fixed points of f and obtain a pseudo-Anosov homeomorphism $\widetilde{f}$ in resulting surfaces. Hence, for $g \geq 3$ and $c = 0, 1, 2, 3, 4$, we have

$$l_{g,c} \leq \log \lambda(\widetilde{f}) \leq \frac{\log 9}{g+1},$$

where $\widetilde{f} \in \mathrm{Mod}(S_{g,c})$.

With Theorem 3.1.1, we have therefore verified

Theorem 3.2.6. *For $g \geq 3$ and $c = 0, 1, 2, 3, 4$,*

$$\frac{\log 2}{12g - 12 + 4c} \leq l_{g,c} \leq \frac{\log 9}{g+1}.$$

Example 3.2.7. For $g \geq 5$, let $\mathcal{T} := \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$ with

$$\mathcal{A} = \bigsqcup_{i=1}^{g-1} a_i, \mathcal{B} = \bigsqcup_{i=1}^{g-1} b_i, \mathcal{C} = \bigsqcup_{i=1}^{g-1} c_i, \text{ and } \mathcal{D} = \bigsqcup_{i=1}^{g-1} d_i$$

as shown in Figure 3.6. Define $f : S_{g,0} \to S_{g,0}$ by

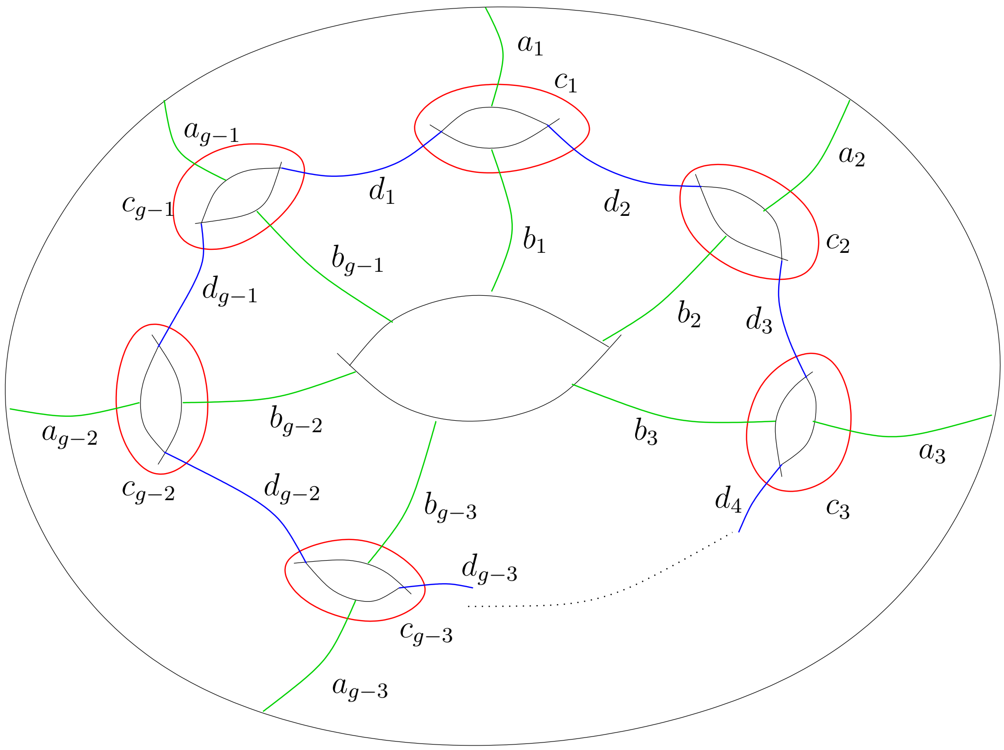

Figure 3.6: A pseudo-Anosov $f \in \text{Mod}(S_{g,0})$.

$$f := \rho \circ T_{d_1} \circ T_{c_1}^{-1} \circ T_{b_1} \circ T_{a_1},$$

where ρ is a finite order element satisfying

$$\rho(a_1) = a_{g-1}, \rho(b_1) = b_{g-1}, \rho(c_1) = c_{g-1}, \rho(d_1) = d_{g-1}, \text{ and}$$
$$\rho(a_i) = a_{i-1}, \rho(b_i) = b_{i-1}, \rho(c_i) = c_{i-1}, \rho(d_i) = d_{i-1}, \text{ for } i = 2, \cdots, g-1.$$

Similarly, Theorem 2.1.5 implies f is pseudo-Anosov, and we construct a $(4g-4) \times (4g-4)$ transition matrix M with respect to the spanning vectors associated with the geodesics in $\mathcal{T}$.

For $g \geq 5$ we have $M^{(g-1)} = N$ as defined in (3.2.1) in Example 3.2.1, where

$$A_1 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 3 & 2 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 3 & 1 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 3 & 2 \\ 1 & 1 & 3 & 0 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, B_2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 3 & 3 \end{pmatrix}, B_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 3 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, D_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, D_3 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 4 & 3 \end{pmatrix}.$$

For $g \geq 5$, the greatest column sum of $M^{(g-1)}$ is 17 and the greatest row sum of $M^{(g-1)}$ is 21, hence

$$\log \lambda(f) \leq \frac{\log 17}{g-1}.$$

For $c = 1$ and 2, we can induce $\widetilde{f} \in \mathrm{Mod}(S_{g,c(g-1)})$ with

$$l_{g,c(g-1)} \leq \log \lambda(\widetilde{f}) \leq \frac{\log 17}{g-1},$$

when $g \geq 5$.

This upper bound and Theorem 3.1.1 imply

Theorem 3.2.8. *For $g \geq 5$ and $c = 1, 2$,*

$$\frac{\log 2}{(12+4c)g - 12 - 4c} \leq l_{g,c(g-1)} \leq \frac{\log 17}{g-1}.$$

3.3 Asymptotic behavior II: $\frac{\log |\chi(S)|}{|\chi(S)|}$

It is rather surprising that $l_{g,n}$ has a different asymptotic behavior if we fix the genus g and let n vary. We restate the Main Theorem more precisely as the following:

Theorem 3.3.1. *For any fixed $g \geq 2$, there is a constant $c_g \geq 1$ depending on g such that*

$$\frac{\log n}{c_g n} < l_{g,n} < \frac{c_g \log n}{n},$$

for all $n \geq 3$.

We will prove the above theorem in the following sections.

3.3.1 Bounding the dilatation from below

Lemma 3.3.2. *For any pseudo-Anosov element $f \in \mathrm{Mod}(S_{g,n})$ equipped with a Markov partition, if $L(f) < 0$, then there is a rectangle R of the Markov partition, such that the interiors of $f(R)$ and R intersect.*

Proof. Since f is a pseudo-Anosov homeomorphism, it has isolated fixed points. Suppose x is an isolated fixed point of f such that one of the following happens:

1. x is a nonsingular fixed point and the local transverse orientation of $\mathcal{F}^u$ is reversed.

2. x is a singular fixed point and no separatrix of $\mathcal{F}^u$ emanating from x is fixed.

A **separatrix** of $\mathcal{F}$ is a maximal arc starting at a singularity and contained in a leaf of $\mathcal{F}$.

Claim. $L_x(f) = +1$.

Let B be a small disk centered at x containing no other fixed point of f. First we show that (in local coordinates) for every $z \in \partial B$, $f(z) - z \neq \alpha z$ for all $\alpha > 0$.

It is easy to verify this in case 1 by choosing local coordinates (ξ_1, ξ_2) around x so that f is given by

$$f(\xi_1, \xi_2) = (-\lambda \xi_1, \frac{-1}{\lambda} \xi_2).$$

In case 2, we choose local coordinates around x such that the separatrices of $\mathcal{F}^s$ emanating from x are sent to rays from 0 through the kth roots of unity in $\mathbb{R}^2$. This means f rotates each of the sectors bounded by these rays through an angle $\frac{2\pi j}{k}$ for some $j = 1, \cdots, k-1$, and so for all $z \in \partial B$ $f(z) - z \neq \alpha z$ for all $\alpha > 0$.

Define a smooth map $h_0 : \partial B \to S^1$ by $h_0(z) = \frac{f(z)-z}{|f(z)-z|}$, so $L_x(f) = \deg(h_0)$ by definition. Let $g : \partial B \to S^1$ be defined by $g(z) = \frac{z}{|z|}$ and $h_1 : S^1 \to S^1$ be defined by $h_1(\frac{z}{|z|}) = \frac{f(z)-z}{|f(z)-z|}$ for $z \in \partial B$, so that $h_0 = h_1 g$. Then

$$L_x(f) = \deg(h_0) = \deg(h_1 g) = \deg(h_1)\deg(g) = \deg(h_1)$$

since $\deg(g) = 1$. Note that h_1 has no fixed point since for all $z \in \partial B$,

$$f(z) - z \neq \alpha z,$$

for all $\alpha > 0$. Therefore $\deg(h_1)$ is the same as the degree of the antipodal map $S^1 \to S^1$ and so $L_x(f) = \deg(h_1) = (-1)^{(1+1)} = +1$.

The assumption of $L(f) < 0$ implies that there exists a fixed point x of f which is in neither of the cases above. In other words, it falls into one of the cases in Figure 3.7. As seen in Figure 3.7, there is a rectangle R of the Markov partition such that the interiors of $f(R)$ and R intersect. $\square$

Recall $\Gamma_S(3)$ is the kernel of homomorphism $\text{Mod}(S) \to H_1(S; \mathbb{Z}/3\mathbb{Z})$. For $S = S_{g,0}$, set

$$\Theta(g) = [\text{Mod}(S) : \Gamma_S(3)].$$

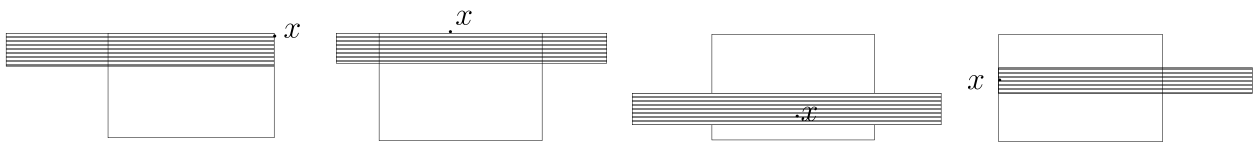

Figure 3.7: The intersection of $f(R)$ and R. R is the underlying rectangle and $f(R)$ is the shaded rectangle.

We need the following consequence of Theorem 2.1.11.

Corollary 3.3.3. *Let $f \in \mathrm{Mod}(S_{g,n})$ be a pseudo-Anosov element and $\widehat{f} \in \mathrm{Mod}(S_{g,o})$ be the induced mapping class obtained by forgetting marked points. There exists a constant $1 \leq \alpha \leq \Theta(g)$ such that $\widehat{f}^{\alpha}$ satisfies exactly one of the following:*

1. *$\widehat{f}^{\alpha}$ restricts to a pseudo-Anosov map on a connected subsurface.*
2. *$\widehat{f}^{\alpha}$ =Id.*
3. *$\widehat{f}^{\alpha}$ is a multitwist.*

Remark. For the first two cases of Lemma 3.3.3, one can find α bounded by a linear function of g, but in case 3, α may be exponential in g.

Theorem 3.3.4. *For $g \geq 2$, given any pseudo-Anosov $f \in \mathrm{Mod}(S_{g,n})$, let α be as in Corollary 3.3.3. Then*

$$\log \lambda(f) \geq \min \left\{ \frac{\log 2}{\alpha(12g-12)} \ , \ \frac{\log(6g+3n-6)}{2\alpha(6g+3n-6)} \right\}.$$

Proof. We will deal with case 1 of Corollary 3.3.3 first.

If $\widehat{f}^{\alpha}$ restricts to a pseudo-Anosov homeomorphism on a connected subsurface $\sum_{g_0,n_0}$ of $S_{g,0}$ of genus g_0 with n_0 boundary components (we have $2g_0 + n_0 \leq 2g$), then Penner's lower bound tells us

$$\log \lambda(\widehat{f}^{\alpha}) \geq \frac{\log 2}{12g_0 - 12 + 4n_0} \geq \frac{\log 2}{12g-12}.$$

Hence $\log \lambda(f) \geq \log \lambda(\widehat{f}) > \dfrac{\log 2}{\alpha(12g-12)}$.

Now suppose we are in either case 2 or 3 of Corollary 3.3.3. If $\widehat{f}^{\alpha}$ is homotopic to the identity or a multitwist, from Proposition 2.5.2, we have
$L(f^{\alpha}) = L(\widehat{f}^{\alpha}) = \chi(S_{g,0}) = 2-2g < 0$. Theorem 2.3.2 tells us that for any pseudo-Anosov f there is a Markov partition with k rectangles, where $k \leq -3\chi(S)$. Recall that the transition matrix M obtained from the rectangles is a $k \times k$ Perron-Frobenius matrix and the Perron-Frobenius eigenvalue $\mu(M)$ equals $\lambda(f)$.

By Lemma 3.3.2, there is a rectangle R such that the interiors of $f^{\alpha}(R)$ and R intersect. This implies that there is a nonzero entry on the diagonal of M^{α}. Applying Proposition 2.3.5, we obtain that $\mu((M^{\alpha})^{2k}) = \mu(M^{2k\alpha})$ is at least k, so we have

$$(\lambda(f))^{2k\alpha} = \lambda(f^{2k\alpha}) = \mu(M^{2k\alpha}) \geq k.$$

One can easily check $\dfrac{\log x}{x}$ is monotone decreasing for $x \geq 3$. Since

$$3 \leq k \leq -3\chi(S) = 6g + 3n - 6,$$

it follows that

$$\log \lambda(f) \geq \frac{\log k}{2\alpha k} \geq \frac{\log(6g+3n-6)}{2\alpha(6g+3n-6)}.$$

$\square$

Remark. Penner's proof in [Pen91] does not use Lefschetz numbers which we used to conclude that $\mu(M^{2k\alpha})$ is at least k, so we obtain a sharper lower bound for $n \gg g$.

3.3.2 An example which provides an upper bound

First, we will construct a pseudo-Anosov $f \in \text{Mod}(S_{2,n})$ for all $n \geq 31$ then we compute its dilatation which gives us an upper bound for $l_{2,n}$. Next, we will generalize the construction to higher genus surfaces.

Let $S_{0,m+2}$ be a genus 0 surface with $m+2$ marked points (i.e. a marked sphere), and recall an example of pseudo-Anosov $\phi \in \text{Mod}(S_{0,m+2})$ defined in [HK06]. We view $S_{0,m+2}$ as a sphere with $s+1$ marked points X circling an unmarked point x and $t+1$ marked points Y circling an unmarked point y, and a single extra marked point z. We can also draw this as a "turnover", as in Figure 3.8. Note that $|X \cap Y| = 1$, $|X| = s+1$, $|Y| = t+1$ and $m = s+t$.

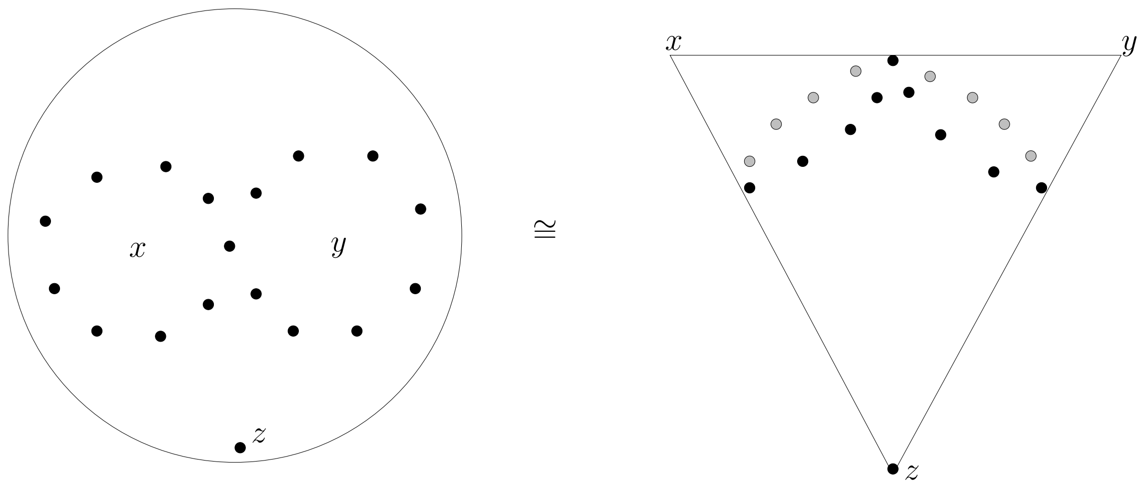

Figure 3.8: Two ways of viewing a marked sphere. Black dots are marked points and the shaded dots on the right are marked points at the back.

We define homeomorphisms α_s, $\beta_t : S_{0,m+2} \longrightarrow S_{0,m+2}$ such that α_s rotates the marked points of X counterclockwise around x and β_t rotates the marked points of Y clockwise around y; see Figure 3.9. Define $\phi_{s,t} := \beta_t \alpha_s$. In [HK06], it is shown that $\phi_{s,t}$ is

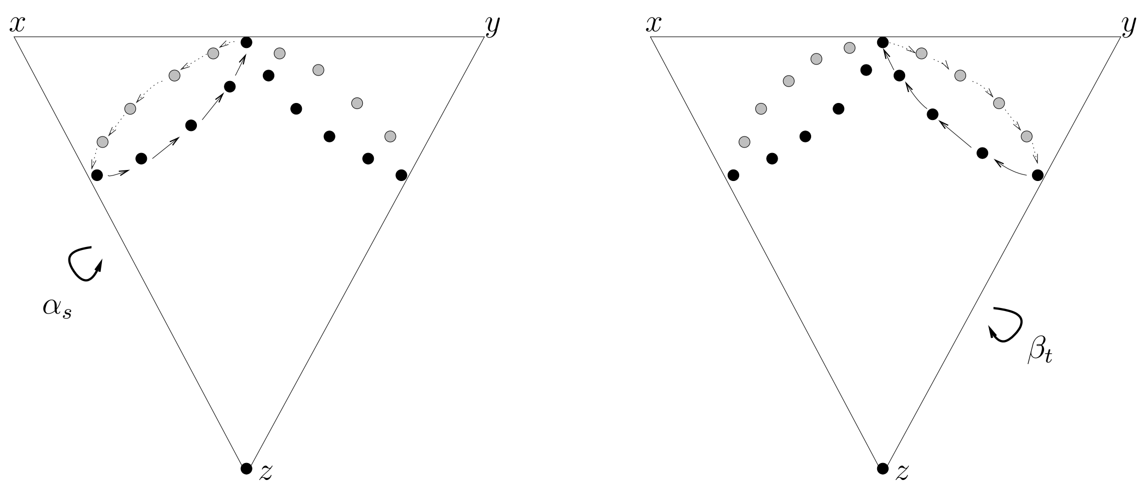

Figure 3.9: Homeomorphisms α_s and β_t.

pseudo-Anosov by checking that it satisfies the criterion of [BH95]. We also note that from this one can check that x, y and z are fixed points of a pseudo-Anosov representative of $\phi_{s,t}$. Moreover, for s, $t \geq 1$ the dilatation of $\phi_{s,t}$ equals the largest root of the polynomial

$$\begin{aligned} T_{s,t}(x) &= x^{t+1}(x^s(x-1)-2) + x^{s+1}(x^{-s}(x^{-1}-1)-2) \\ &= (x-1)x^{(s+t+1)} - 2(x^{s+1}+x^{t+1}) - (x-1). \end{aligned}$$

The dilatation is minimized when $s = \lfloor \frac{m}{2} \rfloor$ and $t = \lceil \frac{m}{2} \rceil$. Let us define $\phi := \phi_{\lfloor \frac{m}{2} \rfloor, \lceil \frac{m}{2} \rceil}$ and its dilatation is the largest root of the polynomial

$$\begin{aligned} T_m(x) &:= T_{\lfloor \frac{m}{2} \rfloor, \lceil \frac{m}{2} \rceil}(x) \\ &= (x-1)x^{(m+1)} - 2\left(x^{\lfloor \frac{m}{2} \rfloor + 1} + x^{\lceil \frac{m}{2} \rceil + 1}\right) - (x-1). \end{aligned}$$

Proposition 3.3.5. *If $m \geq 5$, then the largest real root of $T_m(x)$ is bounded above by $m^{\frac{3}{m}}$.*

Proof. For all m, we have $T_m(1) = -4$. It is sufficient to show that for all $x \geq m^{\frac{3}{m}}$, we have $T_m(x) > 0$. Dividing the inequality by $x^{(m+1)}$, it is equivalent to show

$$(x-1) + x^{-(m+1)} > 2\left(x^{\lfloor \frac{m}{2} \rfloor - m} + x^{\lceil \frac{m}{2} \rceil - m}\right) + x^{-m}.$$

For $m \geq 5$, one can verify the following inequalities hold for all $x \geq m^{\frac{3}{m}}$

(1) $x - 1 > \frac{3\log m}{m} \geq \frac{9}{2m}$,

(2) $x^{\lfloor \frac{m}{2} \rfloor - m} \leq x^{\lceil \frac{m}{2} \rceil - m} \leq \frac{1}{m}$,

(3) $x^{-m} \leq \frac{1}{25m}$.

Therefore,

$$\begin{aligned} (x-1) + x^{-(m+1)} > x - 1 > \frac{9}{2m} > \frac{101}{25m} &= 2(\frac{1}{m} + \frac{1}{m}) + \frac{1}{25m} \\ &\geq 2\left(x^{\lfloor \frac{m}{2} \rfloor - m} + x^{\lceil \frac{m}{2} \rceil - m}\right) + x^{-m}. \end{aligned}$$

□

Remark. Proposition 3.3.5 fails if we try to replace the bound with $c^{\frac{1}{m}}$ where c is any constant.

Remark. Hironaka and Kin [HK06] construct two infinite families of pseudo-Anosovs in $\text{Mod}(S_{0,m})$, with $\phi_{s,t}$ being one of them. Unlike $\phi_{s,t}$, the other family provides the sharp bound on $l_{0,m}$ from Theorem 3.1.3.

Next, we take a cyclic branched cover $S_{2,n}$ of $S_{0,m+2}$ with branched points x, y, and z, where $n = 5(m+1)+1$ (See Figure 3.10.). Define

$$\widetilde{X} := \{\text{marked points around } \widetilde{x}\} \text{ and } \widetilde{Y} := \{\text{marked points around } \widetilde{y}\},$$

so we have $|\widetilde{X} \cap \widetilde{Y}| = 5$, $|\widetilde{X}| = 5(s+1)$ and $|\widetilde{Y}| = 5(t+1)$.

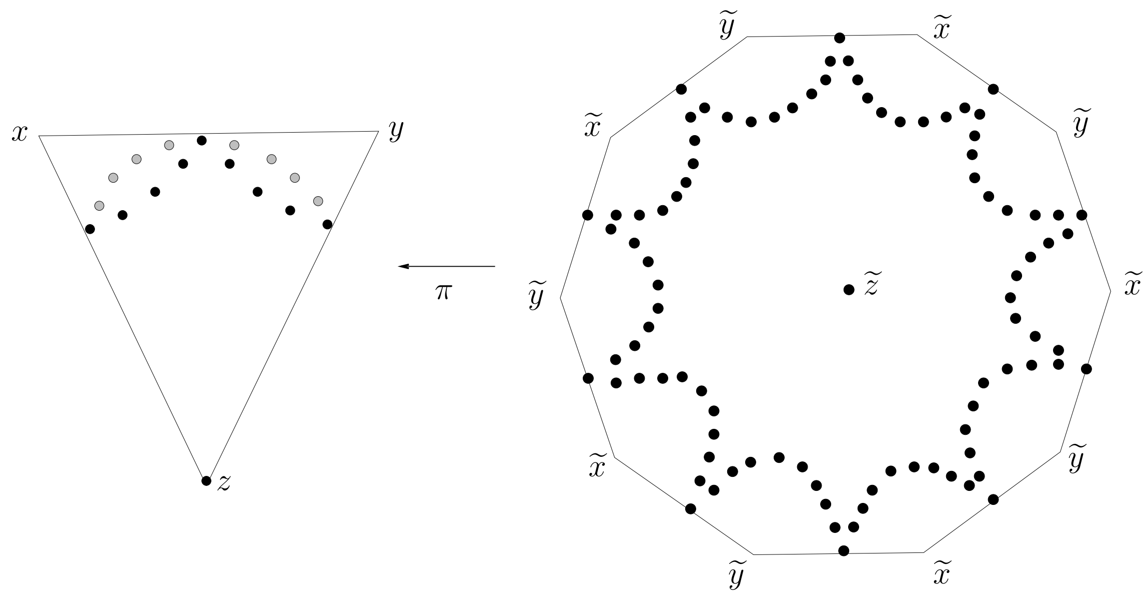

Figure 3.10: π is the covering map. To form $S_{2,n}$ from the decagon, identify the opposite sides. Then π is the quotient by the group generated by rotation of an angle $2\pi/5$.

We lift α_s, β_t to $S_{2,n}$ and call them $\widetilde{\alpha_s}$, $\widetilde{\beta_t}$, so that $\widetilde{\alpha_s}$ rotates the marked points of $\widetilde{X}$ counterclockwise around $\widetilde{x}$ and $\widetilde{\beta_t}$ rotates the marked points of $\widetilde{Y}$ clockwise around $\widetilde{y}$; see Figure 3.11. We define $\psi_{s,t} := \widetilde{\beta_t}\widetilde{\alpha_s}$. It follows that $\psi_{s,t}$ is a lift of $\phi_{s,t}$, and so is pseudo-Anosov with $\lambda(\psi_{s,t}) = \lambda(\phi_{s,t})$. An invariant train track for $\psi_{s,t}$ is obtained by lifting the one constructed in [HK06], and is shown in Figure 3.12 for $s = t = 3$.

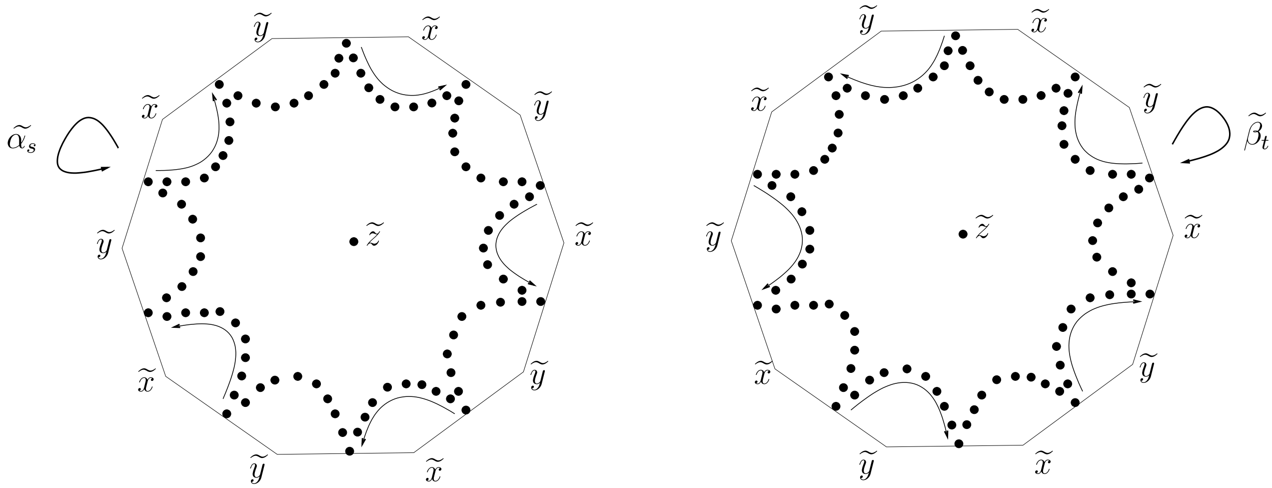

Figure 3.11: Homeomorphisms $\widetilde{\alpha}_s$ and $\widetilde{\beta}_t$.

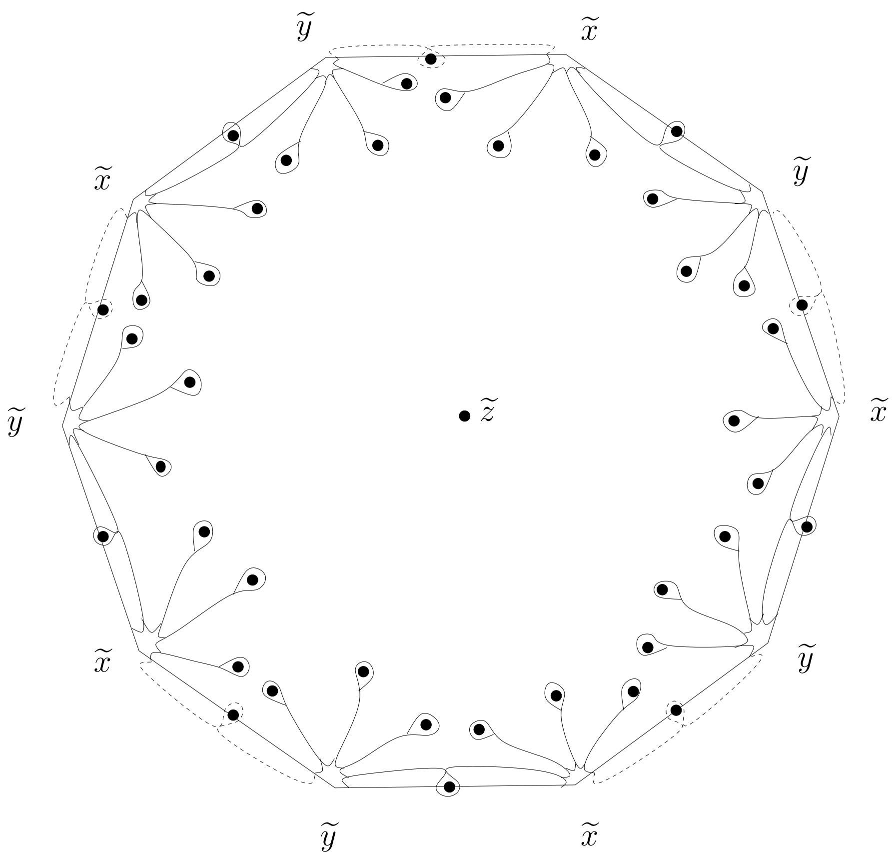

Figure 3.12: A train track for $\psi_{3,3}$

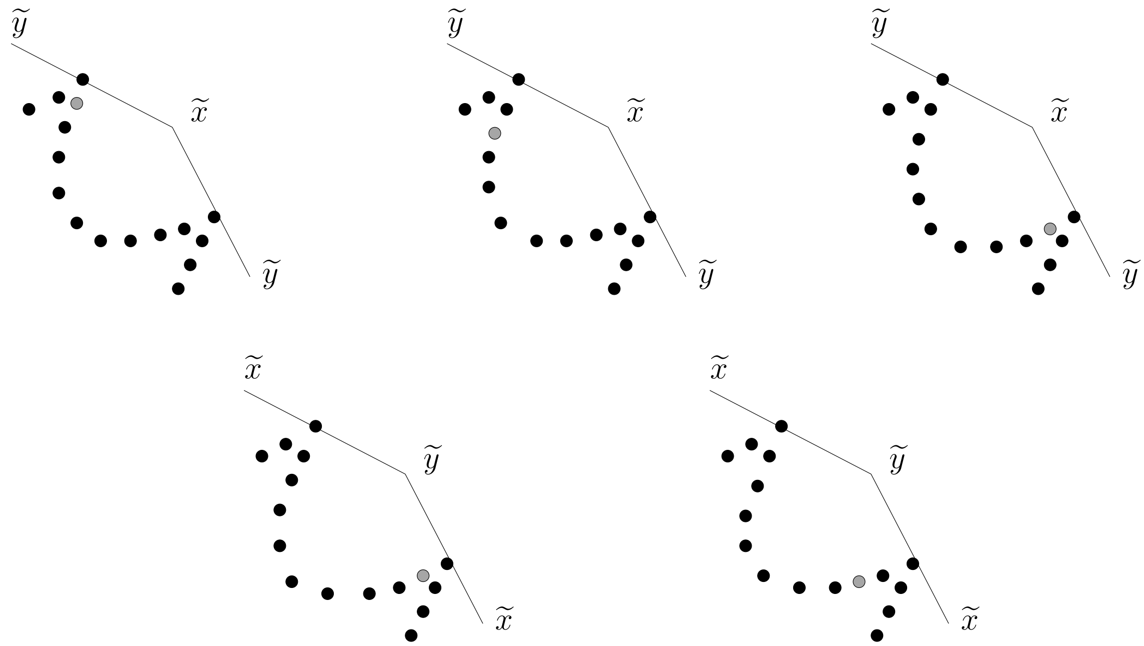

Figure 3.13: We are *not* allowed to add p_1 in the places indicated by a shaded point.

Hence for $n = 5(m+1)+1 \geq 31$, we have constructed a pseudo-Anosov $\psi = \psi_{\lfloor \frac{m}{2} \rfloor, \lceil \frac{m}{2} \rceil} \in \mathrm{Mod}(S_{2,n})$ with $\lambda(\psi) = \lambda(\phi) \leq m^{\frac{3}{m}}$ which implies

$$\log \lambda(\psi) \leq \frac{3 \log m}{m} = \frac{15 \log(n-6) - 15 \log 5}{n-6}.$$

We will now extend ψ so that n can be an arbitrary number ≥ 31. We add an extra marked point p_1 on $S_{2,n}$ between points in $\widetilde{X}$ or $\widetilde{Y}$ *except the places shown in Figure 3.13.*

Without loss of generality we assume p_1 is added in $\widetilde{X}$ to obtain $S_{2,n+1}$ and we define $\psi_1 := \widetilde{\beta}_t \widetilde{\alpha}_s{}' \in \mathrm{Mod}(S_{2,n+1})$ where $\widetilde{\alpha}_s{}'$ is extended from $\widetilde{\alpha}_s$ in the obvious way; see Figure 3.14. One can check that ψ_1 is pseudo-Anosov via the techniques of [BH95]. An invariant train track for ψ_1 is shown in Figure 3.15 and is obtained by modifying the invariant train track for ψ shown in Figure 3.12.

Next, we will show $\lambda(\psi_1) \leq \lambda(\psi)$. Let H (respectively, H_1) be the associated transition matrix of the train track map for ψ (respectively, ψ_1), and let Γ (respectively, Γ_1) be the induced directed graph as described in Section 2.3.2.

From the construction above (i.e. adding p_1), the directed graph Γ_1 is obtained by adding a vertex on the edge going out from some vertex i in Γ (that is, subdividing the

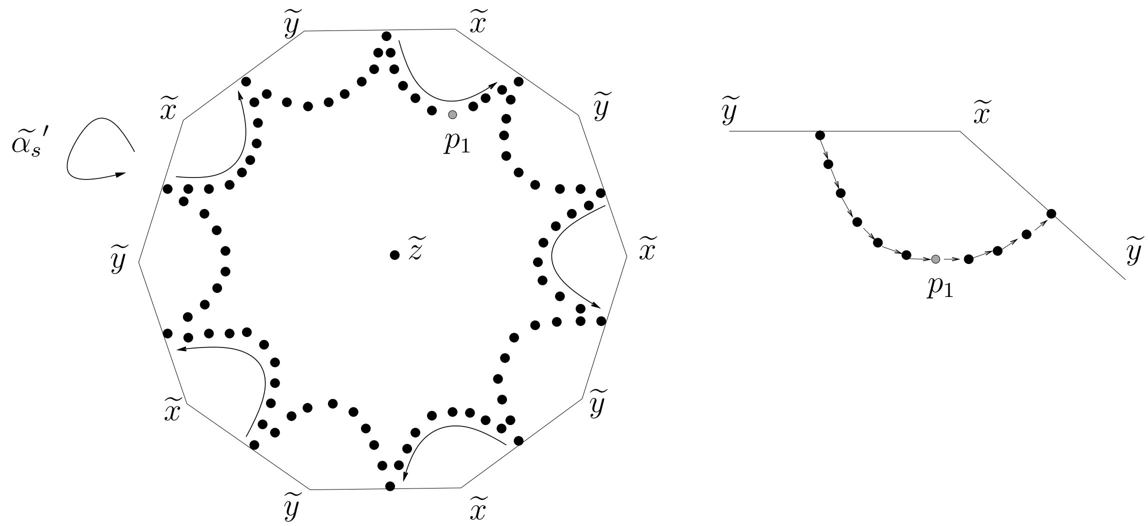

Figure 3.14: The homeomorphism $\widetilde{\alpha_s}'$. The figure on the right is a local picture near the added point p_1.

edge going out from i) where i has exactly one edge coming in and exactly one edge going out. This implies $P_{\Gamma_1}(i, k+1) = P_\Gamma(i,k)$ and

$$\sqrt[k+1]{P_{\Gamma_1}(i,k+1)} \leq \sqrt[k]{P_{\Gamma_1}(i,k+1)} = \sqrt[k]{P_\Gamma(i,k)}$$

for all k. Since H and H_1 are Perron-Frobenius matrices with Perron-Frobenius eigenvalues corresponding to the dilatations of ψ and ψ_1, and Proposition 2.3.6 tells us $\mu(H_1) \leq \mu(H)$, we have $\lambda(\psi_1) = \mu(H_1)$ is no greater than $\lambda(\psi) = \mu(H)$.

We can obtain ψ_2, ψ_3 and ψ_4 by repeating the construction above of adding more marked points without increasing dilatations (i.e. $\lambda(\psi_c) \leq \lambda(\psi)$ for $c = 1, 2, 3, 4$). Since $\frac{\log m}{m} \geq \frac{\log(m+1)}{m+1}$, we need not consider the cases with $c \geq 5$. Therefore, set $f : S_{2,n} \to S_{2,n}$ to be ψ_c, where $n = 5(m+1) + 1 + c$ with $c < 5$, and where $\psi_0 = \psi$. For $n \geq 31$, we have

$$\log \lambda(f) \leq \log \lambda(\psi) < \frac{3 \log m}{m} < \frac{3 \log \left(\frac{n-11}{5}\right)}{\left(\frac{n-11}{5}\right)},$$

where $m = \left\lfloor \frac{n-6}{5} \right\rfloor$.

Theorem 3.3.6. *There exists $\kappa_2 > 0$ such that $l_{2,n} < \dfrac{\kappa_2 \log n}{n}$, for all $n \geq 3$.*

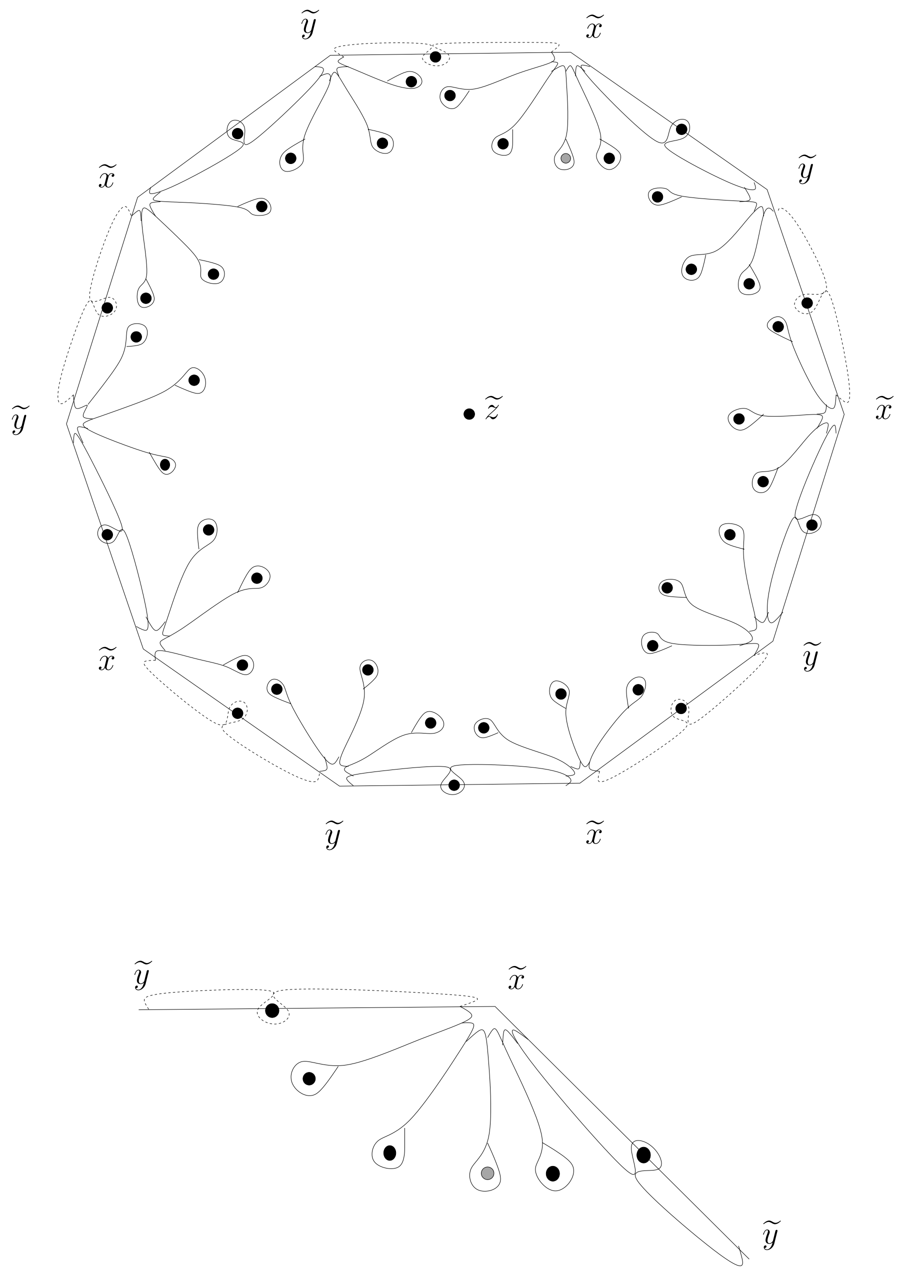

Figure 3.15: A train track for ψ_1. The figure on the bottom is a local picture.

Proof. From the discussion above, for $n \geq 31$,

$$l_{2,n} < \frac{3 \log \left(\frac{n-11}{5}\right)}{\left(\frac{n-11}{5}\right)} < \frac{\kappa_2' \log n}{n},$$

for some κ_2'. For $3 \leq n \leq 30$, let $\kappa_2'' = \max \{l_{2,3}, l_{2,4}, \cdots, l_{2,30}\}$ then

$$l_{2,n} \leq \kappa_2'' = \left(\kappa_2'' \frac{31}{\log 31}\right) \frac{\log 31}{31} < \left(\kappa_2'' \frac{31}{\log 31}\right) \frac{\log n}{n}.$$

Let $\kappa_2 := \max \left\{\kappa_2', \kappa_2'' \frac{31}{\log 31}\right\}$. □

We can generalize our construction and extend to any genus $g > 2$. For any fixed $g > 2$, we define ψ to be a homeomorphism of $S_{g,n}$ in the same fashion with $n = (2g+1)(m+1)+1$ by taking an appropriate branched cover over $S_{0,m+2}$, and we can again extend to arbitrary n by adding c extra marked points and constructing ψ_c. Define $f : S_{g,n} \to S_{g,n}$ to be ψ_c where $n = (2g+1)(m+1)+1+c$. If $n \geq 6(2g+1)+1$, then

$$\begin{aligned} \log \lambda(f) &< \frac{3 \log m}{m}, \text{ where } m = \left\lfloor \frac{n-1}{2g+1} \right\rfloor - 1 \\ &< \frac{3 \log \left(\frac{n-4g-3}{2g+1}\right)}{\left(\frac{n-4g-3}{2g+1}\right)}. \end{aligned}$$

Theorem 3.3.7. *For any fixed $g \geq 2$, there exists $\kappa_g > 0$ such that $l_{g,n} < \dfrac{\kappa_g \log n}{n}$, for all $n \geq 3$.*

Proof. This is similar to the proof of Theorem 3.3.6, where κ_g is defined to be

$$\kappa_g := \max \left\{\kappa_g', \kappa_g'' \frac{12g+7}{\log(12g+7)}\right\}.$$

□

Proof of Theorem 3.3.1. We only need to prove that the lower bounds on $\log \lambda(f)$ of Theorem 3.3.4 are bounded below by $\dfrac{\log n}{\omega_g n}$ for some ω_g depending only on g, then let

$c_g = \max\{\kappa_g, \omega_g\}$. We use the monotone decreasing property of $\dfrac{\log n}{n}$ for $n \geq 3$. Let

$$\omega'_g(\alpha) := \frac{\alpha(12g-12)}{\log 2} \times \frac{\log 3}{3} \geq \frac{\alpha(12g-12)}{\log 2} \times \frac{\log n}{n}$$

and so

$$\frac{\log 2}{\alpha(12g-12)} \geq \frac{\log n}{\omega'_g(\alpha) n}.$$

For $n \geq g-1$,

$$\frac{\log(6g+3n-6)}{2\alpha(6g+3n-6)} \geq \frac{\log 9n}{2\alpha 9n} > \frac{1}{18\alpha} \times \frac{\log n}{n}.$$

For $3 \leq n < g-1$,

$$\frac{\log(6g+3n-6)}{2\alpha(6g+3n-6)} > \frac{\log(9(g-1))}{2\alpha 9(g-1)} > \frac{\log g}{18\alpha g} \times \frac{3}{\log 3} \times \frac{\log n}{n}.$$

Let $\omega_g := \max\left\{\omega'_g(\alpha), 18\alpha, \dfrac{6\alpha g \log 3}{\log g}\right\}$, where $0 \leq \alpha \leq \Theta(g)$. $\square$

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